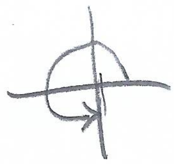
AP Calculus AB
Dr. Paul L. Bailey

Homework 0508g
Friday, May 8, 2020

Problem 1. Let f be the function defined by $f(x) = e^x \cos x$.

(a) Find the average rate of change of f on the interval $0 \leq x \leq \pi$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{-e^{\pi} - 1}{\pi}$$

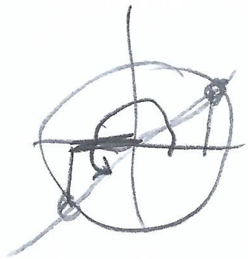


(b) What is the slope of the line tangent to the graph of f at $x = \frac{3\pi}{2}$?

$$f(x) = e^x \cos x$$

$$f'(x) = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$$

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2}(1) = e^{3\pi/2}$$



Problem 1. Let f be the function defined by $f(x) = e^x \cos x$.

(c) Find the absolute minimum value of f on the interval $0 \le x \le 2\pi$. Justify your answer.

Plug all cp's and ep's into f.

$f'(x) = e^x(\cos x - \sin x) = 0 \Rightarrow \cos x = \sin x$

This occurs at $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

$f(0) = 1$

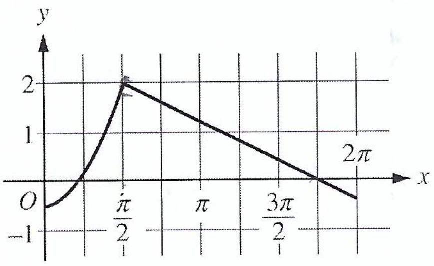
$f(2\pi) = e^{2\pi}$

$f(\frac{\pi}{4}) = e^{\pi/4}\left(\frac{\sqrt{2}}{2}\right)$

$f(\frac{5\pi}{4}) = e^{5\pi/4}\left(-\frac{\sqrt{2}}{2}\right)$

The minimum value is $-e^{5\pi/4} \cdot \frac{\sqrt{2}}{2}$ which occurs at $x = \frac{5\pi}{4}$.

(d) Let g be a differentiable function such that $g\left(\frac{\pi}{2}\right) = 0$. The graph of g', the derivative of g, is shown below.



Graph of g'

Find the value of $\lim_{x \to \pi/2} \frac{f(x)}{g(x)}$, or state that it does not exist. Justify your answer.

Note f and g are continuous at $\frac{\pi}{2}$

We know $g(\frac{\pi}{2}) = 0$ and $f(\frac{\pi}{2}) = e^{\pi/2} \cos\frac{\pi}{2} = 0$.

Dear AP reader I will not write $\cancel{\frac{0}{0}}$

So we use L'Hospital's rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim_{x \to \frac{\pi}{2}} \frac{f'(x)}{g'(x)} = \lim_{x \to \frac{\pi}{2}} \frac{e^x(\cos x - \sin x)}{g'(x)}$$

$$= \frac{e^{\pi/2}(\cos\frac{\pi}{2} - \sin\frac{\pi}{2})}{2} = -\frac{e^{\pi/2}}{2}$$

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Problem 1. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x-2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

$h'(2) = \text{slope}\left(4 + \frac{2}{3}(x-2)\right) = \frac{2}{3}$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression of $a'(x)$. Find $a'(2)$.

$a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 36(4) + 24\left(\frac{2}{3}\right)$

$= 160$

Problem 1. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x-2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x\to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

Since f is cont, we know $1-(f(x))^3 = 0$ when $x=2$.

So, $f(2)^3 = 1$, so $f(2) = 1$.

maybe not

Also, we know $h'(2) = \frac{2}{3}$,

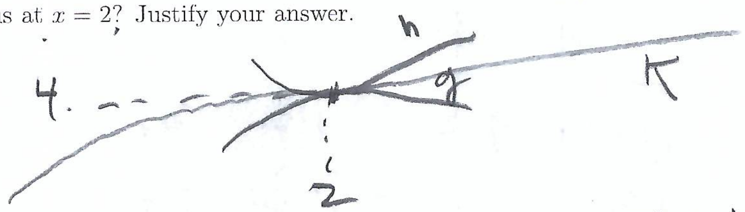
so $h'(x) = \dfrac{2x(1-(f(x))^3) + (x^2-4)\,3f(x)f'(x)}{(1-(f(x))^3)^2}$

Since h is cont,

$4 = h(2) = \lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{2x}{-3f(x)^2 f'(x)} = \dfrac{4}{-3f(2)^2 f'(2)}$ So $f'(2) = \dfrac{1}{-3f(2)^2}$

$= -\dfrac{1}{3}$

(d) It is known that $g(x) \le h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \le k(x) \le h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.



Since $g(x) \le K(x) \le h(x)$ and $4 = g(2) \le K(2) \le h(2) = 4$ we know $K(2) = 4$.

$\lim_{x\to 2} g(x) = 4 = \lim_{x\to 2} h(x)$,

By the Squeeze Theorem, we know

$\lim_{x\to 2} K(x) = 4$.

Since $\lim_{x\to 2} K(x) = K(2)$, K is continuous.

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Homework 0508i
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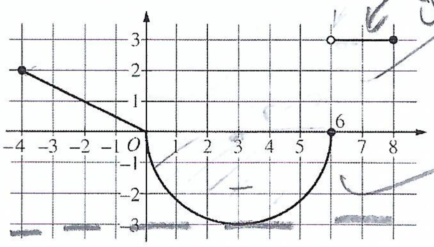
Problem 1. The function g is defined on the closed interval $[-4, 8]$. The graph of g consists of two linear pieces and a semicircle, as show in the figure below.

$g(7) = 3$

$-\frac{(3)^2\pi}{2}$

$y = -3$



Graph of g

using graph of g, how would you find $\int_0^7 g(t)$ = signed area under

Let f be the function defined by $f(x) = 3x + \int_0^x g(t)\,dt$.

(a) Find $f(7)$ and $f'(7)$.

$f(7) = 3(7) + \boxed{\int_0^7 g(t)\,dt}$

$= 21 + -\frac{9\pi}{2} + 3 = 24 - \frac{9\pi}{2}$

$f'(7) = 3 + g(7) = 6$

$f'(x) = 3 + \frac{d}{dx}\left(\int_0^x g(t)\,dt\right) = 3 + g(x)$

(b) Find the value of x in the closed interval $[-4, 3]$ at which f attains its maximum value. Justify your answer.

slope of $f > 0$ | slope of $f < 0$

$f' > 0$ to $f' < 0$

$3 + g(x) > 0$ to $3 + g(x) < 0$

$g(x) > -3$ to $g(x) < -3$

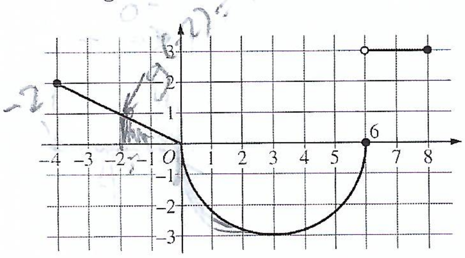
$g(x) \geq -3$ for all $x \in [-4, 3]$

$f'(x) \geq 0$ for all $x \in [-4, 3]$

f is increasing for all $x \in [-4, 3]$

abs max must occur at $x = 3$

Problem 1. The function g is defined on the closed interval $[-4, 8]$. The graph of g consists of two linear pieces and a semicircle, as show in the figure below.



Graph of g

Let f be the function defined by $f(x) = 3x + \int_0^x g(t)\,dt$.

(c) For each of $\lim_{x\to 0^-} g'(x)$ and $\lim_{x\to 0^+} g'(x)$, find the value or state that it does not exist.

$g'(x) =$ slope of g

$$\lim_{x\to 0^-} g'(x) = -\frac{2}{4} = \cancel{-\frac{1}{2}}$$

$$\lim_{x\to 0^+} g'(x) = \text{DNE} = +\infty$$

(d) Find $\lim_{x\to -2} \dfrac{f(x)+7}{e^{3x+6}-1}$.

continuous

$$\lim_{x\to -2} f(x)+7 = f(-2) + 7 = \underbrace{\left(3(-2) + \int_0^{-2} g(t)\,dt\right)}_{f(-2)} + 7$$

$$= -6 + -\int_{-2}^{0} g(t)\,dt + 7$$

$$= 1 - \frac{(2)(1)}{2} = \boxed{1 - 1 = 0}$$

$$\lim_{x\to -2} e^{3x+6} - 1 = e^{3(-2)+6} - 1 = e^0 - 1 = 0$$

Use L'Hôpital's rule

$$\lim_{x\to -2} \frac{f(x)+7}{e^{3x+6}-1} = \lim_{x\to -2} \frac{f'(x)}{3e^{3x+6}}$$

$$= \lim_{x\to -2} \frac{3+g(x)}{3e^{3x+6}}$$

$$= \frac{3+g(-2)}{3e^{3(-2)+6}} = \boxed{\frac{3+1}{3}}$$

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Question 5

(a) The average rate of change of f on the interval $0 \le x \le \pi$ is $\dfrac{f(\pi) - f(0)}{\pi - 0} = \dfrac{-e^{\pi} - 1}{\pi}$.

1 : answer

(b) $f'(x) = e^x \cos x - e^x \sin x$

$$f'\left(\frac{3\pi}{2}\right) = e^{3\pi/2} \cos\left(\frac{3\pi}{2}\right) - e^{3\pi/2} \sin\left(\frac{3\pi}{2}\right) = e^{3\pi/2}$$

The slope of the line tangent to the graph of f at $x = \dfrac{3\pi}{2}$ is $e^{3\pi/2}$.

$2 : \begin{cases} 1 : f'(x) \\ 1 : \text{slope} \end{cases}$

(c) $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow x = \dfrac{\pi}{4}, x = \dfrac{5\pi}{4}$

x	$f(x)$
0	1
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} e^{\pi/4}$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}} e^{5\pi/4}$
2π	$e^{2\pi}$

The absolute minimum value of f on $0 \le x \le 2\pi$ is $-\dfrac{1}{\sqrt{2}} e^{5\pi/4}$.

$3 : \begin{cases} 1 : \text{sets } f'(x) = 0 \\ 1 : \text{identifies } x = \frac{\pi}{4}, x = \frac{5\pi}{4} \\ \quad \text{as candidates} \\ 1 : \text{answer with justification} \end{cases}$

(d) $\displaystyle\lim_{x \to \pi/2} f(x) = 0$

Because g is differentiable, g is continuous.

$\displaystyle\lim_{x \to \pi/2} g(x) = g\left(\frac{\pi}{2}\right) = 0$

By L'Hospital's Rule,

$$\lim_{x \to \pi/2} \frac{f(x)}{g(x)} = \lim_{x \to \pi/2} \frac{f'(x)}{g'(x)} = \frac{-e^{\pi/2}}{2}.$$

$3 : \begin{cases} 1 : g \text{ is continuous at } x = \frac{\pi}{2} \\ \quad \text{and limits equal } 0 \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Note: max $1/3$ [1-0-0] if no limit notation attached to a ratio of derivatives

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Question 6

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

(c) Because h is differentiable, h is continuous, so $\lim_{x \to 2} h(x) = h(2) = 4$.

Also, $\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim_{x \to 2} \left(x^2 - 4\right) = 0$, we must also have $\lim_{x \to 2} \left(1 - (f(x))^3\right) = 0$.

Thus $\lim_{x \to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim_{x \to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim_{x \to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x \to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus $f'(2) = -\frac{1}{3}$.

$4 : \begin{cases} 1 : \lim_{x \to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$

(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x \to 2} g(x) = g(2) = 4$ and $\lim_{x \to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x \to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

1 : continuous with justification

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Question 3

(a) $f(7) = 3 \cdot 7 + \int_0^7 g(t)\,dt = 21 - \frac{9\pi}{2} + 3 = 24 - \frac{9\pi}{2}$ $f'(7) = 3 + g(7) = 3 + 3 = 6$	$2 : \begin{cases} 1 : f(7) \\ 1 : f'(7) \end{cases}$
(b) On the interval $-4 \le x \le 3$, $f'(x) = 3 + g(x)$. Because $f'(x) \ge 0$ for $-4 \le x \le 3$, f is nondecreasing over the entire interval, and the maximum must occur when $x = 3$.	2 : answer with justification
(c) $\lim\limits_{x \to 0^-} g'(x) = -\frac{1}{2}$ $\lim\limits_{x \to 0^+} g'(x)$ does not exist.	$2 : \begin{cases} 1 : \text{left-hand limit} \\ 1 : \text{right-hand limit} \end{cases}$
(d) $\lim\limits_{x \to -2} (f(x) + 7) = -6 + \int_0^{-2} g(t)\,dt + 7 = 0$ $\lim\limits_{x \to -2} \left(e^{3x+6} - 1\right) = 0$ Using L'Hospital's Rule, $\lim\limits_{x \to -2} \frac{f(x) + 7}{e^{3x+6} - 1} = \lim\limits_{x \to -2} \frac{f'(x)}{3e^{3x+6}} = \frac{3 + g(-2)}{3} = \frac{3+1}{3} = \frac{4}{3}.$	$3 : \begin{cases} 1 : \text{limits equal 0} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$ Note: max 1/3 [1-0-0] if no limit notation attached to a ratio of derivatives

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