RESEARCH SUMMARY

PAUL L. BAILEY

1. Motivation

The initial responsibility of algebra is the solution of polynomial equations. The quadratic formula for producing the roots of a degree two polynomial was known in ancient times. During the European Renaissance, a formula for computing the roots of a cubic polynomial was discovered. The search began for solutions of higher degree polynomial equations.

Early in the nineteenth century, Galois gave necessary and sufficient conditions that the roots of a polynomial with rational coefficients be expressible by an algebraic formula. His criterion introduced a group of permutations of its roots, now known as the Galois group of the polynomial. This motivated the eventual definition of abstract group. Once abstracted, it was natural to ask if anything new had been introduced by the abstraction, which led to an unsolved problem in mathematics, the *Inverse Galois Problem*; is every finite group the Galois group of a polynomial with rational coefficients?

Meanwhile, others were pushing the frontiers of calculus. It was realized that phenomena hidden in the world of the real line were illuminated by considering complex plane, and that additional information is obtained by adding a single point ∞ to the plane to obtain the Riemann sphere.

In an attempt to solve integrals such as $\int \frac{dx}{\sqrt{x^3+cx+d}}$, which were known as elliptic integrals, Abel discovered that this antiderivative is best understood when viewed as the inverse of a function whose domain is the set of points (z, w) which satisfy the equation $w^2 = z^3 + cz + d$, where z and w are complex variables. Such a set of points became known as an *elliptic curve*, which naturally maps onto the Riemann sphere by sending (z, w) to z. Later, Riemann generalized this by considering sets of points defined by any polynomial in two variables; such a set became known as a *Riemann surface*.

To obtain information about families of elliptic curves, *modular curves* were developed; these are Riemann surfaces whose points correspond to elliptic curves and the maps between them. It is known that functions between objects induce functions between associated *moduli spaces* (spaces whose points correspond to objects). Thus functions between elliptic curves induce functions between modular curves, a process that forms infinite sequences of modular curves.

Hurwitz spaces are moduli spaces of ramified covers of the Riemann sphere. Michael Fried's *Modular Towers* are towers of Hurwitz spaces which simultaneously generalize the classical towers of modular curves and test our penetration of the inverse Galois problem. My research studies the maps between Hurwitz spaces, such as those which constitute a Modular Tower.

Date: March 9, 2003.

2. History

2.1. **Simple Branching.** In 1891, Hurwitz [Hu91] considered the moduli space of ramified covers of the Riemann sphere with degree n and simple branching; that is, exactly one ramification point of index two over each branch point. The monodromy group of such a cover is S_n, with ramification coming from the conjugacy class of transpositions. In particular, it was shown that these spaces are connected. We now denote Hurwitz's spaces by $\mathcal{H}(S_n, \boldsymbol{C}_{2^r})^{\mathrm{ab}}$, where C_2 is the conjugacy class of transposition, and r is the number of branch points.

2.2. **Absolute Hurwitz Spaces.** In 1977, Fried [Fr77] proposed using a generalization of this idea, with any group G replacing S_n and any tuple $\boldsymbol{C}$ of conjugacy classes of ramification from G replace transpositions, for application to the Inverse Galois Problem. Denote the moduli space of such covers by $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{ab}}$. We now call this an *absolute Hurwitz space.*

Each point on an absolute Hurwitz space corresponds to a strong equivalence class of covers; we say that $\varphi_1 : Y_1 \to \mathbb{P}^1$ is strongly equivalent to $\varphi_2 : Y_2 \to \mathbb{P}^1$ if there exists an isomorphism $\xi : Y_1 \to Y_2$ such that $\varphi_1 = \varphi_2 \circ \xi$.

Let r denote the number of conjugacy classes in $\boldsymbol{C}$; this is the number of branch points for the covers being parameterized, and is called the *rank* of the tuple. The rank is equal to the dimension of the Hurwitz space.

As is common with moduli spaces, the initial question about $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{ab}}$ is whether or not it is connected.

2.3. **Reduced Hurwitz Spaces.** In 1987, Fried [Fr87] discusses the moduli space of weak equivalence classes of covers; this is a *reduced Hurwitz space*, denoted $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{ab,rd}}$.

We say that $\varphi_1 : Y_1 \to \mathbb{P}^1$ is weakly equivalent to $\varphi_2 : Y_2 \to \mathbb{P}^1$ if there exists an isomorphism $\xi : Y_1 \to Y_2$ and a linear fractional transformation $\alpha \in \mathrm{PSL}_2(\mathbb{C})$ such that $\alpha \circ \varphi_1 = \varphi_2 \circ \xi$. Thus reduced Hurwitz spaces are formed by modding out by the action of $\mathrm{PSL}_2(\mathbb{C})$; this cuts the dimension by three. When the rank $r = 4$, reduced Hurwitz spaces have dimension one, and are quotients of the upper half plane by a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ which naturally cover the j-line, $\mathbb{P}^1_j$.

2.4. **Inner Hurwitz Spaces.** In 1991, Fried and Volklein [FV91] studied moduli spaces of static covers. A static cover is a normal ramified cover $\varphi : Y \to X$ together with an isomorphism $\tau : G \to \mathrm{Aut}(\varphi)$. Denote the moduli space by $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$. This is called an *inner Hurwitz space.*

If G is centerless, these spaces have the property that the rational points on them correspond to regular realizations of G as a Galois group over $\mathbb{Q}$. [FV91] shows that every group is covered by a centerless group.

2.5. **Modular Towers.** In 1995, Fried [Fr95] introduced Modular Towers, which are towers of Hurwitz spaces. This construction generalizes of towers of modular curves.

The open modular curves $Y_1(n)$ and $Y_0(n)$ are, respectively, $\mathcal{H}(D_n, \boldsymbol{C}_{2^4})^{\mathrm{in,rd}}$ and $\mathcal{H}(D_n, \boldsymbol{C}_{2^4})^{\mathrm{ab,rd}}$. The group cover $D_{p^{k+1}} \to D_{p^k}$ induces a cover of Riemann surfaces $Y_1(p^{k+1}) \to Y_1(p^k)$, producing a tower of modular curves.

To generalize this, consider that the map $D_{p^{k+1}} \to D_{p^k}$ has these properties:

(1) $\ker = \mathbb{Z}/p$;

(2) it is a *Frattini cover.*

A homomorphism $f : H \to G$ is a Frattini cover if any lift of any set of generators for G generates H. Indeed, $D_{p^{k+1}} \to D_{p^k}$ is versal for covers of D_{p^k} among Frattini covers with elementary p-group kernel. Every group G admits such a cover ${}_p^1\tilde{G}$, its *universal elementary p-Frattini cover.* Replace D_p with any group G, and replace the homomorphisms $D_{p^{k+1}} \to D_{p^k}$ with ${}_p^{k+1}\tilde{G} \to {}_p^k\tilde{G}$. This produces a sequence $\mathbf{MT}_p(G, \boldsymbol{C})^{\mathrm{in,rd}}$ of Riemann surfaces

$$\cdots \mathcal{H}({}_p^{k+1}\tilde{G}, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathcal{H}({}_p^k\tilde{G}, \boldsymbol{C})^{\mathrm{in,rd}} \to \cdots \to \mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathbb{P}^1_j.$$

Implicit in this construction is the fact that conjugacy classes of elements whose order is prime to p lift uniquely through a p-Frattini cover. Call $\mathcal{H}({}_p^k\tilde{G}, \boldsymbol{C})^{\mathrm{in,rd}}$ the k^{th} *level* of the Modular Tower. The main conjecture for reduced rank four Modular Towers is that the genus of every component becomes unbounded as the level increases.

2.6. **Level One of $\mathbf{MT}_2(A_5, \boldsymbol{C}_{3^4})^{\mathrm{in,rd}}$.** The main example of [BF02] is level one of the Modular Tower $\mathbf{MT}(A_5, \boldsymbol{C}_{3^4})^{\mathrm{in,rd}}$, where $\boldsymbol{C}_{3^4}$ contains four conjugacy classes of three cycles in A_5. This investigation started after I developed a computer program in [GAP] to compute the group ${}_1^2\tilde{A}_5$, the Nielsen class $\mathrm{Ni}({}_1^2\tilde{A}_5, \boldsymbol{C}_{3^4})^{\mathrm{in}}$, and the action of the braid group on this Nielsen class, showing that that $\mathcal{H}({}_1^2\tilde{A}_5, \boldsymbol{C}_{3^4})^{\mathrm{in}}$ consists of two components of genus 12 and 9. We developed tools such as the *shift-incidence matrix* and explored *spin separation* to explain these phenomena and prove these statements.

2.7. **Level One of $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$.** My dissertation [Ba02] investigates the Modular Tower $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$, where $\boldsymbol{C}_{3^2_\pm}$ contains one pair each of the conjugacy classes of three cycles in A_4.

Certain configurations of the branch points give *Harbater-Mumford* covers, which are necessarily defined over $\mathbb{R}$, producing real points on the Hurwitz space. If $p = 2$, these are the only points which lie in projective systems of real points up the tower, and lay at the center of computations.

Given a ramified cover, [Ba02] develops its *Nielsen graph*, which dictates which covers can factor through the given one. Classical generators for the base space of the cover lift to an embedded realization of the graph in the covering space; this is a *branch cycle design*, and it produces classical generators for the covering space. Using branch cycle designs as platforms and real points as ladders, [Ba02] ascends to the first level of the Modular Tower $\mathbf{MT}_2(A_4, \boldsymbol{C}_{3^2_\pm})^{\mathrm{in,rd}}$.

3. Construction of Hurwitz Spaces

3.1. **Topological Covers $\leftrightarrow$ Subgroups of π_1.** A continuous function $\varphi : Y \to X$ induces a homomorphism $\varphi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$, where $x_0 = f(y_0)$, by $[\gamma] \mapsto f \circ \gamma$. If φ is a topological cover, this map is injective, producing a subgroup $\varphi_*(\pi_1(Y)) \leq \pi_1(X)$. Equivalent covers produce conjugate subgroups.

This process has an inverse; namely, if $H \leq \pi_1(X, x_0)$, define Y_H to be the set of paths in X based at x_0 modulo fixed endpoint homotopy and the relation that $\gamma_1 \sim \gamma_2$ if they have the same endpoint and $[\gamma_1\gamma_2^{-1}] \in H$. This has a natural topology, and a map $\varphi_H : Y_H \to X$ given by sending the equivalence class of a point to its endpoint; this map is a topological cover. Let $y_0 \in Y_H$ be the equivalence class of the trivial loop. Then $\varphi_{H*} : \pi_1(Y_H, y_0) \to \pi_1(X, x_0) = H$.

3.2. **Ramified Covers $\leftrightarrow$ Nielsen Tuples.** Let $X = \mathbb{P}^1$ and $\varphi : Y \to X$ be a ramified cover; it is a nonconstant holomorphic map between compact connected Riemann surfaces. There are finitely many points in Y where this map is ramified, and their images in X are the *branch points* of the cover. Let $B = \{x_1, \dots, x_r\}$ denote the branch points. Let $X^\circ = X \smallsetminus B$, $Y^\circ = Y \smallsetminus \varphi^{-1}(B)$, and $\varphi^\circ = \varphi \restriction_{Y^\circ}$. Then $\varphi^\circ : Y^\circ \to X^\circ$ is a ramified cover. Conversely, given a topological cover of a punctured sphere, we can fill in the missing points in a unique way to obtain a ramified cover.

A *classical loop* in X° about $x \in X$ is a loop based at x_0 which is homotopic in x° to a loop of the form $\lambda = \alpha\delta\alpha^{-1}$, such that

- **(a)** δ is a circle around x, based at $u \in X$, which is null homotopic in $X^\circ \cup \{x\}$;
- **(b)** α is an injective path in $X^\circ \smallsetminus U$ from x_0 to u.

A *bouquet* of classical loops in X° with respect $(\boldsymbol{x}, x_0)$ is a tuple $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$ of loops in X based at x_0 such that

- **(a)** λ_i is a classical loop about x_i;
- **(b)** $\lambda_i(t_1) = \lambda_j(t_2) \Rightarrow t_1, t_2 \in \{0, 1\}$ for $i \neq j$;
- **(c)** there exists a circle around x_0 which intersects each path exactly once in the given order.

Notice that such loops generate $\pi_1(X, x_0)$, and that the concatenation of the loops in a bouquet is homotopic on the sphere to a trivial loop.

A *classical generator* of $\pi_1(X^\circ, x_0)$ is the homotopy class of a classical loop, and a *classical tuple* is a tuple of homotopy classes of a bouquet.

The action of $\pi_1(X^\circ, x_0)$ on the fiber $\varphi^{-1}(x_0)$ produces a permutation representation $T_\varphi : \pi_1(X^\circ, x_0) \to S_n$, where $n = \deg(\varphi)$. Let $\boldsymbol{\lambda}$ be a classical tuple for $(\boldsymbol{x}, x_0)$, and let $\boldsymbol{g} = (g_1, \dots, g_r)$, where $g_i = T_\varphi(\lambda_i)$. Then

- **(a)** $G = \langle \boldsymbol{g} \rangle$ is a transitive subgroup of S_n;
- **(b)** $\Pi \boldsymbol{g} = 1$.

A *Nielsen tuple* of degree n and rank r is an element of S_n^r satisfying these conditions.

Given a bouquet $\boldsymbol{\lambda}$ and a Nielsen tuple $\boldsymbol{g}$, produce a permutation representation of the fundamental group by mapping $\boldsymbol{\lambda}$ to $\boldsymbol{g}$. Pull back the stabilizer of 1 in $G = \langle \boldsymbol{g} \rangle$ to obtain a subgroup of $\pi_1(X, x_0)$. This produces a ramified cover of $\mathbb{P}^1$, and provides an inverse for this process.

3.3. **Nielsen Classes.** Let $G \leq S_n$ and let $\boldsymbol{C}$ be a tuple of conjugacy classes from G. The *total Nielsen class* of $(G, \boldsymbol{C})$ is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} = \{\boldsymbol{g} \in G^r \mid \langle \boldsymbol{g} \rangle = G,\ \Pi \boldsymbol{g} = 1,\ \boldsymbol{g} \vDash \boldsymbol{C}\},$$

where $\boldsymbol{g} \vDash \boldsymbol{C}$ means that some rearrangement of the entries of $\boldsymbol{g}$ are in the conjugacy classes of $\boldsymbol{C}$. The *inner Nielsen class* is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}} = \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{to}} / \mathrm{Inn}(G);$$

these correspond to equivalence classes of static covers. The *absolute Nielsen class* is

$$\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{ab}} = \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{ab}} / \mathrm{Abs}(G),$$

where $\mathrm{Abs}(G)$ is the group of automorphism of G which preserve the conjugacy class of a one point stabilizer; these correspond to equivalence classes of covers.

3.4. **Braid action on Nielsen Tuples.** Set

$$\mathcal{U}^r = \{(x_1, \dots, x_r) \in (\mathbb{P}^1)^r \mid x_i = x_j \Rightarrow i = j\}.$$

The group S_r acts discretely on $\mathcal{U}^r$ by permuting the slots; let $\mathcal{U}_r$ denote the quotient space. Then $\mathcal{U}_r$ is the parameter space whose points correspond to unordered sets of r distinct points from $\mathbb{P}^1$.

The *Hurwitz monodromy group* is the fundamental group

$$H_r = \pi_1(\mathcal{U}_r).$$

It is a quotient of the Artin braid group, freely generated by $Q_1, \dots, Q_{r-1}$ modulo the relations

- **(a)** $Q_i Q_j = Q_j Q_i$ if $|i - j| > 1$;
- **(b)** $Q_i Q_{i+1} Q_i = Q_{i+1} Q_i Q_{i+1}$;
- **(c)** $Q_1 \cdots Q_{r-1} Q_{r-1} \cdots Q_1 = 1$.

The Hurwitz monodromy group acts on the Nielsen class on the right via the formula

$$(g_1, \dots, g_i, g_{i+1}, \dots, g_r)Q_i = (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_r).$$

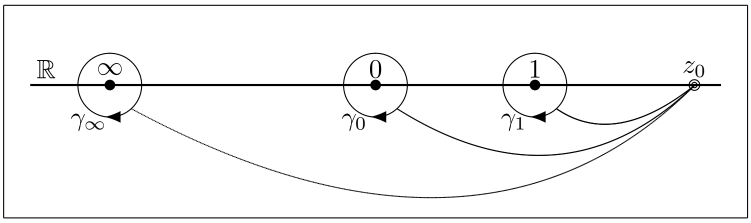
We call this *braid action*; it corresponds to continuous deformation of the cover induced by motion of its branch points.

3.5. **Hurwitz Spaces.** Each orbit of the action of H_r on $\mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$ produces a topological cover of $\mathcal{U}_r$. Specifically, let $\boldsymbol{g} \in \mathrm{Ni}(G, \boldsymbol{C})^{\mathrm{in}}$, and let O denote the orbit of $\boldsymbol{g}$ under the action of H_r. The stabilizer of $\boldsymbol{g}$ is an index $|O|$ subgroup of $H_r = \pi_1(\mathcal{U}_r, \underline{\boldsymbol{x}})$, which produces a cover $\mathcal{H}_O \to \mathcal{U}_r$. The points in the fiber of $\underline{\boldsymbol{x}}$ correspond to the elements of O.

Let $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$ be the collection of these components; this is an *inner Hurwitz space*. Each point on $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in}}$ corresponds to a unique equivalence class of static covers with automorphism group G and ramification in $\boldsymbol{C}$.

3.6. **Reduced Hurwitz Spaces.** The action of $\mathrm{PSL}_2(\mathbb{C})$ produces an equivalence relation on the classical tuples; call the equivalence classes *reduced classical tuples*. The kernel of the action of H_r on reduced classical tuples is generated the group $\hat{K}_4 = \langle Q_1 Q_3^{-1}, (Q_1 Q_2 Q_3)^2 \rangle$. The *reduced Nielsen class* is the quotient of the Nielsen class under the action of this group. Let $\bar{M}_4 = H_4/\hat{K}_4$; then $\bar{M}_4 \cong \mathrm{PSL}_2(\mathbb{Z})$. Reduced Hurwitz spaces can be computed from the action of $\bar{M}_4$ on the reduced Nielsen class.

Let $\gamma_0 = Q_1 Q_2$, $\gamma_1 = Q_1 Q_2 Q_3$, and $\gamma_\infty = Q_2$. The action of these braids on the reduced Nielsen class is equivalent to the action of the following paths in $\mathbb{P}^1_j$ on the fiber in a reduced Hurwitz space.



Primary paths for cover of $\mathcal{J}_4$.

This produces a branch cycle description for the cover $\mathcal{H}(G, \boldsymbol{C})^{\mathrm{in,rd}} \to \mathbb{P}^1_j$.

4. Results and Mysteries

4.1. **Branch Cycle Designs.** Let $\psi : Z \xrightarrow{\xi} Y \xrightarrow{\varphi} X$ be a factored ramified cover of compact Riemann surfaces, where Y and X both have genus zero. Each of these covers admits a branch cycle description. Given two of them, we would like to be able to compute the third. My dissertation [Ba02] produces algorithms to do exactly this. Moreover, I have written [GAP] programs which implement these algorithms.

This is accomplished through the use of a *Nielsen graph* embedded in Y to form a *branch cycle design*. This branch cycle design is determined by (and, conversely, determines) a branch cycle description, and hence a cover. It produces an explicit bouquet in Y written in terms of lifts of generators for $\pi_1(X)$.

The main example of [Ba02] uses a sequence of groups

$$1 \leftarrow A_3 \leftarrow A_4 \leftarrow O_4 \leftarrow U_4,$$

where $U_4 = \frac{1}{2}\tilde{A}_4$. This produces a sequence of ramified covers

$$\mathbb{P}^1_j \leftarrow \mathcal{H}(A_3, \boldsymbol{C})^{\mathrm{in,rd}} \leftarrow \mathcal{H}(A_4, \boldsymbol{C})^{\mathrm{in,rd}} \leftarrow \mathcal{H}(O_4, \boldsymbol{C})^{\mathrm{in,rd}} \leftarrow \mathcal{H}(U_4, \boldsymbol{C})^{\mathrm{in,rd}},$$

where $\boldsymbol{C}$ is two pairs of conjugacy classes of three cycles. Now compute the Nielsen class at the first stage, apply the specific paths of section 3.6 to produce a branch cycle description for the first cover, and use branch cycle designs to find generators for $\pi_1(\mathcal{H}(A_3, \boldsymbol{C}))$. Repeat this process up the tower.

4.2. **Future Directions.** The method of branch cycle design ascent was used in [Ba02] to show that $\mathcal{H}(U_4, \boldsymbol{C})^{\mathrm{in,rd}}$ contains to Harbater-Mumford components. These are the only components capable of supporting rational points at higher levels. Thus, we would like to understand their fields of definition, and understand to what extent we can explicitly identify the elliptic curves.

Let $\mathcal{E}$ denote an elliptic curve component of $\mathcal{H}(U_4, \boldsymbol{C})^{\mathrm{in,rd}}$, and let $\mathcal{R}$ denote the genus zero component of $\mathcal{H}(O_4, \boldsymbol{C})^{\mathrm{in,rd}}$ which it covers. My dissertation finds the branch points of $\mathcal{E} \to \mathcal{R}$ up to $\mathrm{PSL}_2(\mathbb{C})$ equivalence, whence it produces the j-invariant of this curve. This identifies the complex structure of the curve, but not its arithmetic nature. The next task here is to introduce tools to detect this arithmetic information.

Our initial examples of Modular Towers yielded many interesting phenomena. Yet, mysteries revealed themselves and other directions remain to be explored. We list some of these.

- **(a)** Vary p. The examples of [BF02] and [Ba02] choose $p = 2$, which implies that real points on the moduli space live over points given by Harbater-Mumford tuples.
- **(b)** Vary r. In particular, if $r = 5$, the reduced Hurwitz spaces are surfaces, and the classification of surfaces comes into play.
- **(c)** Increase r in one tower. If a branch cycle description does not lift through a group homomorphism $H \to G$, we can add a branch point to force a lift. This will produce a map between Hurwitz spaces which is a fiber bundle as opposed to a topological cover.
- **(d)** Explore other maps between Hurwitz spaces. For example, an A_5 Galois cover $\psi : Z \to X$ automatically produces an A_4 cover $\xi : Z \to Y$ by setting Y to Z modulo the action of $A_4 \leq A_5$. This in turn induces a map between the corresponding Hurwitz spaces.

References

[Ba02] Bailey, P.L., *Incremental Ascent of a Modular Tower via Branch Cycles Designs*, Ph.D. Dissertation from University of California, Irvine (2002)

[BF02] Bailey, P.L., and Fried, M.D., *Hurwitz Monodromy, Spin Separation, and Higher Levels of a Modular Tower*, Proceedings of Symposia in Pure Mathematics, Vol. 70 (2002)

[Bi75] Birman, Joan S., *Braids, Links, and Mapping Class Groups*, Annals of Mathematics Studies 82, Princeton University Press (1974)

[DF90] Debes, P. and Fried, M. D., *Rigidity and Real Residue Class Fields*, Acta Arithmetica, Vol. 56 (1990) pp. 291-323

[DF94] Debes, P. and Fried, M. D., *Nonrigid Constructions in Galois Theory*, Pacific Journal of Mathematics, Vol. 163, No. 1 (1994) pp. 81-122

[DF99] Debes, P. and Fried, M. D., *Integral Specialization of Families of Rational Functions*, Pacific Journal of Mathematics, Vol. 190, No. 1 (1999) pp. 45-85

[Fr77] Fried, Michael D., *Fields of Definitions of Functions Fields and Hurwitz Families - Groups as Galois Groups*, Communications in Algebra Vol. 5 (1977) pp. 17-82

[Fr87] Fried, Michael D., *Arithmetic of* 3 *and* 4 *Branch Point Covers*, Seminaire de Theorie des Nombres, Paris (1987) pp. 87-117

[Fr95] Fried, Michael D., *Introduction to Modular Towers: Generalizing the Relation between Dihedral Groups and Modular Curves*, Proceedings AMS-NSF Summer Conference, Vol. 186 (1995) pp. 111-171

[FV91] Fried, M. D. and Volklein, H., *The Inverse Galois Problem and Rational Points on Moduli Spaces*, Ann. of Math., Vol 290 (1991) pp. 771-800

[GAP] Schönert, Martin, et al., *Groups, Algorithms, Programming*, Lehrstuhl D für Mathematik, Rheinisch Westfaälische Technische Hochschule, Aachen, Germany, fifth edition (1995)

[Hu91] Hurwitz, A., *Ueber Riemannsche Flaechen mit gegebenen Verzweigungspunkten*, Math. Ann., Vol. 39 (1891) pp. 1-61

[Ri85] Ribes, L., *Frattini covers of Profinite Groups*, Archiv der Math., Vol. 44 (1985) pp. 390-396

[Vo96] Volklein, Helmut, *Groups as Galois Groups*, Cambridge Studies in Advanced Mathematics 53, Cambridge University Press (1996)

22046 Lakeland Ave. Lake Forest CA 92630
E-mail address: pbailey@math.uci.edu