Problem 1. Functions f, g, and h are twice-differentiable functions with $g(5) = h(5) = 1$. The line $y = 1 - \frac{5}{3}(x - 5)$ is tangent to both the graph of g at $x = 5$ and the graph of h at $x = 5$.

(a) Find $g'(5)$.

$$g'(5) = -\frac{5}{3}$$

(b) Let b be the function given by $b(x) = 2x^2 g(x)$. Write an expression for $b'(x)$. Find $b'(5)$.

$$b'(x) = 4xg(x) + 2x^2 g'(x)$$

$$b'(5) = 20(1) + 50\left(-\frac{5}{3}\right)$$

$$= -\frac{190}{3}$$

(c) Let w be the function given by $w(x) = \frac{3h(x) - x}{2x + 1}$. Write an expression for $w'(x)$. Find $w'(5)$.

$$w'(x) = \frac{(3h'(x) - 1)(2x+1) - 2(3h(x) - x)}{(2x+1)^2}$$

$$w'(5) = \frac{\left(3\left(-\frac{5}{3}\right) - 1\right)(11) - 2(3 - 5)}{11^2}$$

$$= -\frac{62}{11^2}$$

$$\frac{d}{dx}G(u(x)) \quad \boxed{\frac{dG}{dx} = \frac{dG}{du}\frac{du}{dx}}$$

Problem 1 (continued). Functions f, g, and h are twice-differentiable functions with $g(5) = h(5) = 1$. The line $y = 1 - \frac{5}{3}(x-5)$ is tangent to both the graph of g at $x = 5$ and the graph of h at $x = 5$.

(d) Let $M(x) = \frac{d}{dx}\left[\int_0^{2x} g(t)\,dt\right]$. Write an expression for $M'(x)$. Find $M'(2.5)$.

Let $G(u) = \int_0^u g(t)\,dt$. Then $\frac{dG}{du} = g(u)$ by FTC.

Let $u = 2x$: So $M(x) = \frac{d}{dx}G(u) = \frac{dG}{du}\frac{du}{dx} = 2g(u)$.

$= 2g(2x)$.

So $M'(x) = \frac{d}{dx}2g(2x) = 4g'(2x)$

So $M'(2.5) = 4g'(5) = 4\left(-\frac{5}{3}\right)$

$= -\frac{20}{3}$

(e) Let $M(x) = \frac{d}{dx}\left[\int_0^{2x} g(t)\,dt\right]$. It is known that $c = 2.5$ satisfies the conclusion of the Mean Value Theorem applied to $M(x)$ on the interval $1 \le x \le 4$. Use $M'(2.5)$ to find $g(8) - g(2)$.

$$-\frac{20}{3} = M'(2.5) = M'(c) = \frac{M(4) - M(1)}{4-1}$$

$$= \frac{2g(8) - 2g(2)}{3}$$

So $2(g(8) - g(2)) = -20$

So $\boxed{g(8) - g(2) = -10}$

Problem 1 (continued). Functions f, g, and h are twice-differentiable functions with $g(5) = h(5) = 1$. The line $y = 1 - \frac{5}{3}(x-5)$ is tangent to both the graph of g at $x = 5$ and the graph of h at $x = 5$.

(f) The function g satisfies $g(x) = \dfrac{x + 5\cos\left(\frac{\pi}{5}x\right)}{3 - \sqrt{f(x)}}$ for $x \neq 5$. It is known that $\lim_{x\to 5} g(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 5} g(x)$ to find $f(5)$ and $f'(5)$. Show the work that leads to your answers.

Since the numerator and denominator at continuous,

We see $\lim_{x\to 5} x + 5\cos\left(\frac{\pi}{5}x\right) = 5 + 5\cos(\pi) = 0$.

So, $0 = \lim_{x\to 5} 3 - \sqrt{f(x)} = 3 - \sqrt{f(5)} \Rightarrow f(5) = 9$.

Now, since g is also continuous,

$$1 = g(5) = \lim_{x\to 5} \frac{x + 5\cos\left(\frac{\pi}{5}x\right)}{3 - \sqrt{f(x)}} = \lim_{x\to 5} \frac{1 - \pi\sin\left(\frac{\pi}{5}x\right)}{-\frac{f'(x)}{2\sqrt{f(x)}}}$$

$$= \frac{1-0}{-\frac{f'(5)}{2\sqrt{9}}}.$$ So $\boxed{f'(5) = -6}$

(g) It is known that $h(x) \leq g(x)$ for $4 < x < 6$. Let k be a function satisfying $h(x) \leq k(x) \leq g(x)$ for $4 < x < 6$. Is k continuous at $x = 5$? Justify your answer.

WTS: ① $f(5) = 1$ ② $\lim_{x\to 5} k(x) = 1$

$1 = h(5) \leq k(5) \leq g(5) = 1$, thus $k(5) = 1$.

Also, $1 = \lim_{x\to 5} h(x)$ and $1 = \lim_{x\to 5} g(x)$, which we knowing since handy are continuous!

so by the Squeeze Theorem,

$\lim_{x\to 5} k(x) = 1$.

Thus $\lim_{x\to 5} k(x) = k(5)$, and k is continuous at $x = 5$.

① Some pts partial sol.

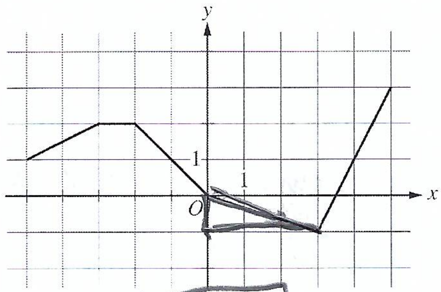
Zero if you run out of time.

② Use only 1 side of a page.

Problem 2. Let f be the function defined by $f(x) = \sin(\pi x) + \ln(2 - x)$. Let g be a twice differentiable function. The table below gives values of g and its derivative g' at selected values of x. Let h be the function whose graph, consisting of five line segments, is shown in the figure below.

x	$g(x)$	$g'(x)$
−5	10	−3
−4	5	−1
−3	2	4
−2	3	1
−1	1	−2
0	0	−3

g



Graph of h

(a) Find the slope of the line tangent to the graph of f at $x = 1$.

$$f'(x) = \pi\cos(\pi x) + \frac{-1}{2-x}$$

So $f'(1) = \boxed{-\pi - 1}$

(b) Let k be the function defined by $k(x) = h(f(x) + 2)$. Find $k'(1)$.

$$k'(x) = h'(f(x)+2)(f'(x))$$

So $k'(1) = h'(f(1)+2)f'(1)$

$= h'(2)f'(1)$

$= \left(-\frac{1}{3}\right)(-\pi - 1) = \frac{\pi+1}{3}$

$f(1) = \sin(\pi) + \ln(2-1) = 0$

Problem 2. Let f be the function defined by $f(x) = \sin(\pi x) + \ln(2 - x)$. Let g be a twice differentiable function. The table below gives values of g and its derivative g' at selected values of x. Let h be the function whose graph, consisting of five line segments, is shown in the figure below.

x	$g(x)$	$g'(x)$
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–1	1	–2
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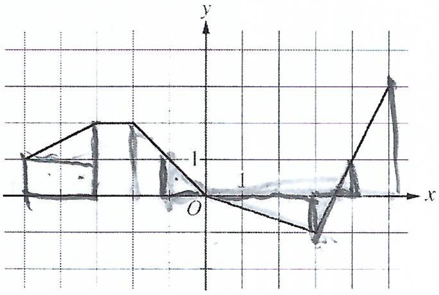

Graph of h

(c) Evaluate $\int_{-5}^{-1} g'(x)\,dx$.

By FTC, $\int_{-5}^{-1} g'(x)\,dx = g(-1) - g(-5)$

$= 1 - 10 = -9$

x_i Δx

(d) Rewrite $\lim_{n\to\infty} \sum_{k=1}^{n} \left(h\left(-1 + \frac{5k}{n}\right)\right) \cdot \frac{5}{n}$ as a definite integral in terms of $h(x)$ with a lower bound of $x = -1$. Evaluate the definite integral.

Take interval [a, b] break into n pieces equal size.

$\boxed{\Delta x = \frac{b-a}{n}} \quad \boxed{x_i = x_0 + i\Delta x}$

x_0 $x_0 + \Delta x$

$x_0 = -1$ total width = $b - a$

$(b-a) = 5$ so $b = x_n = 4$

Get $\int_{-1}^{4} h(x)\,dx = \cancel{3 + 2 + 2 - \frac{1}{2}\left(\frac{7}{2}\right)(1) + \frac{9}{4}}$

I think.

$= -1$

$\cancel{= 7 - \frac{7}{4} + \frac{9}{4}}$

$\cancel{= \frac{1}{2} - \frac{7}{4} = -\frac{5}{4}}$

$\cancel{= 7 - \frac{7}{4} = \frac{21}{4}}$

$\cancel{= \frac{3}{2}}$

Problem 2 (continued). Let f be the function defined by $f(x) = \sin(\pi x) + \ln(2 - x)$. Let g be a twice differentiable function. The table below gives values of g and its derivative g' at selected values of x. Let h be the function whose graph, consisting of five line segments, is shown in the figure below.

x	$g(x)$	$g'(x)$
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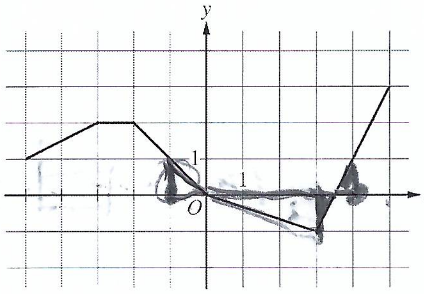

Graph of h

$$\int_{-1}^{4} h(x)\,dx = \frac{1}{2} - \frac{3}{2} = -1$$

(e) What is the fewest number of horizontal tangents $g(x)$ has on the interval $-5 < x < 0$? Justify your answer.

Since g' is continuous, IVT says g' has a zero between $x=-4$ and $x=-3$ and also between $x=-2$ and $x=-1$.

So, g' has at least 2 zeros,

So g has at least 2 horizontal tangents.