AP CALCULUS AB
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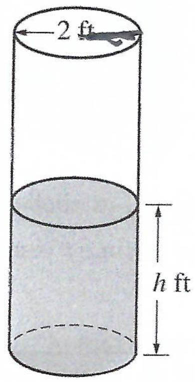
Homework 0510
Sunday, May 10, 2020

Problem 1. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as show in the figure below.



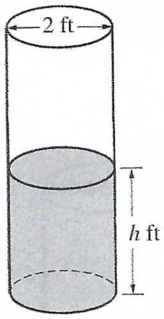
The water drains out through a valve (not shown) at thebottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds.

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

$V = \pi r^2 h = \pi h$ choose $\frac{dV}{dt}$ when $h=4$.

$$\frac{dV}{dt} = \pi \frac{dh}{dt} = \pi\left(-\frac{1}{10}\sqrt{h}\right)\Big|_{h=4} = -\frac{\pi}{5}\,\frac{ft^3}{sec}$$

Problem 1 (continued). A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as show in the figure below.



The water drains out through a valve (not shown) at thebottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

$$\frac{d^2h}{dt^2} = -\frac{1}{10}\,\frac{1}{2\sqrt{h}}\,\frac{dh}{dt}$$

$$= \frac{1}{200} > 0.$$

Since the derivative of $\frac{dh}{dt}$ is positive, $\frac{dh}{dt}$ is increasing.

(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

We know $h(0) = 5$.

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$

$$2\sqrt{h} = -\frac{t}{10} + C$$

$$2\sqrt{5} = C$$

So,

$$2\sqrt{h} = -\frac{t}{10} + 2\sqrt{5}$$

$$\sqrt{h} = \sqrt{5} - \frac{t}{20}$$

$$h = \left(\sqrt{5} - \frac{t}{20}\right)^2$$

Problem 2. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^{\text{deg}}$C) at time $t = 0$, and the internal temperature of the potato is greater than 27^{deg}C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the interval temperature of the potato at time $t = 3$.

$$\left.\frac{dH}{dt}\right|_{t=0} = -\frac{1}{4}(91-27) = -\frac{64}{4} = -16 \ {}^{\circ}\text{C}/\text{min}$$

$$L(t) = -16(t) + 91$$

$$L(3) = -48 + 91 = 43^{\circ}\text{C}$$

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$$\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H-27).$$

Since $H > 27$, $\frac{d^2H}{dt^2}$ is positive,

so H is concave up,

So $L(3)$ is an underestimate.

Problem 2. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^{\text{deg}}$C) at time $t = 0$, and the internal temperature of the potato is greater than 27$^{\text{deg}}$C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(c) For $t < 0$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\frac{dG}{dt} = -(G-27)^{2/3}$$

$$\int \frac{dG}{(G-27)^{2/3}} = -\int dt$$

$$\begin{array}{r} 91 \\ 27 \\ \hline 64 \end{array}$$

$$3(G-27)^{1/3} = -t + C$$

$$\cancel{G(0) = 91 \Rightarrow \quad 3\,3 = -91 + C}$$

$G(0)=91$ $\quad 3(64)^{1/3} = C \Rightarrow C = 12$

So $\quad (G-27)^{1/3} = 4 - \frac{t}{3}$

$$G - 27 = \left(\frac{12-t}{3}\right)^3$$

$$\boxed{G = 27 + \left(\frac{12-t}{3}\right)^3}$$

$\cancel{G(0) = 27+}$

So $G(3) = 27 + 27 = 54$

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Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$2 : \begin{cases} 1 : \dfrac{dV}{dt} = \pi \dfrac{dh}{dt} \\ 1 : \text{answer with units} \end{cases}$

(b) $$\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$$

Because $\dfrac{d^2h}{dt^2} = \dfrac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3 : \begin{cases} 1 : \dfrac{d}{dh}\left(-\dfrac{1}{10}\sqrt{h}\right) = -\dfrac{1}{20\sqrt{h}} \\ 1 : \dfrac{d^2h}{dt^2} = -\dfrac{1}{20\sqrt{h}} \cdot \dfrac{dh}{dt} \\ 1 : \text{answer with explanation} \end{cases}$

(c) $$\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad \text{and uses initial condition} \\ 1 : h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables