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LONG + LIVE + MATH

High School Math Solution

Algebra II

Student Edition
Volume 2

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Mathematics is so much more than memorizing rules. It is learning to reason, to make connections, and to make sense of the world. We believe in Learning by Doing(TM)—you need to actively engage with the content if you are to benefit from it. The lessons were designed to take you from your intuitive understanding of the world and build on your prior experiences to then learn new concepts. My hope is that these instructional materials help you build a deep understanding of math.

Sandy Bartle Finocchi, Senior Academic Officer

You have been learning math for a very long time—both in school and in your interactions in the world. You know a lot of math! In this course, there's nothing brand new. It all builds on what you already know. So, as you approach each activity, use all of your knowledge to solve problems, to ask questions, to fix mistakes, and to think creatively.

Amy Jones Lewis, Director of Instructional Design

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Barry Malkin, CEO, Carnegie Learning

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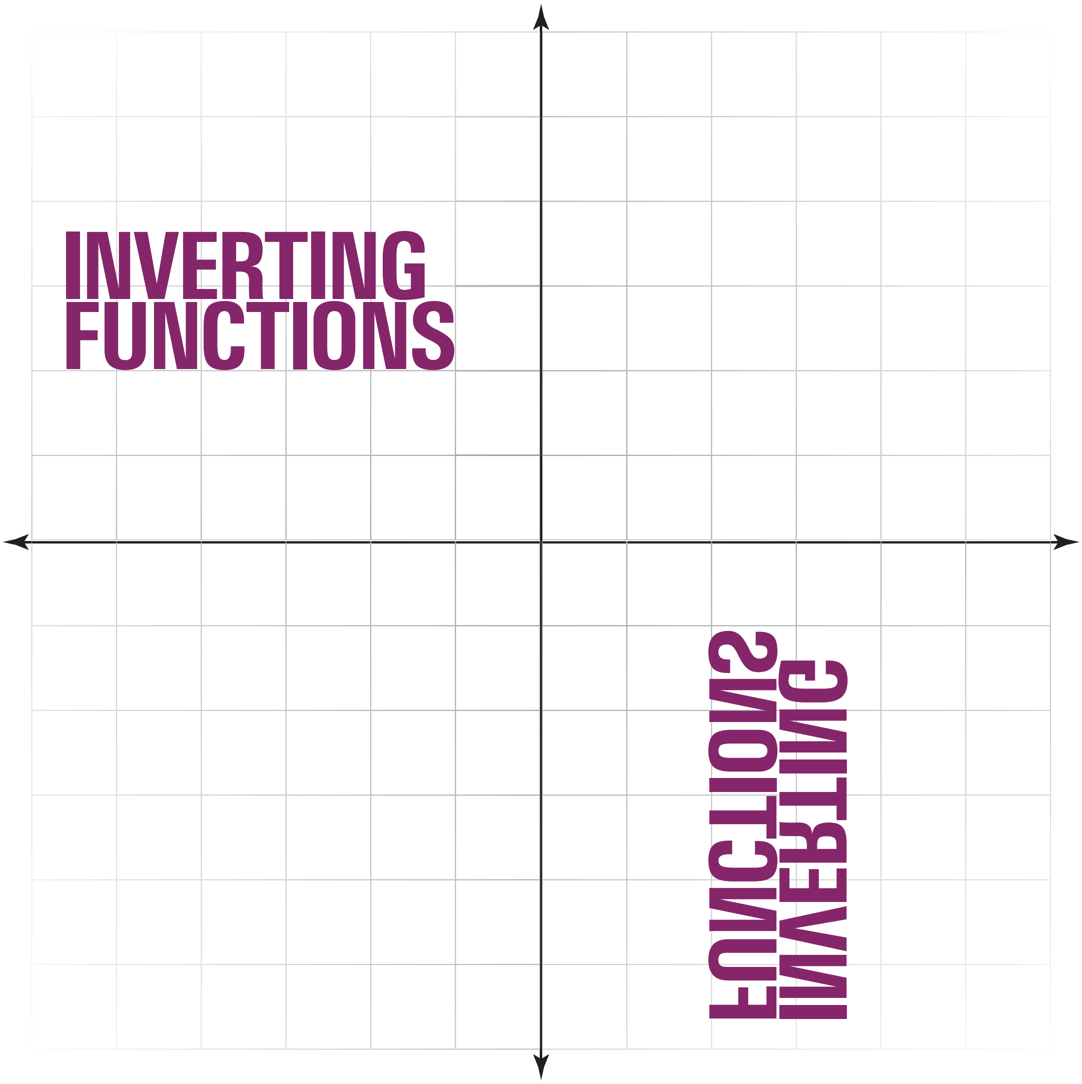

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The lessons in this module build on your experiences with power functions, exponential functions, and function transformations. You will define new functions—the square root function and the cube root function—from the inverses of the degree-2 and the degree-3 power functions. You will define the logarithmic function as the inverse of an exponential function. You will solve real-world and mathematical problems with each of the types of functions that you learn. At the end of the module, you will have the opportunity to flex your creativity as you apply your knowledge of exponentials to create and analyze images.

TOPIC 1

Radical Functions

The most basic radical function, $y = \sqrt{x}$, climbs steeply at first, then more shallowly. This breaching humpback has a similar trajectory.

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Module 3: Inverting Functions

TOPIC 1: RADICAL FUNCTIONS

This topic presents opportunities for students to explore radical functions, rewrite radical expressions, and solve radical equations. Using patty paper, students switch the *x*- and *y*-axes of power functions and generate their inverses, recognizing that they are transposing the axes when they are inverting a function. Students then shift their focus from the graphical representation of inverses to the algebraic representation. With an understanding of radical functions, students then consider radical expressions and equations.

Where have we been?

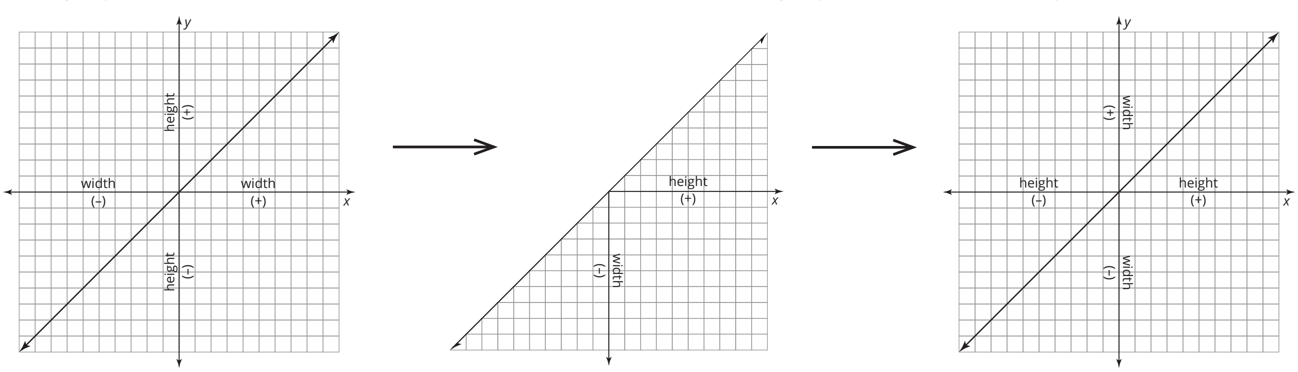
Beginning in middle school, students have been solving equations using the Properties of Equality. They solved for unknown values in the Pythagorean Theorem by taking the square root of both sides. They have experience with reflecting functions across the line $y = x$ to determine an inverse graphically and transposing x and y in an equation to determine an inverse algebraically. They have also used exponential functions to reason why $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ and have extracted perfect squares from under radicands.

Where are we going?

Students will use their understanding of function inverses throughout this module. They will explore the inverses of exponential functions, which introduces them to logarithmic functions. Radical functions are used extensively to model real-world problems, particularly in physics and in other applications, including medical dosage, wind speed, pendulums, and centrifugal force.

Graphs of Inverses

The graph of the inverse of a function is a reflection of the graph across the line $y = x$.



The inverse of the linear function $f(x) = x$ is the same line.

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Transpositions

The word *transpose* means to switch two or more items. The word combines the Latin prefix *trans-*, meaning "across" or "over" and *ponere*, meaning "to put" or "place." The word *interchange* means the same thing as transpose.

Like many words, *transpose* is used in different ways in different fields:

- In music, the word transpose is most often used to mean rewriting a song in a different key—either higher or lower.
- In biology, a transposable element is a sequence of DNA that can move from one location to another in a gene.
- Magicians use transposition when they make two objects appear to switch places.

Keep an eye out for the word *transpose* in these lessons.

Talking Points

Composition of functions can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Let $g(x) = x^2 - 5$. If $f(g(x)) = \sqrt{x^2 + 4}$, what is $f(x)$?

The entire expression $x^2 - 5$ is input to the function f, and the output is $\sqrt{x^2 + 4}$, so what does f do to the input to produce the output?

If you add 9 to $x^2 - 5$, and then take the square root of that, you get the correct output, so the function f adds 9 and takes the square root of its input.

Thus, $f(x) = \sqrt{x + 9}$.

Key Terms

inverse of a function

The inverse of a function is the set of all ordered pairs (y, x), or $(f(x), x)$.

radical function

The inverses of power functions with exponents greater than or equal to 2, such as the square root function and the cube root function, are called radical functions.

composition of functions

The process of evaluating one function inside of another function is called the composition of functions. For two functions f and g, the composition of functions uses the output of $g(x)$ as the input of $f(x)$.

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1

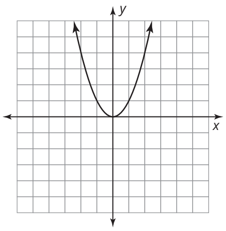
Strike That, Invert It

Inverses of Power Functions

Warm Up

Determine whether each graph represents a function. Explain your reasoning.

1.



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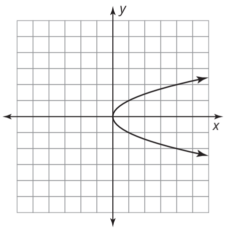

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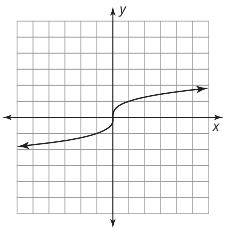

Learning Goals

- Sketch the graphs of the inverses of power functions.
- Use the Vertical Line Test to determine whether an inverse relation is a function.
- Use graphs to determine whether a function is invertible.
- Use the Horizontal Line Test to determine whether a function is invertible.
- Generalize about inverses of even- and odd-degree power functions.

Key Terms

- inverse of a function
- invertible function
- Horizontal Line Test

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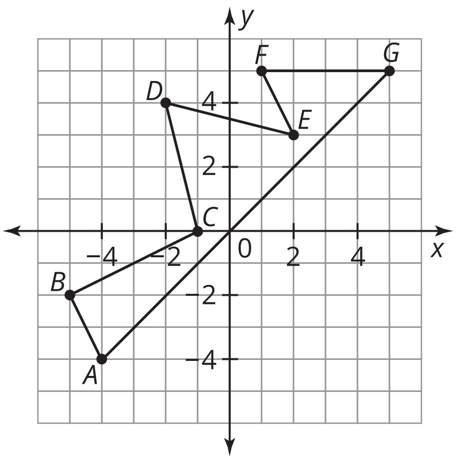
You have analyzed the graphs of power functions, like $y = x^2$, $y = x^3$, and $y = x^4$. What do the graphs of these functions look like when the independent and dependent variables are switched?

GETTING STARTED

Reflect on It



Consider the polygon drawn on the coordinate plane.

1. **Trace the polygon and the axes onto a piece of patty paper.**

   a. **Describe how you can you use your tracing to show a reflection of the polygon across the line $y = x$.**

   b. **Describe the location of the x- and y-axis on your patty paper in the reflection.**

2. **Draw the reflection of the polygon across $y = x$ on the same coordinate plane shown.**

3. **Write the coordinates of the named vertices of the polygon after a reflection across $y = x$.**

4. **Compare the coordinates of the vertices of the original polygon with the coordinates you wrote in Question 3.**

5. **Predict what the graph of the function $f(x) = x^2$ would look like when reflected across $y = x$. Explain your reasoning.**

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ACTIVITY

1.1 Switching x and y

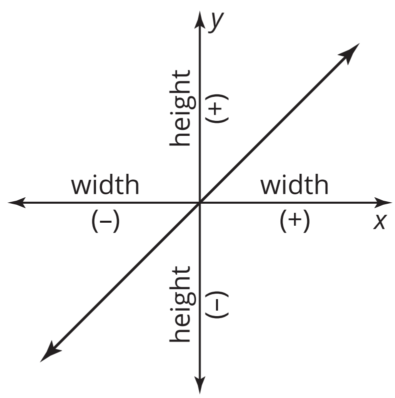
The graphs located at the end of this lesson show these 6 power functions.

$L(x) = x \qquad Q(x) = x^2 \qquad C(x) = x^3 \qquad F(x) = x^4 \qquad V(x) = x^5 \qquad S(x) = x^6$

1. Trace each graph onto a separate piece of patty paper and label the axes as shown on the original graph. Label each function to help you identify them.

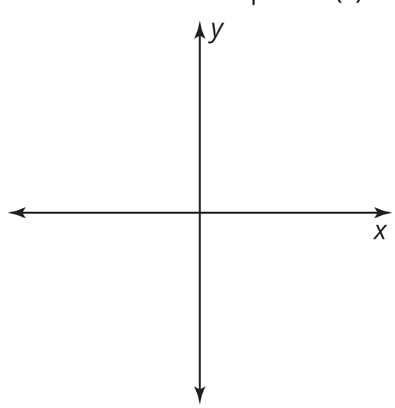
2. The graph of the linear function $L(x) = x$ models the width of a square as the independent quantity and the height of the square as the dependent quantity.

$L(x) = x$



a. Use the patty paper image of $L(x)$ to transform the graph so that it shows the height as the independent quantity on the horizontal axis and the width as the dependent quantity on the vertical axis. Then sketch the resulting graph and label the axes.

Transformed Graph of $L(x)$



Remember:

A power function is a polynomial function of the form $P(x) = ax^n$, where n is a non-negative integer.

Think about:

Which part or parts of each graph don't make sense in terms of the quantities in each situation?

Ask yourself:

How do you know when you've got the correct graph?

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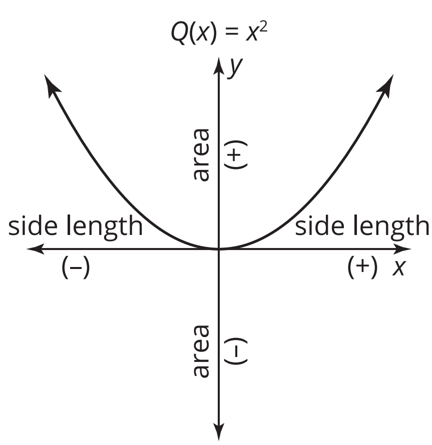
b. **Describe the transformations you used to transpose the independent and dependent quantities.**

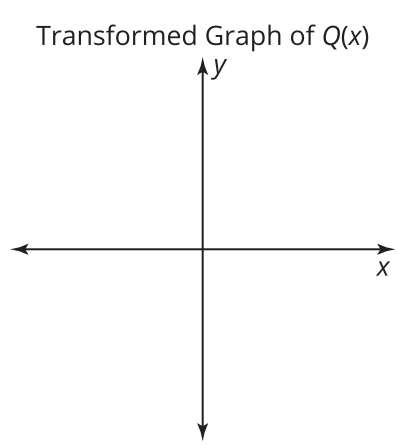
c. **Is the resulting graph a function? Explain your reasoning.**

d. **Compare the graph of $L(x) = x$ to the resulting graph. Interpret both graphs in terms of the width and height of a square.**

3. **The graph of the quadratic function $Q(x) = x^2$ models the side length of a square as the independent quantity and the area of the square as the dependent quantity.**

a. **Use the patty paper image of $Q(x)$ to transform the graph so that it shows the area as the independent quantity on the horizontal axis and the side length as the dependent quantity on the vertical axis. Then sketch the resulting graph and label the axes.**





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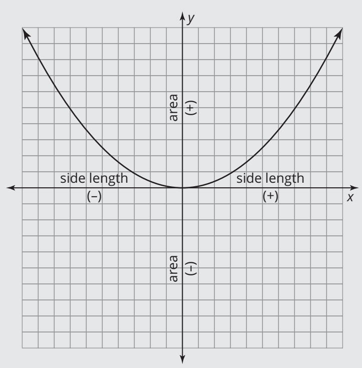
b. **Describe the transformations you used to transpose the independent and dependent quantities.**

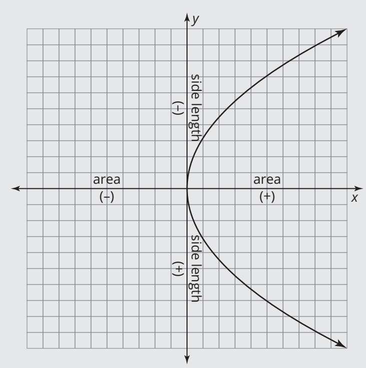
c. **Is the resulting graph a function? Explain your reasoning.**

d. **Cole used an incorrect strategy to transpose the independent and dependent quantities. Describe why Cole's strategy is incorrect.**

Cole

I can rotate the graph 90° clockwise to transpose the independent and dependent quantities.





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e. **Compare the graph of $Q(x) = x^2$ to the transformed graph you sketched. Interpret both graphs in terms of the side length and area of a square.**

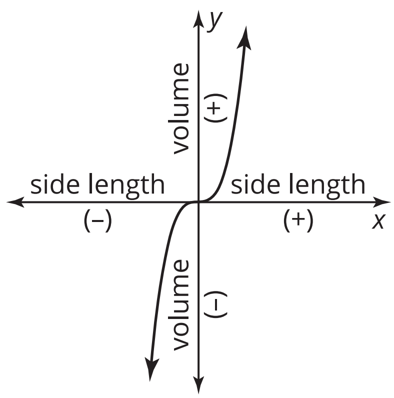
Ask yourself:

What units are used to describe area?

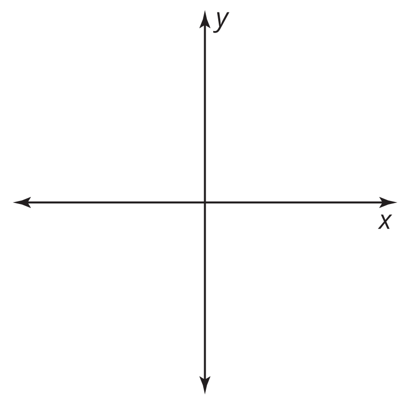
4. **The graph of the cubic function $C(x) = x^3$ models the side length of a cube as the independent quantity and the volume of the cube as the dependent quantity.**

   a. **Use the patty paper image of $C(x)$ to transform the graph so that it shows the volume as the independent quantity on the horizontal axis and the side length as the dependent quantity on the vertical axis. Then sketch the resulting graph and label the axes.**

$C(x) = x^3$



Transformed Graph of $C(x)$



   b. **Describe the transformations you used to transpose the independent and dependent quantities.**

   c. **Is the resulting graph a function? Explain your reasoning.**

   d. **Compare the graph of $C(x) = x^3$ to the resulting graph. Interpret both graphs in terms of the side length and volume of a cube.**

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ACTIVITY

1.2 Inverses of Power Functions

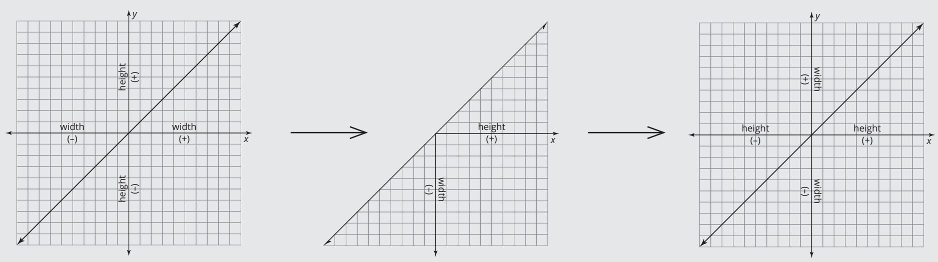
By transforming the graphs you traced in the previous activity, you were able to see and sketch the inverses of the functions $L(x) = x$, $Q(x) = x^2$, and $C(x) = x^3$.

Deanna discovered a way to use just one reflection to transpose the independent and dependent quantities.

A function f is the set of all ordered pairs (x, y), or $(x, f(x))$, where for every value of x there is one and only one value of y, or $f(x)$. The inverse of a function is the set of all ordered pairs (y, x), or $(f(x), x)$.

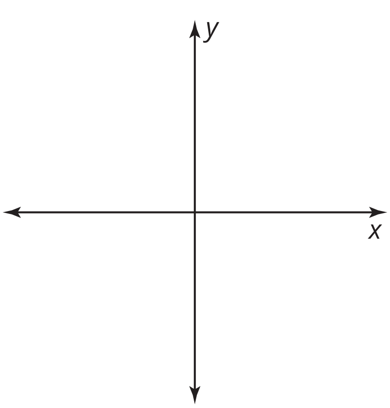
Deanna

I can reflect the graph across the line $y = x$ by folding it diagonally to switch the independent and dependent quantities.

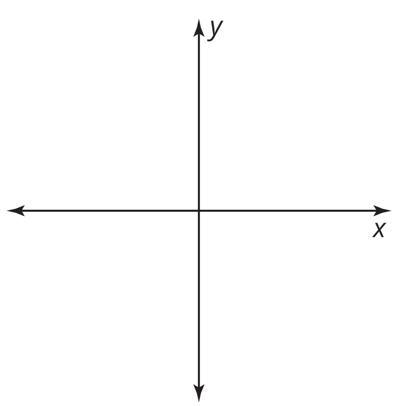


1. Use your traced graphs and Deanna's strategy to sketch the graphs of the inverses of $F(x) = x^4$, $V(x) = x^5$, and $S(x) = x^6$.

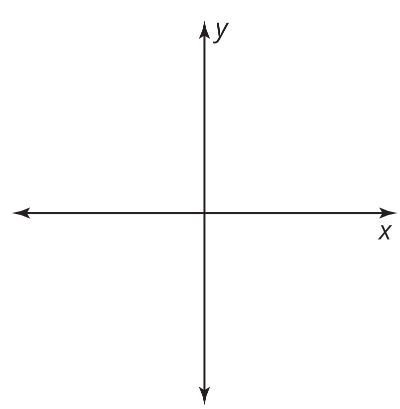
inverse of $F(x) = x^4$



inverse of $V(x) = x^5$



inverse of $S(x) = x^6$



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If the inverse of a function f is also a function, then f is an **invertible function**, and its inverse is written as $f^{-1}(x)$.

Ask yourself:

Is there a pattern here?

2. Which of the 6 power functions that you explored are invertible functions? Explain your reasoning.

You know that the Vertical Line Test can be used to determine whether the graph of a relation represents a function. The **Horizontal Line Test** is a visual method to determine whether a function has an inverse that is also a function. To apply the Horizontal Line Test, consider all the horizontal lines that could be drawn on the graph of the function. If any of the horizontal lines intersect the graph of the function at more than one point, then the inverse of the function is not a function.

3. If a graph passes both the Horizontal Line Test and the Vertical Line Test, what can you conclude about the graph?

4. If a graph passes the Vertical Line Test but not the Horizontal Line Test, what can you conclude about the graph?

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NOTES

TALK the TALK

Power Lines

1. **How does the graph of a power function and the graph of its inverse demonstrate symmetry? Explain your reasoning.**

2. **Determine whether each function is an invertible function.**

   a. $L(x) = -x$

   b. $Q(x) = -x^2$

   c. $C(x) = -x^3$

   d. $F(x) = -x^4$

   e. $V(x) = -x^5$

   f. $S(x) = -x^6$

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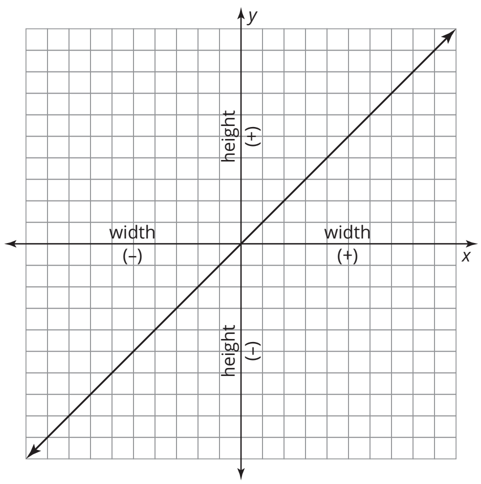
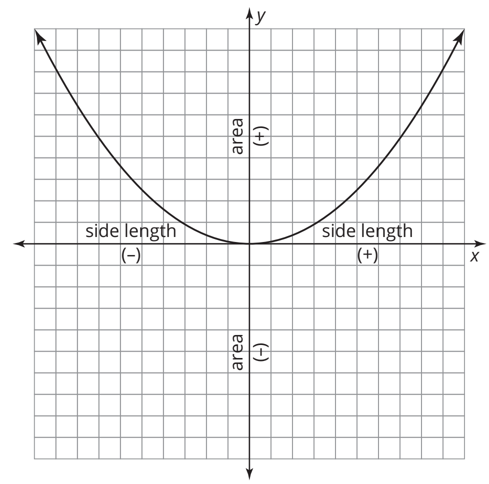
Power Functions

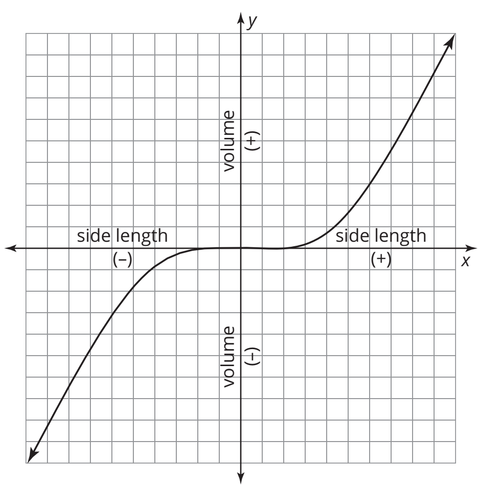
$L(x) = x$



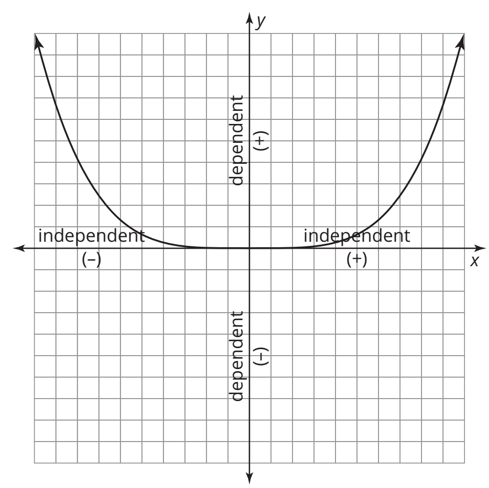
$Q(x) = x^2$



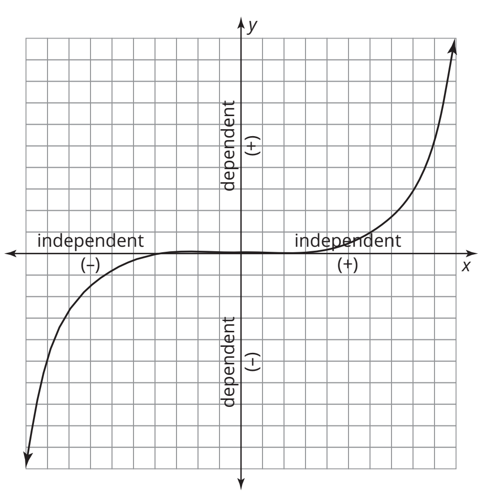
$C(x) = x^3$



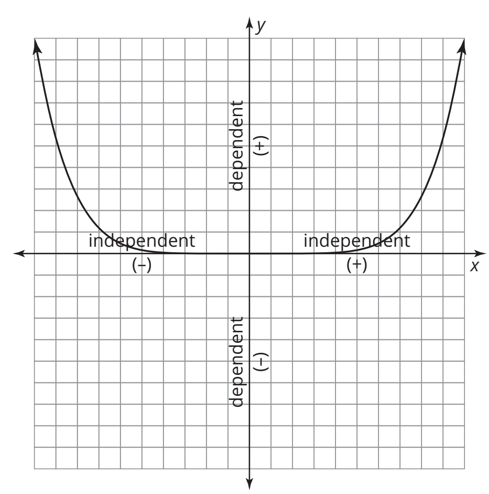
$F(x) = x^4$



$V(x) = x^5$



$S(x) = x^6$



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Assignment

Write

Describe how to create the inverse of a function graphically.

Remember

The inverse of a function is the set of all ordered pairs (y, x), or $(f(x), x)$. If the inverse of a function is also a function, the function is said to be an invertible function, and its inverse is written as $f^{-1}(x)$. To apply the Horizontal Line Test, consider all the horizontal lines that could be drawn on the graph of the function. If any of the horizontal lines intersect the graph of the function at more than one point, then the inverse of the function is not a function.

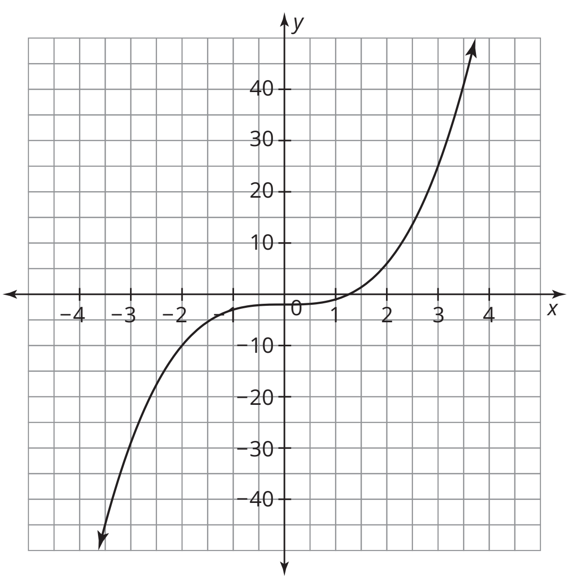
Practice

1. Consider the power function, $f(x) = x^7$.
   a. Sketch the graph of $f(x)$.
   b. Is $f(x)$ invertible? Explain your reasoning.
   c. If $f(x)$ is invertible, sketch the graph of $f^{-1}(x)$.

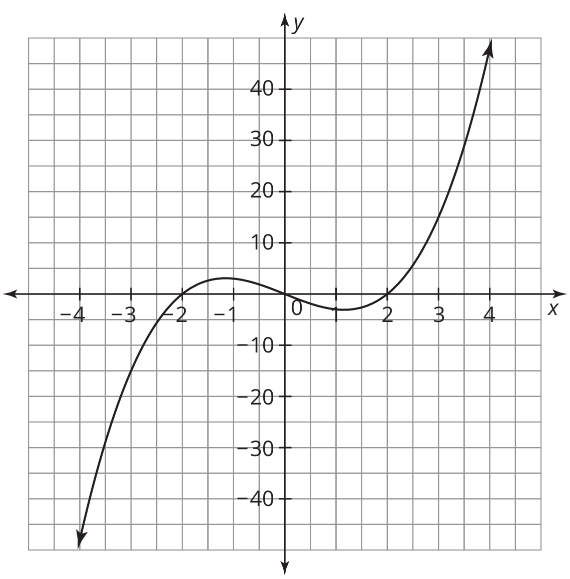
2. Consider the power function, $g(x) = x^8$.
   a. Sketch the graph of $g(x)$.
   b. Is $g(x)$ invertible? Explain your reasoning.
   c. If $g(x)$ is invertible, sketch the graph of $g^{-1}(x)$.

3. Determine whether the inverse of each graphed function is a function. Explain your reasoning.

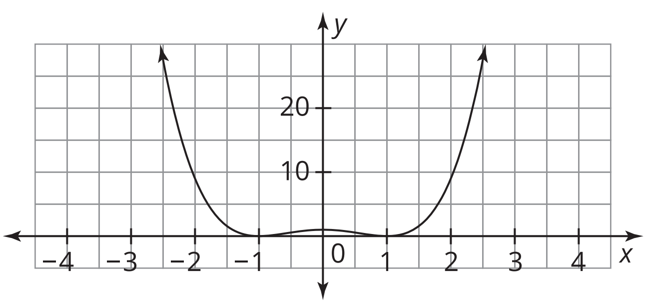
a.



b.



c.



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Stretch

1. Consider the function $f(x) = (x - 1)^5 + 2$.
   a. Sketch the graph of $f(x)$.
   b. Is $f(x)$ invertible? Explain your reasoning.
   c. If $f(x)$ is invertible, sketch the graph of $f^{-1}(x)$.

Review

1. Yolanda and Mahika are refinishing the hardwood floors in an old estate property. Mahika can refinish the floors in 100 hours by herself. Working together, it takes Yolanda and Mahika 60 hours to refinish all the floors. Write and solve an equation to determine how long it would take Yolanda to refinish the floors by herself. Show your work.
2. Eliza is catering a children's party. She made 130 ounces of an organic apple juice blend that is 70% juice. Her client tells her she wants the juice blend to be 80% juice. Write and solve an equation to determine how much juice Eliza must add to the blend she made to have the correct percent of juice for the client. Show your work.
3. Consider the function $f(x) = -\frac{3}{x}$.
   a. Complete the table.
   b. Use the table to graph the function.
   c. Describe the domain, range, and end behavior of the function. Determine all of the asymptotes of the function. Explain your reasoning.

x	$f(x)$
-9	
-6	
-3	
-1	
$-\frac{1}{3}$	
0	
$\frac{1}{3}$	
1	
3	
6	
9	

4. Use long division to determine the quotient of $(x^3 - 4x^2 + 2x + 5) \div (x - 2)$.

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2

Such a Rad Lesson

Radical Functions

Warm Up

Solve each equation for x.

1. $y = x^2 - 1$

2. $y = \frac{1}{4}x^2$

3. $y = (x + 4)^3$

Learning Goals

- Restrict the domain of $f(x) = x^2$ to graph the square root function.
- Determine equations for the inverses of power functions.
- Identify characteristics of square root and cube root functions, such as domain and range.
- Use composition of functions to determine whether two functions are inverses of each other.
- Solve real-world problems using the square root and cube root functions.

Key Terms

- square root function
- cube root function
- radical function
- composition of functions

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You have explored the graphs of the inverses of power functions. How can you write the inverse of a power function algebraically? How can you write the inverse of any power function using the notation $f^{-1}(x)$?

GETTING STARTED

The Root of the Matter

You know that the inverse of a power function defined by the set of all points (x, y), or (x, $f(x)$) is the set of all points (y, x), or ($f(x)$, x). Thus, to determine the equation of the inverse of a power function, you can transpose x and y in the equation and solve for y.

Worked Example

Determine the inverse of the power function $f(x) = x^2$, or $y = x^2$.

First, transpose x and y.

$$y = x^2 \longrightarrow x = y^2$$

Then, solve for y.

$$\sqrt{x} = \sqrt{y^2}$$
$$y = \pm\sqrt{x}$$

The inverse of $f(x) = x^2$ is $y = \pm\sqrt{x}$.

1. **Why must the symbol ± be written in front of the radical to write the inverse of the function $f(x) = x^2$?**

2. **Notice that the inverse of the function $f(x) = x^2$ is not written with the notation $f^{-1}(x)$. Explain why not.**

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ACTIVITY
2.1 The Square Root Function

You can use the Horizontal Line Test to recognize that the function $f(x) = x^2$ is not an invertible function. Are there any conditions for which the function $f(x) = x^2$ could pass the Horizontal Line Test?

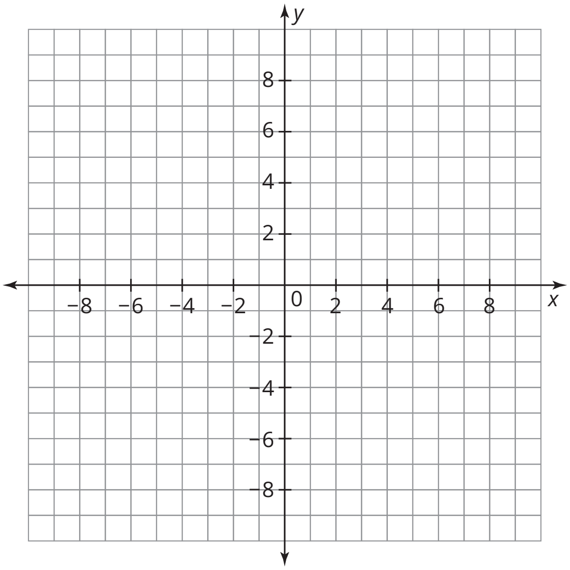
Ask yourself:

How does each point (x, y) of the function map to the inverse?

1. The table shows several coordinates of the function $f(x) = x^2$.

a. Use the ordered pairs in the table and what you know about inverses to graph the function and the inverse of the function, $y = \pm\sqrt{x}$. Explain your reasoning.

x	$f(x) = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9



b. What point or points do the two graphs have in common? Why?

2. Describe the key characteristics of each graph:

Function: $f(x) = x^2$

Domain: ____________________

Range: ____________________

***x*-intercept(s):** ______________

***y*-intercept(s):** ______________

Inverse of $f(x)$: $y = \pm\sqrt{x}$

Domain: ____________________

Range: ____________________

***x*-intercept(s):** ______________

***y*-intercept(s):** ______________

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The graph in Question 1 shows that every positive real number has 2 square roots—a positive square root and a negative square root. For example, 9 has 2 square roots, because $3^2 = 9$ and $(-3)^2 = 9$. The two square roots of 9 are 3 and -3.

When you restrict the domain of the power function $f(x) = x^2$ to values greater than or equal to 0, the inverse of the function is called the **square root function** and is written as:

$$f^{-1}(x) = \sqrt{x}, \text{ for } x \geq 0.$$

3. Draw dashed line segments between the plotted points on the function for the restricted domain $x \geq 0$ and the corresponding inverse points.

a. List the ordered pairs of the points you connected.

b. List the ordered pairs of the points that you did not connect. Explain why these points are not connected.

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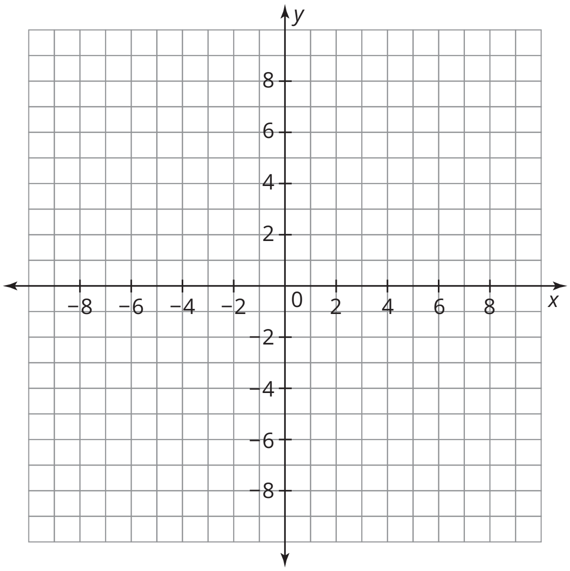
4. Graph the square root function $f^{-1}(x) = \sqrt{x}$ by restricting the domain of $f(x) = x^2$.



Ask yourself:

Does restricting the domain of the function restrict the range of the inverse?

5. Describe the key characteristics of each function:

Function: $f(x) = x^2$, for $x \geq 0$	**Inverse function: $f^{-1}(x) = \sqrt{x}$**
Domain: ____________________	**Domain:** ____________________
Range: ____________________	**Range:** ____________________
***x*-intercept(s):** ______________	***x*-intercept(s):** ______________
***y*-intercept(s):** ______________	***y*-intercept(s):** ______________

6. Does the inverse function $f^{-1}(x) = \sqrt{x}$ have an asymptote? Explain your reasoning.

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You've explored the relationship between the function $f(x) = x^2$ and its inverse, both with a domain restriction and without a domain restriction.

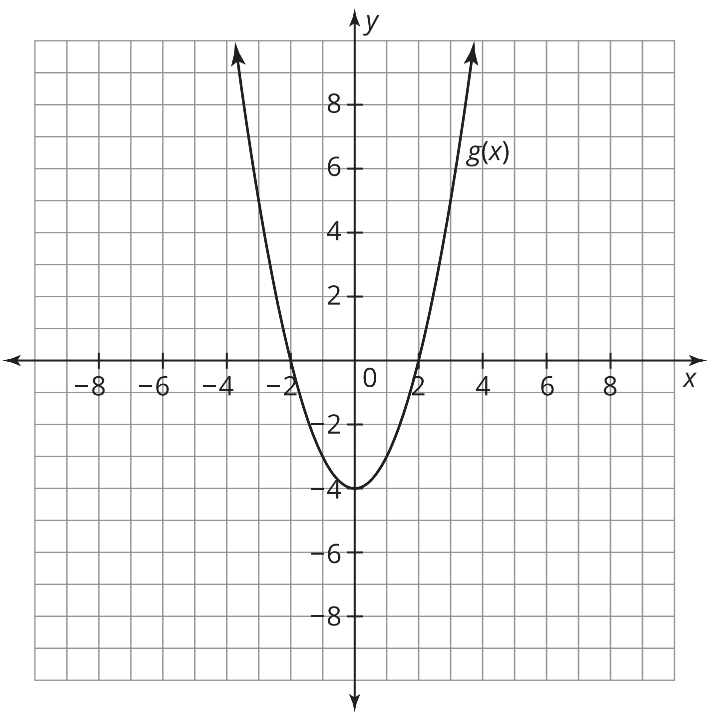
7. Make a conjecture about the relationship between the domain and range of a quadratic function and its inverse.

ACTIVITY 2.2

Domain Restrictions

Let's look at more quadratic functions to explore domain restrictions and the relationship between the domain and range of a quadratic function and its inverse.

1. Consider the function $g(x) = x^2 - 4$, shown on the coordinate plane.



a. How is $g(x)$ transformed from the basic quadratic function $f(x) = x^2$?

b. Write the equation for the inverse of $g(x)$ and sketch its graph.

c. Is the inverse of $g(x)$ a function? Explain your reasoning.

d. How is the inverse of $g(x)$ transformed from the basic square root relation $y = \pm\sqrt{x}$?

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e. **List the domain and range of $g(x)$ and the inverse of $g(x)$.**

Function: $g(x) = x^2 - 4$ **Inverse of $g(x)$: $y =$ ____________**

Domain: ____________ **Domain:** ____________

Range: ____________ **Range:** ____________

f. **What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is not restricted?**

g. **How can you restrict the domain of $g(x)$ so that its inverse is also a function?**

h. **List the domain and range for both the quadratic function with the domain restriction and the inverse function.**

Function: $g(x) = x^2 - 4$ **Inverse of $g(x)$: $g^{-1}(x) =$ ____________**

Domain restriction: ____________

Domain: ____________ **Domain:** ____________

Range: ____________ **Range:** ____________

i. **What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is restricted?**

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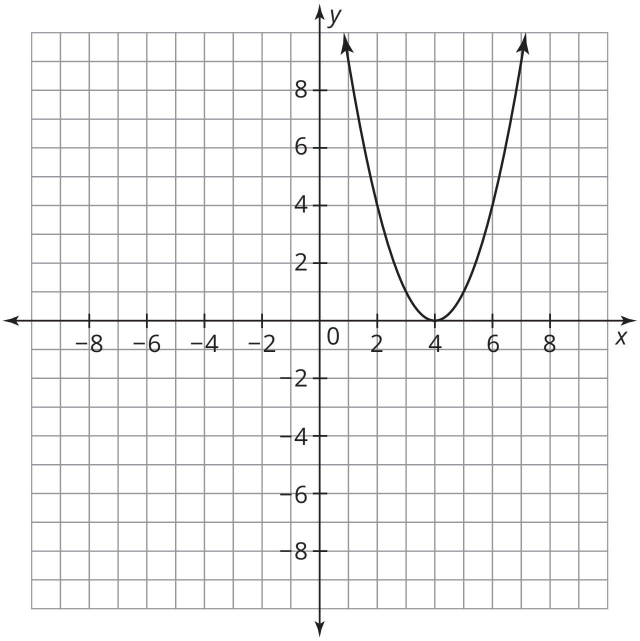
2. **Consider the function $h(x) = (x - 4)^2$, shown on the coordinate plane.**



a. **How is $h(x)$ transformed from the basic quadratic function $f(x) = x^2$?**

b. **Write the equation for the inverse of $h(x)$ and sketch its graph.**

c. **Is the inverse of $h(x)$ a function? Explain your reasoning.**

d. **How is the inverse of $h(x)$ transformed from the basic square root relation $y = \pm\sqrt{x}$?**

e. **List the domain and range of $h(x)$ and the inverse of $h(x)$.**

Function: $h(x) = (x - 4)^2$	**Inverse of $h(x)$: $y =$ ____________**
Domain: ____________	**Domain:** ____________
Range: ____________	**Range:** ____________

f. **What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is not restricted?**

g. **How can you restrict the domain of $h(x)$ so that its inverse is also a function?**

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h. **List the domain and range for both the quadratic function with the domain restriction and the inverse function.**

Function: $h(x) = (x - 4)^2$ **Inverse of** $h(x)$: $h^{-1}(x) =$ ____________

Domain restriction: ____________

Domain: ____________ **Domain:** ____________

Range: ____________ **Range:** ____________

i. **What conclusion can you make about the relationship between the domain and range of a quadratic function and its inverse when the domain is restricted?**

3. **When the domain of a quadratic function is restricted to create an inverse function, what is the lower bound of the domain? Explain your reasoning.**

4. **Complete the table to describe the effect of each transformation on the inverse of the quadratic function.**

Transformation of Quadratic Function, $f(x)$	Transformation of Inverse Function, $f^{-1}(x)$
translation up D units	
translation down D units	
translation right C units	
translation left C units	

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5. **Write the equation for the inverse of each quadratic function and identify the appropriate domain restrictions. Then, describe the domain and range of each function and its inverse without graphing the functions.**

a. $f(x) = x^2 - 2$

b. $f(x) = (x + 2)^2$

ACTIVITY 2.3

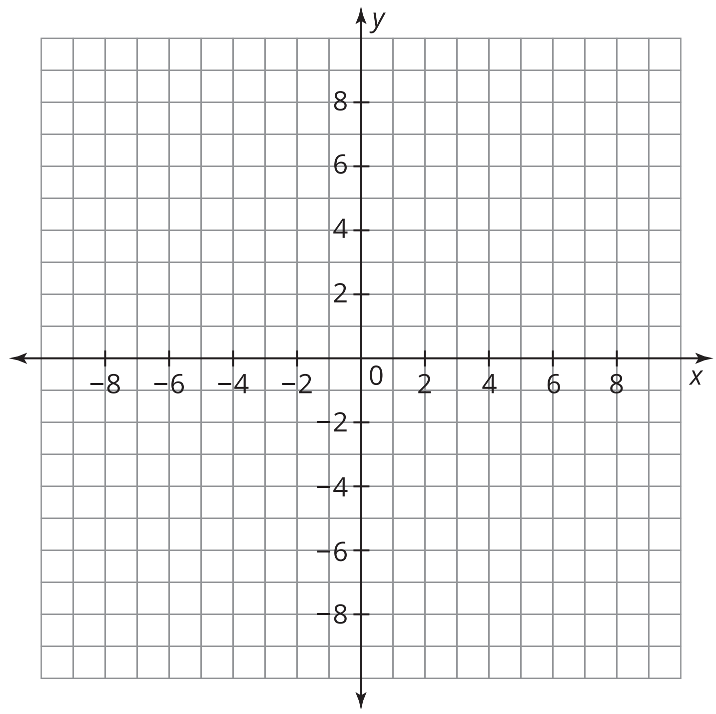
The Cube Root Function

The **cube root function** is the inverse of the power function $f(x) = x^3$ and can be written as $f^{-1}(x) = \sqrt[3]{x}$.

1. The table shows several coordinates of the function $c(x) = x^3$.

a. Use these points to graph the function and the inverse of the function, $c^{-1}(x)$.

x	$c(x) = x^3$
−2	−8
−1	−1
0	0
1	1
2	8



b. Explain how you determined the coordinates for the points on the inverse of the function.

c. What point or points do the two graphs have in common? Why?

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2. Why is the symbol ± not written in front of the radical to write the inverse of the function $c(x) = x^3$?

3. Why do you not need to restrict the domain of the function $c(x) = x^3$ to write the inverse with the notation $c^{-1}(x)$?

4. Describe the key characteristics of each function:

Function: $c(x) = x^3$	Inverse function: $c^{-1}(x) = \sqrt[3]{x}$
Domain: ______	Domain: ______
Range: ______	Range: ______
x-intercept(s): ______	x-intercept(s): ______
y-intercept(s): ______	y-intercept(s): ______

5. Does the inverse function $c^{-1}(x) = \sqrt[3]{x}$ have an asymptote? Explain your reasoning.

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The inverses of power functions with exponents greater than or equal to 2, such as the square root function and the cube root function, are called **radical functions**. Radical functions are used in many areas of science, including physics and computer science.

ACTIVITY 2.4

Inverse by Composition

You know that when the domain is restricted to $x \geq 0$, the function $f(x) = \sqrt{x}$ is the inverse of the power function $g(x) = x^2$. You also know that the function $h(x) = \sqrt[3]{x}$ is the inverse of the power function $q(x) = x^3$.

The process of evaluating one function inside of another function is called the **composition of functions**. For two functions f and g, the composition of functions uses the output of one function as the input of the other. It is expressed as $f(g(x))$ or $g(f(x))$.

Worked Example

To write a composition of the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ when the domain of $g(x)$ is restricted to $x \geq 0$, substitute the value of one of the functions for the argument, x, of the other function.

$f(x) = \sqrt{x}$ $\qquad$ $g(x) = x^2$

$f(g(x)) = \sqrt{x^2} = x$, for $x \geq 0$

You can write the composition of these two functions as $f(g(x)) = x$ for $x \geq 0$.

1. Determine $g(f(x))$ for the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ for $x \geq 0$.

If $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

2. Are $f(x)$ and $g(x)$ inverse functions? Explain your reasoning.

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3. **Algebraically determine whether the functions in each pair are inverses. Show your work.**

   a. $h(x) = \sqrt[3]{x}$ and $q(x) = x^3$

   b. $k(x) = 2x^2 + 5$ and $j(x) = -2x^2 - 5$

4. **Mike said that all linear functions are inverses of themselves because $f(x) = x$ is the inverse of $g(x) = x$.**

   Is Mike correct? Explain your reasoning.

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ACTIVITY
2.5

Pendulums

Acceleration due to gravity is the force acting on an object falling freely under the influence of Earth's gravitational pull.

Acceleration due to gravity is approximately equal to 9.8 $\frac{m}{s^2}$.

The time it takes for one complete swing of a pendulum depends on the length of the pendulum and the acceleration due to gravity.

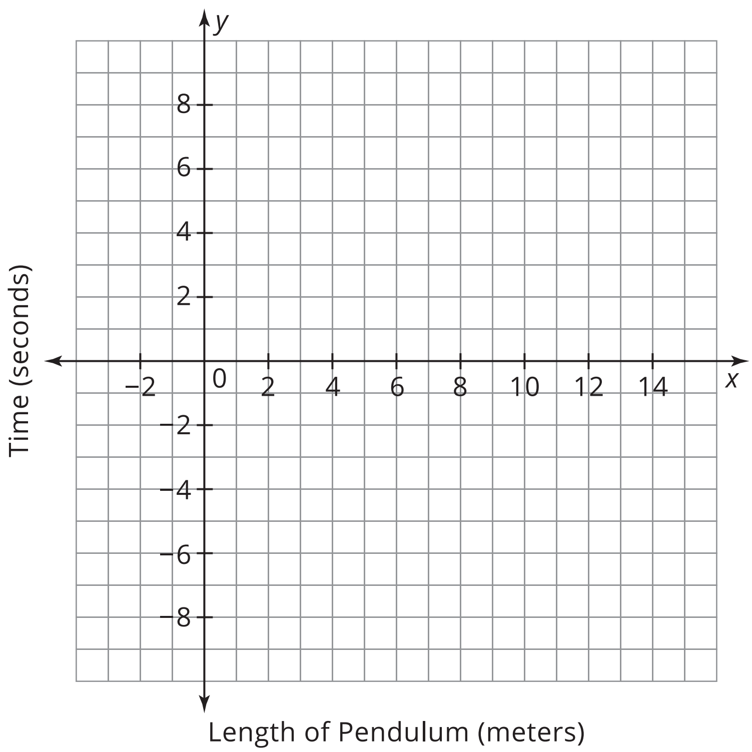
The formula for the time it takes a pendulum to complete one swing is $T = 2\pi\sqrt{\frac{L}{g}}$, where T is time in seconds, L is the length of the pendulum in meters, and g is the acceleration due to gravity in meters per second squared.

1. Write a function $T(L)$ that represents the time of one pendulum swing.

Think about:

Notice that the time for one swing does not depend on the mass of the pendulum.

2. Use technology to sketch a graph of the function $T(L)$.



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3. Describe the characteristics of the function, such as its domain, range, and intercepts. Explain your reasoning.

4. How long does it take for one complete swing when the length of the pendulum is 0.5 meter?

5. The base of a typical grandfather clock is a rectangular prism that contains the pendulum and chimes. The pendulum completes a full swing in 2 seconds. The hood can be any prism that contains the face of the clock and sits on top of the base. Design your own grandfather clock that can stand in a room that has a ceiling height of 2.5 meters. The clockface should have a diameter of at least 0.4 meter. The hood and the base together should take up less than 1 cubic meter of space.

Be sure to include in your design:

- **The shape and dimensions of the hood, including the clockface.**
- **The dimensions of the base, including the pendulum.**

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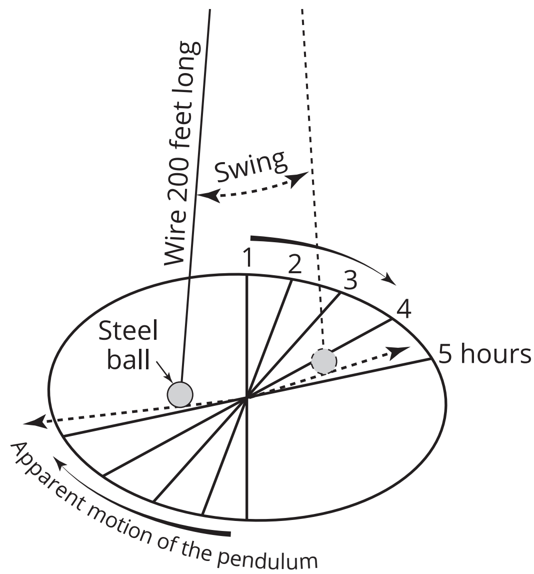
6. Many museums display what is known as a Foucault pendulum. As a Foucault pendulum swings back and forth throughout the day, the Earth's rotation causes it to appear to move in a circular direction. If the pendulum takes 1 day to come back to its initial position on the circle, approximately how many full pendulum swings occur each day?



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ACTIVITY 2.6

Centrifugal Force

The Rotor is a popular amusement park ride shaped like a cylindrical room. Riders stand against the circular wall of the room while the room spins. When The Rotor reaches the necessary speed, the floor drops out and centrifugal force leaves the riders pinned up against the wall.

The minimum speed (measured in meters per second) required to keep a person pinned against the wall during the ride can be determined with the function $s(r) = 4.95\sqrt{r}$, where r is the radius of The Rotor measured in meters.

1. An amusement park designed a rotor ride with a radius of 2 meters. At what speed does it need to spin?

2. The same park decided to build a larger rotor ride with a radius of 4 meters. At what speed does it need to spin?

3. Designers at another park have a motor that could spin a rotor ride at 6 meters per second. What is the length of the radius of this ride?

4. The designers estimate that they must allow 60 cm (0.6 meter) of space along the wall for each person on the ride. For each of the designs from Questions 1 and 2, determine the maximum number of people who should be allowed on the ride at one time.

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5. **Write an algebraic function, *n*(*r*), to determine the number of people a rotor ride can hold based on its radius.**

6. **Designers at yet another amusement park have a motor that could spin a rotor ride at 7.7 meters per second. Based on the equation you wrote in Question 5, how many people should be allowed on the ride at one time?**

7. **Planners at Flash Amusement Park want to build a new indoor rotor ride. They have a room that is 10 meters wide by 10 meters long with a 3-meter-high ceiling.**

   a. **Design a rotor ride that can fit in the room.**

   b. **At what speed does the motor of the rotor ride you designed need to spin? How many people should be allowed on the ride at one time?**

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NOTES

TALK the TALK

Masters of the Inverse

You know that the inverse of an invertible function $f(x)$ can be represented using the notation $f^{-1}(x)$.

1. **Analyze Brent's equation and explain why it is incorrect.**

Brent

$$f^{-1}(x) = \frac{1}{f(x)}$$

2. **How does knowing the domain, range, intercepts, and other key characteristics of a power function help you determine those characteristics for the function's inverse? Explain your reasoning.**

3. **When a function has an asymptote, does its inverse have an asymptote? If so, describe the location of the asymptote for the function's inverse.**

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NOTES

4. **Complete the table.**

Function	Domain	Range	x-intercept(s)	y-intercept(s)
$f(x) = x^2$, for $x \geq 0$				
$f^{-1}(x) = \sqrt{x}$				
$g(x) = x^3$				
$g^{-1}(x) = \sqrt[3]{x}$				
$h(x) = x^4$, for $x \geq 0$				
$h^{-1}(x) = \sqrt[4]{x}$				

5. **Determine whether each statement is true or false. If false, explain the error in the statement.**

   a. **Inverses of all radical functions are functions.**

   b. **The inverses of even-degree power functions are always functions.**

   c. **The inverses of odd-degree power functions are always functions.**

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Assignment

Write

Provide an example of each term.

1. square root function
2. cube root function
3. radical function
4. composition of functions

Remember

Radical functions are inverses of even-degree power functions if the domain of the even-degree power function is restricted to $x \geq 0$.

For two functions f and g, the composition of functions uses the output of one function as the input of the other. If $f(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

Practice

Brian has a new beehive. The number of bees in the hive after x weeks can be modeled by the function $b(x) = 36x^2$ for $1 \leq x \leq 30$.

1. Determine the corresponding range of $b(x)$ for the given domain. Describe what the domain and range represent in this problem.
2. Sketch the function $b(x)$ with the given domain restrictions.
3. Use the function $b(x)$ to predict the bee population after 10 weeks.
4. Use the function $b(x)$ to predict the bee population after 20 weeks.
5. Write the inverse function $b^{-1}(x)$.
6. Use compositions to verify that $b(x)$ and $b^{-1}(x)$ are inverse functions. Show your work.
7. Determine the domain and range of $b^{-1}(x)$. Describe what the domain and range represent in this problem.
8. Sketch the graph of the inverse function $b^{-1}(x)$.
9. Use the inverse function to determine when the bee population will be 25,000.

Stretch

1. The number of ants in an ant population after x days can be modeled by the function $a(x) = 20x^3$ for $1 \leq x \leq 45$.
   a. Determine the corresponding range of $a(x)$ for the given domain. Describe what the domain and range represent in this problem.
   b. Use the function $a(x)$ to predict the ant population after 15 days.
   c. Write the inverse function $a^{-1}(x)$.
   d. Use compositions to verify that $a(x)$ and $a^{-1}(x)$ are inverse functions. Show your work.
   e. Determine the domain and range of $a^{-1}(x)$. Describe what the domain and range represent in this problem.
   f. Use the inverse function to determine when the ant population will be 895,000.

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2. Consider the functions $f(x) = \sqrt{x}$, $g(x) = \sqrt{x + 4}$, and $h(x) = \sqrt{x} + 4$.

   a. Complete the table.

x	$f(x)$	$g(x)$	$h(x)$
0			
4			
8			
16			

   b. Graph $f(x)$, $g(x)$, and $h(x)$ on a coordinate plane.

   c. How do the graphs of $g(x)$ and $h(x)$ compare to the graph of $f(x)$? Explain your reasoning.

Review

1. Consider the power function, $f(x) = x^5$.
   a. Sketch the graph of $f(x)$.
   b. Is $f(x)$ invertible? Explain your reasoning.
   c. If $f(x)$ is invertible, sketch the graph of $f^{-1}(x)$.
2. Write a rational function with vertical asymptote $x = -3$ and a horizontal asymptote at $y = -5$. Sketch a graph of the function.
3. Use synthetic division to divide the given polynomials.
   a. $(x^3 + 13x^2 + 40x + 26) \div (x + 9)$
   b. $(x^4 - 8x^3 + 10x^2 + 2x + 4) \div (x - 2)$

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3

Making Waves

Transformations of Radical Functions

Warm Up

1. Describe the similarities and differences between the graphs of $f(x)$ and $g(x)$.

$f(x) = h(x - 5)$

$g(x) = h(x) + 5$

Learning Goals

- Graph transformations of radical functions.
- Analyze transformations of radical functions using transformational function form.
- Describe transformations of radical functions using algebraic, graphical, and verbal representations.
- Generalize the effects of transformations on power functions and their inverses.

You have transformed many different function types. How does the transformational function form apply to the transformations of radical functions?

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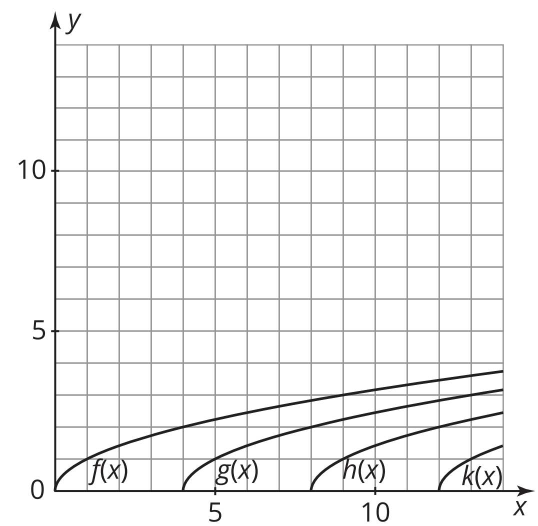
A Sea Change

A group of art students had the idea to use transformations of radical functions to create a logo for the Radical Surfing School.

To start, they graphed the function $f(x) = \sqrt{x}$, for $0 \le x \le 14$, and shifted copies of the curve to create the waves $g(x)$, $h(x)$, and $k(x)$.



Remember:

The transformation function form is $g(x) = Af(B(x - C)) + D$.

1. What value(s) in the transformation function form were changed to create these curves? Explain your reasoning.

Ask yourself:

Do the transformations of $f(x)$ shown on the graph take place inside the function or outside the function?

2. Do the graphs of $g(x)$, $h(x)$, and $k(x)$ have the same domain as $f(x)$? Explain your reasoning.

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ACTIVITY 3.1

Transformations of Radical Functions

Consider the transformations of the radical function $f(x) = \sqrt{(x)}$ graphed by the art students in the previous activity.

1. Devin, Stuart, and Kristen each wrote an equation for one of the functions that was added to the graph, first in terms of $f(x)$ and then in terms of x.

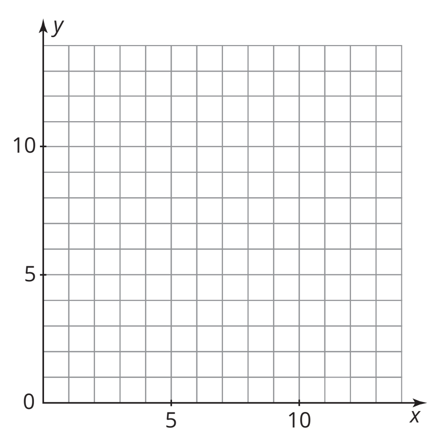
- **Devin's equation:** $g(x) = f(x) - 4$
  $= \sqrt{x} - 4$
- **Stuart's equation:** $h(x) = f(x - 8)$
  $= \sqrt{x - 8}$
- **Kristen's equation:** $k(x) = f(x + 12)$
  $= \sqrt{x + 12}$

Explain why each student's equation is either correct or incorrect. If it is incorrect, write the correct equation, first in terms of $f(x)$ and then in terms of x. Finally, state the domain of each.

The students decide that reflecting each curve, $g(x)$, $h(x)$, and $k(x)$, across the respective lines where $x = C$ will make them look more like waves crashing on the beach.

2. Consider the graph of the reflections.

a. Graph the resulting functions $f'(x)$, $g'(x)$, $h'(x)$, and $k'(x)$. Write each function, first in terms of its transformation of $f(x)$, $g(x)$, $h(x)$, or $k(x)$, and then in terms of x. Finally, state the domain of each.



Remember:

You can use the prime symbol (') to indicate that a function is a transformation of another function.

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b. Describe how you can use the transformation function form to determine the equations of the new functions.

c. How did the domain of each transformed function change as a result of the reflection across $x = C$?

d. Why does your graph show only 3 curves when the original graph had 4? Explain your reasoning.

Suppose the students wanted to reflect the 3 new waves $g'(x)$, $h'(x)$, and $k'(x)$ across the line $y = 0$.

3. Consider the equations for the reflected functions.

a. Describe how you can use transformation function form to determine the equations of the reflected functions.

Use the double prime symbol (**"**) to indicate each transformed function.

b. Write three new functions using transformational form to represent each reflection of $g'(x)$, $h'(x)$, and $k'(x)$, and then each in terms of x. Finally, write the domain of each transformed function.

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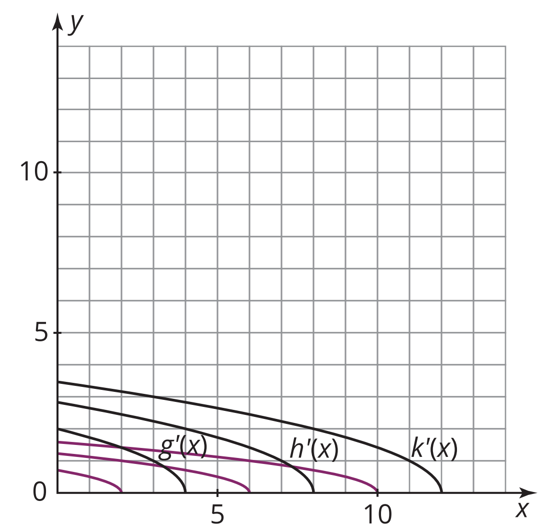
The art students want to add waves below the 3 waves, as shown. These waves are copies of $g'(x)$, $h'(x)$, and $k'(x)$, except half as high and shifted to the left 2 units.

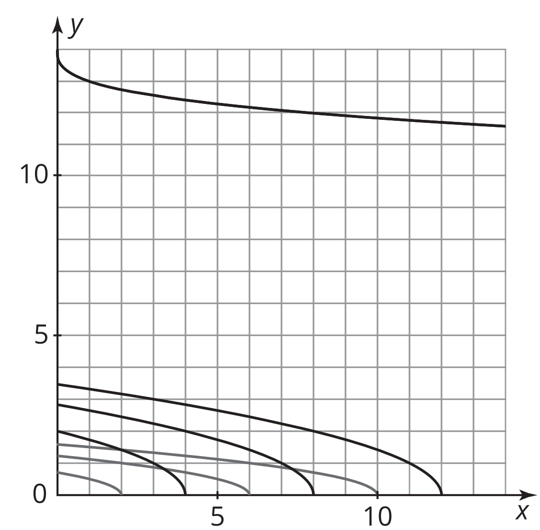


4. Consider the equations for the waves.

a. Write 3 new functions $q(x)$, $r(x)$, and $s(x)$ in terms of $g'(x)$, $h'(x)$, and $k'(x)$ to create the waves that the art students want. Make sure to write the domains of each transformed function.

b. Describe how you used what you know about the transformation function form to determine your answer to part (a).

The art students want to add some clouds to the top of the logo. For the clouds, they use the inverses of cubic functions. They start with the function $c(x) = -\sqrt[3]{x} + 14$.



5. Transform this function and write 2 more equations to create the clouds the students want. Graph the results.

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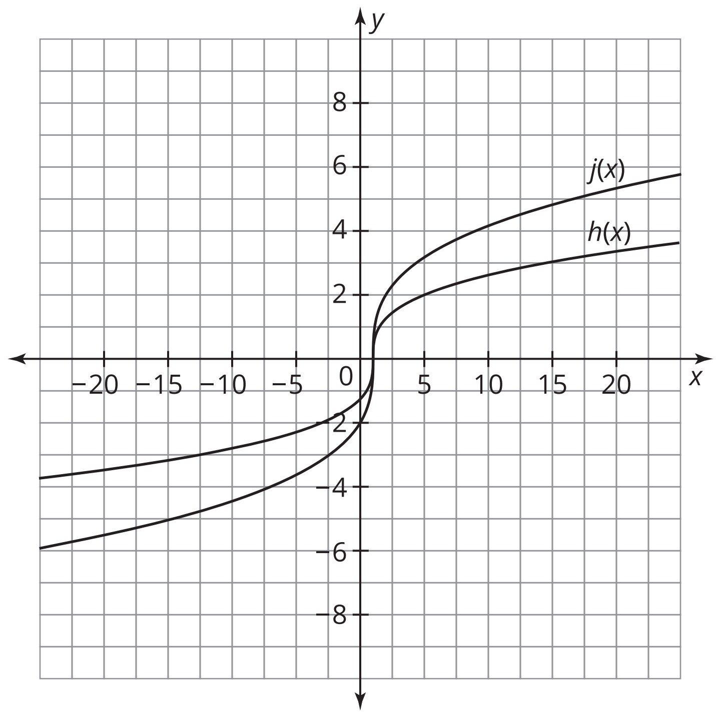
The text shown, for example, follows the curve $f(x) = -x^2$.

Your Text Here

In many graphic design programs, a trace path can be created. A trace path is an invisible line or curve that acts as the baseline of text that is added to the design. When you insert text on a trace path, the text follows the line or curve.

6. The art students are experimenting with different radical function graphs to use as trace paths for the surfing school's name: Radical Surfing School. They have narrowed their trace paths down to 2 choices. The graphs of the functions are shown.

$h(x) = \sqrt[3]{2(x - 1)}$ $\qquad\qquad$ $j(x) = 2\sqrt[3]{x - 1}$



a. Use technology to sketch the graph of the function $\sqrt[3]{x}$ and list its domain, range, and x-intercept and y-intercept.

Domain: ____________________

Range: ____________________

***x*-intercept:** ________________

***y*-intercept:** ________________

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b. Compare and contrast the graphs of the functions *h*(*x*) and *j*(*x*) and their equations. What do you notice?

c. Compare the effects of increasing the *A*-value with increasing the *B*-value in a radical function. What do you notice?

d. State the domain of each transformed function.

7. Choose one of the cube root functions as a trace path for the title of the surfing school. Or, write a different radical function to use as a trace path. Graph the function on the coordinate plane in Question 5, and write the title of the school on the trace path.

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NOTES

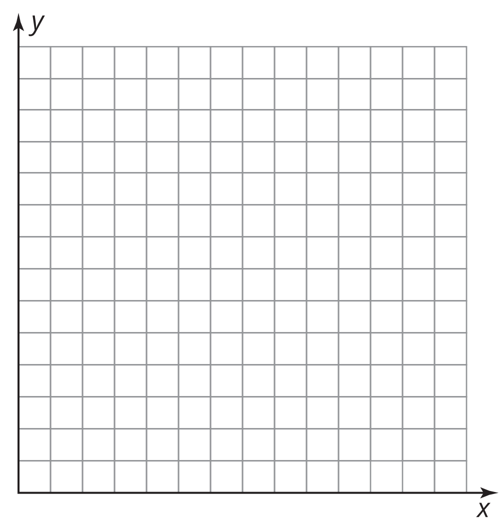
TALK the TALK

Now it's your turn to design a logo!

Logo to Town

1. Use technology to design a logo that includes the transformations of $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{x}$.

- **Graph at least two transformations of each function.**
- **Write each function you graphed first in terms of its transformations of $f(x)$ or $g(x)$, and then in terms of x.**
- **State the domain of each transformed function.**



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Assignment

Write

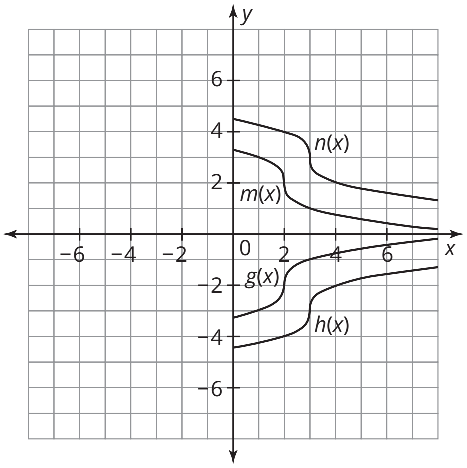
Describe how transformations affect the domain and range of radical functions.

Remember

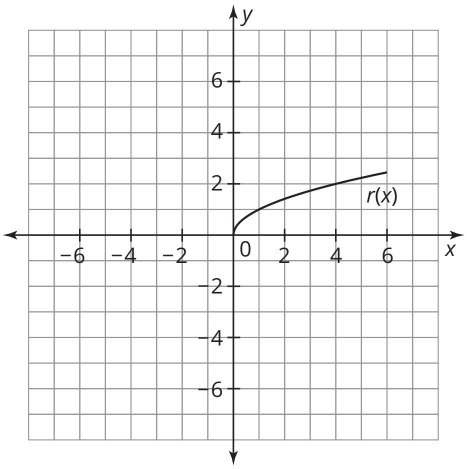
For the square root function and cube root function respectively, the transformation function can be written as $s(x) = A\sqrt{B(x - C)} + D$ or $c(x) = A\sqrt[3]{B(x - C)} + D$. Changes to the A- or D-values dilate, translate, or reflect a function vertically. Changes to the B- or C-values dilate, translate, or reflect a function horizontally.

Practice

1. Brandon, a graphic designer, designed the logo shown for the Lazy Y Ranch. Each curve in the design is a transformation of the cube root function $f(x) = \sqrt[3]{x}$ with a restricted domain.
   a. Describe each of the four transformations of $f(x) = \sqrt[3]{x}$ that were used to create the four functions in the design.
   b. Write each function used in the design. For each function, write the domain as an inequality.



2. Brandon is working on a logo for a publishing company. He starts by graphing the function $r(x) = \sqrt{x}$ with the restricted domain $0 \le x \le 6$. He plans to add the graphs of 5 more functions to complete the design.
   a. The next function Brandon adds is $s(x)$, which is the square root function after a vertical stretch by a factor of 2 and a translation 1 unit up. Write the function $s(x)$ and graph $s(x)$ with the domain $0 \le x \le 6$.
   b. Next, Brandon adds the function $t(x)$, which is the square root function after a vertical stretch by a factor of 3 and a translation 2 units up. Write the function $t(x)$ and graph $t(x)$ with the domain $0 \le x \le 6$.
   c. To complete the design, Brandon adds the functions $r'(x)$, $s'(x)$, and $t'(x)$ which are reflections of the original 3 functions across the y-axis. Write the functions $r'(x)$, $s'(x)$, and $t'(x)$ and graph each function with the domain $-6 \le x \le 0$.



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Stretch

1. Consider the functions $f(x) = \sqrt[3]{x^4}$ and $g(x) = x\sqrt[3]{x}$.
   a. Determine $f(4)$ and $g(4)$ in decimal form. Show your work.
   b. Determine $f(-8)$ and $g(-8)$ in decimal form. Show your work.
   c. Rewrite $f(x)$ to show that it is equivalent to $g(x)$.

Review

1. Consider the function $f(x) = 4x^2$.
   a. Determine the domain and range of $f(x)$.
   b. Write the inverse function $f^{-1}(x)$.
   c. Determine the domain and range of $f^{-1}(x)$.
2. Algebraically determine whether $f(x) = 2x^3 + 15$ and $g(x) = \sqrt[3]{\frac{x - 15}{2}}$ are inverses. Show your work.
3. Consider the basic rational function $f(x) = \frac{1}{x}$. Explain how the graph of $g(x) = \frac{5}{x + 10} + 3$ compares to the graph of $f(x)$.
4. Consider the rational function $\frac{x}{x^3 - 5x^2 + 6x}$. Determine any vertical and horizontal asymptotes and any removable discontinuities of the graph of $f(x)$. Explain your reasoning.
5. Identify the number of real zeros of $x^3 - 7x^2 + 14x - 8 = 0$. Explain your reasoning.

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4

Keepin' It Real

Rewriting Radical Expressions

Warm Up

Extract roots to rewrite each radical expression.

1. $\sqrt{48}$

2. $\sqrt{27}$

3. $\sqrt{32}$

Learning Goals

- Extract roots from radicals.
- Rewrite radicals as powers that have rational exponents.
- Rewrite powers that have rational exponents as radicals.
- Rewrite radicals by extracting roots.
- Multiply, divide, add, and subtract radicals.

You know how to rewrite radical expressions that have only numbers in the radicand. How do you rewrite radical expressions that have variables in the radicand?

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GETTING STARTED

The Power of Positivity

You know how to use the properties of powers to rewrite expressions.

1. Match each expression to an equivalent expression in the box. For each given expression, $a \neq 0$.

a. $a^3 \cdot a$

b. $\frac{a^3}{a^7}$

c. $a^4 \cdot a^{-4}$

d. $(ab^2)^2$

e. $(a^2)^2$

f. $\frac{a^6}{a^6}$

g. $(a^8\, b^4)^{\frac{1}{2}}$

h. $a^2 \cdot a^{-6}$

1
a^4
a^{-4}
a^2b^4
a^4b^2

2. Jamal says that the expression $\frac{a^6}{a^6}$ is equivalent to 1 because any number, except 0, divided by itself is 1. Brittany says $\frac{a^6}{a^6}$ is equal to 1 because $a^{6-6} = a^0$, and anything to the zero power, except zero, equals 1. Who's correct? Explain your reasoning.

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3. Consider each expression in Question 1. If a and b are real numbers, what do you know about the value of each expression? Explain your reasoning.

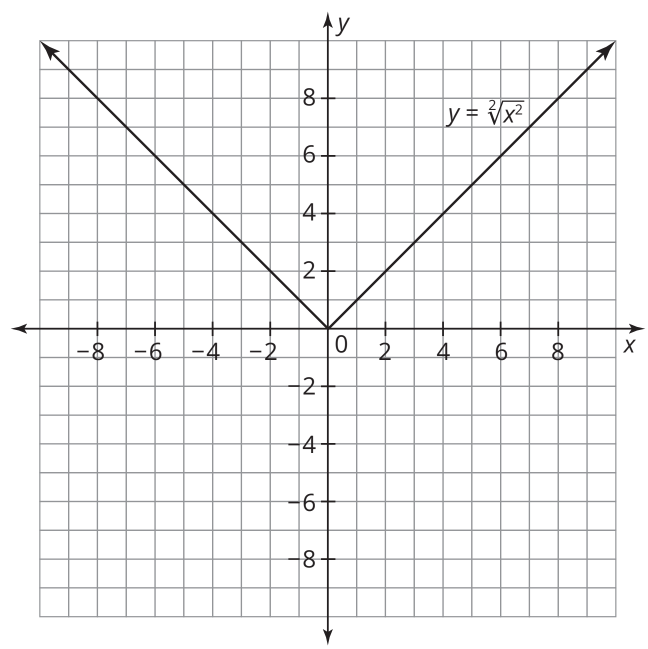
ACTIVITY 4.1 Extracting Variables from Radicals

Previously, you have rewritten radicals by extracting roots involving numbers. In this lesson, you will explore how to extract roots for expressions of the form $\sqrt[n]{x^n}$. To determine how to extract a variable from a radical, let's consider several different values of *n*.

1. For each value of *n* for the expression $\sqrt[n]{x^n}$, analyze the table and graph. Identify the function family associated with the graph and write the corresponding equation.

a. Let $n = 2$.

x	$x^n = x^2$	$\sqrt[n]{x^n} = \sqrt[2]{x^2}$
−2	4	2
−1	1	1
0	0	0
1	1	1
2	4	2



Function family of the graph: ____________________

Equation of the graph: ____________________

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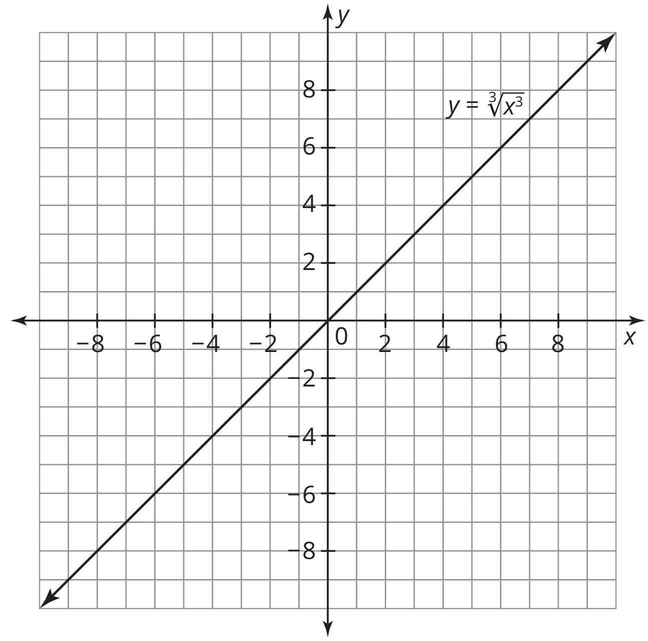
b. Let $n = 3$.

x	$x^n = x^3$	$\sqrt[n]{x^n} = \sqrt[3]{x^3}$
−2	−8	−2
−1	−1	−1
0	0	0
1	1	1
2	8	2

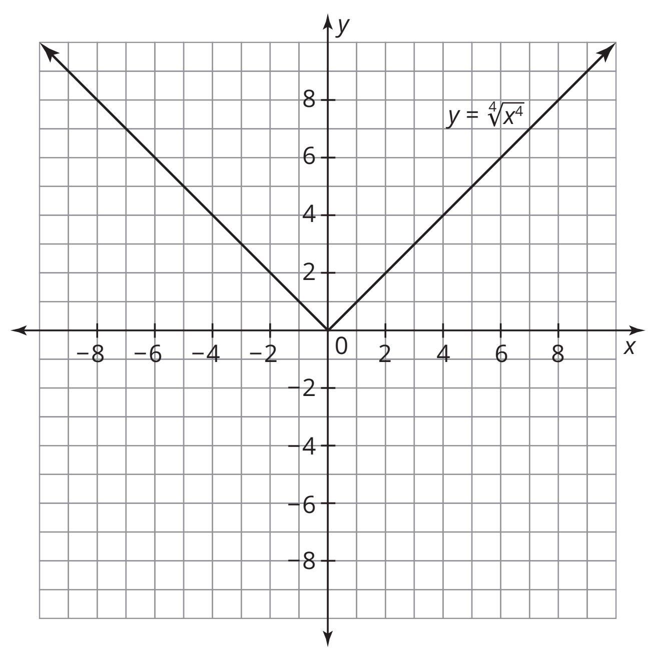


Function family of the graph: ____________________

Equation of the graph: ____________________

c. Let $n = 4$.

x	$x^n = x^4$	$\sqrt[n]{x^n} = \sqrt[4]{x^4}$
−2	16	2
−1	1	1
0	0	0
1	1	1
2	16	2



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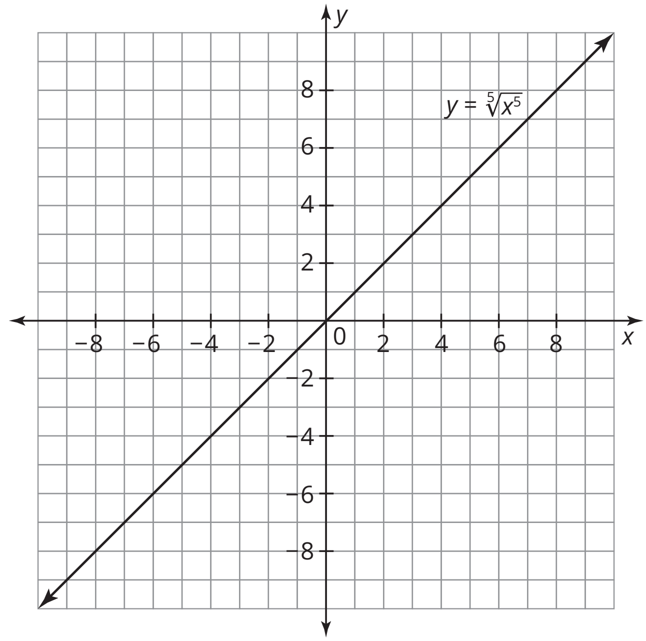
Function family of the graph: ____________________

Equation of the graph: ____________________

d. Let $n = 5$.

x	$x^n = x^5$	$\sqrt[n]{x^n} = \sqrt[5]{x^5}$
−2	−32	−2
−1	−1	−1
0	0	0
1	1	1
2	32	2



Function family of the graph: ____________________

Equation of the graph: ____________________

e. Analyze the representations for each value of n. What do you notice?

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To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as:

$$\sqrt[n]{x^n} = \begin{cases} |x|, & \text{when } n \text{ is even} \\ x, & \text{when } n \text{ is odd} \end{cases}$$

One way to say $\sqrt[7]{x^7}$ is "the seventh root of x to the seven."

2. Explain why $\sqrt[7]{x^7} = |x|$ is incorrect, for real values of x.

3. Asia and Melissa shared their work for extracting the root from $\sqrt{x^4}$, for real values of x.

Asia

$$\sqrt{x^4} = |x^2|$$

Melissa

$$\sqrt{x^4} = \sqrt{x^2 \cdot x^2}$$
$$= x^2$$

Who's correct? Explain your reasoning.

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ACTIVITY

4.2

Rewriting Radicals as Powers

In a previous course, you have learned how to rewrite a radical as a power with a rational exponent and to rewrite a power with a rational exponent as a radical. You can use the rules of exponents to verify the relationship between powers and roots.

Worked Example

You can rewrite an expression in radical form as an expression with a rational exponent. Solve the equation $\sqrt{x} = x^a$ for a, given $x \geq 0$, to determine the exponential form of $\sqrt{x}$.

$$\sqrt{x} = x^a$$

Square each side of the equation.

$$(\sqrt{x})^2 = (x^a)^2$$

The bases are the same, so set the exponents equal to each other, and solve for a.

$$x = x^{2a}$$

$$1 = 2a$$

$$a = \frac{1}{2}$$

The exponential form of the square root of x, given $x \geq 0$, is x to the one-half power.

$$\sqrt{x} = x^{\frac{1}{2}}, \text{ given } x \geq 0$$

1. Why is the restriction "given $x \geq 0$" stated at the beginning of the worked example?

2. How do you know when the initial x-value can be any real number or when the initial x-value should be restricted to a subset of the real numbers?

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3. Rewrite each expression in radical form as an expression with a rational exponent.

a. The cube root of *x*

b. The cube root of *x* squared

4. Complete the cells in each row. In the last column, write "$x \geq 0$" or "all real numbers" to describe the restrictions that result in equal terms for each row.

Radical Form	Radical to a Power Form	Exponential Form	Restrictions
$\sqrt[4]{x^2}$	$(\sqrt[4]{x})^2$		
		$x^{\frac{3}{4}}$	
		$x^{\frac{2}{5}}$	
$\sqrt[5]{x}$			

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You can rewrite a radical expression $\sqrt[n]{x^a}$ as an exponential expression $x^{\frac{a}{n}}$:
- For all real values of x if the index n is odd.
- For all real values of x greater than or equal to 0 if the index n is even.

ACTIVITY 4.3

Extracting Roots to Rewrite Radicals

You can extract roots to rewrite radicals, using radicals or powers.

The root of a product is equal to the product of the roots of each factor: $\sqrt[p]{a^m b^n} = \sqrt[p]{a^m} \cdot \sqrt[p]{b^n}$.

Worked Example

Extract the roots and rewrite $\sqrt[3]{8x^6}$ using radicals and using powers.

Using Radicals

$$\begin{aligned}\sqrt[3]{8x^6} &= \sqrt[3]{2^3 \cdot x^6}\\ &= \sqrt[3]{2^3 \cdot (x^2)^3}\\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{(x^2)^3}\\ &= 2x^2\end{aligned}$$

Using Powers

$$\begin{aligned}\sqrt[3]{8x^6} &= (8x^6)^{\frac{1}{3}}\\ &= (2^3 \cdot x^6)^{\frac{1}{3}}\\ &= 2^{\frac{3}{3}} \cdot x^{\frac{6}{3}}\\ &= 2^1 \cdot x^2\\ &= 2x^2\end{aligned}$$

1. Which method do you prefer?

2. Embry and Devon shared their work for extracting roots from $\sqrt[4]{j^8k^4}$.

Embry

$$\begin{aligned}\sqrt[4]{j^8k^4} &= \sqrt[4]{j^8} \cdot \sqrt[4]{k^4}\\ &= \sqrt[4]{(j^2)^4} \cdot \sqrt[4]{k^4}\\ &= j^2|k|\end{aligned}$$

Devon

$$\begin{aligned}\sqrt[4]{j^8k^4} &= (j^8k^4)^{\frac{1}{4}}\\ &= (j^8)^{\frac{1}{4}}(k^4)^{\frac{1}{4}}\\ &= j^{\frac{8}{4}}k^{\frac{4}{4}}\\ &= j^2|k|\end{aligned}$$

Explain why it is not necessary to use the absolute value symbol around j^2 and why it is necessary to use the absolute value symbol around k.

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3. **Betty, Wilma, and Rose each extracted roots and rewrote the radical $\sqrt{x^2y^2}$.**

Betty

$$\sqrt{x^2y^2} = \sqrt{x^2 \cdot y^2}$$
$$= \sqrt{x^2} \cdot \sqrt{y^2}$$
$$= |x| \cdot |y|$$

Wilma

$$\sqrt{x^2y^2} = \sqrt{x^2 \cdot y^2}$$
$$= \sqrt{x^2} \cdot \sqrt{y^2}$$
$$= |xy|$$

Rose

$$\sqrt{x^2y^2} = \sqrt{x^2 \cdot y^2}$$
$$= \sqrt{x^2} \cdot \sqrt{y^2}$$
$$= xy$$

Who's correct? Explain you reasoning.

4. **Rewrite each radical involving a quotient.**

a. $\sqrt{\dfrac{m^4}{n^6}}$

b. $\sqrt[3]{\dfrac{j^{12}}{k^3}}$

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For some radicals, you may not be able to extract the entire radicand.

5. Bernadette and Cris extracted the roots from $\sqrt[3]{16x^8}$.

Bernadette

$$\sqrt[3]{16x^8} = 2^{\frac{4}{3}} \cdot x^{\frac{8}{3}}$$
$$= 2^{\frac{3}{3}} \cdot 2^{\frac{1}{3}} \cdot x^{\frac{6}{3}} \cdot x^{\frac{2}{3}}$$
$$= 2 \cdot \sqrt[3]{2} \cdot x^2 \cdot \sqrt[3]{x^2}$$
$$= 2x^2\sqrt[3]{2x^2}$$

Cris

$$\sqrt[3]{16x^8} = \sqrt[3]{2^4 \cdot x^8}$$
$$= \sqrt[3]{2^3x^6} \cdot \sqrt[3]{2x^2}$$
$$= 2x^2\sqrt[3]{2x^2}$$

a. In the last line of work, why was $2x^2$ not extracted from the radical?

b. Compare and contrast the methods.

6. Rewrite each radical by extracting all possible roots, and write the final answer in radical form.

When the power and root are equal and even numbers, remember to use absolute value for the principal root.

a. $\sqrt{16x^6}$

b. $-\sqrt{8v^3}$

c. $\sqrt{d^3f^4}$

d. $\sqrt{h^4j^6}$

e. $\sqrt{25a^2b^8c^{10}}$

f. $\sqrt[4]{81x^5y^{12}}$

g. $\sqrt[3]{(x+3)^9}$

h. $\sqrt{(x+3)^2}$

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ACTIVITY 4.4 Operating with Radicals

Arianna and Heidi multiplied $\sqrt{18a^2} \cdot 4\sqrt{3a^2}$ and extracted all roots.

1. Compare Arianna's and Heidi's solution methods. Explain the difference in their solution methods.

Arianna

$$\begin{aligned}\sqrt{18a^2} \cdot 4\sqrt{3a^2} &= 4\sqrt{54a^4}\\ &= 4\sqrt{9 \cdot 6 \cdot a^4}\\ &= 4 \cdot \sqrt{9} \cdot \sqrt{6} \cdot \sqrt{a^4}\\ &= 4 \cdot 3 \cdot \sqrt{6} \cdot a^2\\ &= 12a^2\sqrt{6}\end{aligned}$$

Heidi

$$\begin{aligned}\sqrt{18a^2} \cdot 4\sqrt{3a^2} &= \sqrt{9 \cdot 2 \cdot a^2} \cdot 4 \cdot \sqrt{3 \cdot a^2}\\ &= \sqrt{9} \cdot \sqrt{2} \cdot \sqrt{a^2} \cdot 4 \cdot \sqrt{3} \cdot \sqrt{a^2}\\ &= 3 \cdot \sqrt{2} \cdot |a| \cdot 4 \cdot \sqrt{3} \cdot |a|\\ &= 12a^2\sqrt{6}\end{aligned}$$

You have seen that the root of a product is equal to the product of the roots of each factor. This concept applies to quotients also. The root of a quotient is equal to the quotient of the roots of the dividend and divisor: $\sqrt[p]{\frac{a^m}{b^n}} = \frac{\sqrt[p]{a^m}}{\sqrt[p]{b^n}}$.

Worked Example

Extract the roots to rewrite the quotient $\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$.

$$\begin{aligned}\frac{\sqrt{18a^2}}{4\sqrt{3a^2}} &= \frac{\sqrt{6 \cdot \cancel{3} \cdot \cancel{a^2}}}{4\sqrt{\cancel{3a^2}}}\\ &= \frac{\sqrt{6}}{4}\end{aligned}$$

2. Would you get the same quotient if you extracted the roots from each radical first and then divided out common factors? Explain your reasoning.

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3. Jackie shared her solution for extracting roots and rewriting the quotient $\frac{\sqrt{25bc}}{\sqrt[3]{b^2c^2}}$, given $b > 0$ and $c > 0$. Explain why Jackie's work is incorrect.

Jackie

$$\frac{\sqrt{25bc}}{\sqrt[3]{b^2c^2}} = \frac{\sqrt{25bc}}{\sqrt[3]{b^2c^2}}$$
$$= \frac{\sqrt{25}}{\sqrt[3]{bc}}$$
$$= \frac{5}{\sqrt[3]{bc}}$$

Ask yourself:

Why are the restrictions $b > 0$ and $c > 0$, instead of $b \geq 0$ and $c \geq 0$?

4. Perform each operation and extract all roots. Write your final answer in radical form.

a. $2\sqrt{x} \cdot \sqrt{x} \cdot 5\sqrt{x}$, given $x \geq 0$

b. $2(\sqrt[3]{k})(\sqrt[3]{k})$

c. $7\sqrt{h}(3\sqrt{h} + 4\sqrt{h^3})$, given $h \geq 0$

d. $\sqrt{a} \cdot \sqrt[3]{a}$, given $a \geq 0$

e. $(n)(\sqrt[3]{4n})(\sqrt[3]{2n^2})$

f. $\sqrt{\frac{4x^4}{x^2}}$, given $x \neq 0$

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In some cases, you can rewrite the sum or difference of two terms as one term.

5. Consider how Grace, Diane, Ron, and Sheila rewrote the sum or difference of their original expression as one term.

Grace

$2\sqrt{x} + 6\sqrt{x} = 8\sqrt{x}$

Diane

$16\sqrt[3]{x^2} - 10\sqrt[3]{x^2} = 6\sqrt[3]{x^2}$

Ron

$0.1\sqrt{x} + 3.6\sqrt[3]{x} = 3.7\sqrt[3]{x}$

Sheila

$\sqrt{x} + \sqrt{y} = 2\sqrt{xy}$

a. Explain why Grace and Diane were each able to rewrite their original expression as one term.

b. Explain why Ron's answer is incorrect.

c. Explain why Sheila's answer is incorrect.

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When adding or subtracting radicals, you can combine like terms and write the result using fewer terms.

For example, the two terms, $3\sqrt{x}$ and $\sqrt{x}$ are like terms because their radicand and index, $\sqrt{x}$, are the same. The coefficients do not have to be the same.

On the other hand, the terms $-8\sqrt[3]{x^4}$ and $7\sqrt[3]{x}$ are not like terms. The indices are the same, 3, but the radicands are different, x^4 and x. Consider the exponential form, $x^{\frac{4}{3}}$ and $x^{\frac{1}{3}}$, notice that the bases are the same, the denominators in the exponent are the same, but the numerators in the exponents are different.

Worked Example

To determine the sum or difference of like radicals, add or subtract the coefficients.

$$3\sqrt{x} + \sqrt{x} = 4\sqrt{x}\text{, given } x \geq 0$$

You can also write an equivalent expression using powers.

$$3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{1}{2}}\text{, given } x \geq 0$$

6. Larry considered whether or not $4\sqrt{x}$ and $-5x^{\frac{1}{2}}$ are like terms, given $x \geq 0$.

Larry

They are not like terms because they are not written in the same form.

Explain the error in Larry's reasoning.

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7. Combine like terms, if possible, and write your final answer in radical form.

a. $\sqrt{y} - \sqrt{y}$, given $y \geq 0$

b. $9\sqrt{a} + 5\sqrt{b}$, given $a \geq 0$, $b \geq 0$

c. $2\sqrt{x} + \sqrt{x} + 5\sqrt{x}$, given $x \geq 0$

d. $7\sqrt{h} - 4.1\sqrt{h} + 2.4\sqrt{h}$, given $h \geq 0$

e. $3\sqrt{t}\,(\sqrt{t} - 8\sqrt{t}) + 4t$, given $t \geq 0$

f. $5\sqrt{g} + 2\sqrt[3]{g}$, given $g \geq 0$

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NOTES

Talk the Talk

Time to Operate!

1. **Complete the graphic organizer. Write two radicals whose sum, difference, product, and quotient are each equivalent to $6\sqrt[3]{x}$.**

Sum	Difference

$6\sqrt[3]{x}$

Product	Quotient

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NOTES

2. Identify, describe, and correct any error(s), if possible. If there are no errors, write Correct next to the problem.

a. $\sqrt{4x^2} = \sqrt{4} \cdot \sqrt{x^2}$
$= \sqrt{2^2} \cdot \sqrt{x^2}$
$= 2|x|$

b. $\sqrt[3]{3x^3} = (3x^3)^{\frac{1}{2}}$
$= |x|\sqrt{3}$

c. $\sqrt[4]{16x^8y^{12}} = \sqrt[4]{16} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{y^{12}}$
$= 4|x^2y^3|$

d. $\sqrt{81x^2} = (81x^2)^{\frac{1}{2}}$
$= 9|x|$

e. $\sqrt{16x^4y} = (16x^4y)^{\frac{1}{2}}$
$= 4|x^2|y$

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Assignment

Write

Explain two different instances when the expression $\sqrt[n]{x^n}$ can be written as x.

Remember

To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as $|x|$, when n is even, or x, when n is odd.

A radical expression $\sqrt[n]{x^a}$ can be rewritten as an exponential expression for $x^{\frac{a}{n}}$ all real values of x if the index n is odd, and for all real values of x greater than or equal to 0 if the index n is even.

Practice

1. Analyze Luke's and Elena's incorrect work. Identify the error(s) and correctly rewrite each radical after extracting all possible roots.

Luke

$$\sqrt{9a^5b^2} = \sqrt{9}\cdot\sqrt{a^5}\cdot\sqrt{b^2}$$
$$= \sqrt{9}\cdot\sqrt{a^4}\cdot\sqrt{a}\cdot\sqrt{b^2}$$
$$= 3a^2b\sqrt{a}$$

Elena

$$\sqrt[3]{24x^3y^6} = \sqrt[3]{24}\cdot\sqrt[3]{x^3}\cdot\sqrt[3]{y^6}$$
$$= \sqrt[3]{8}\cdot\sqrt[3]{3}\cdot|x|\cdot y^2$$
$$= 2\cdot 1\cdot|x|\cdot y^2$$
$$= 2|x|y^2$$

2. Extract all possible roots to rewrite each expression.
   a. $\sqrt{50a^3b^4}$
   b. $\sqrt[4]{48x^5y^{16}}$
   c. $\sqrt{(x-5)^2}$

3. Analyze Leland's and Kata's incorrect work. Identify the error(s) and correctly rewrite each radical after performing each operation and extracting all possible roots.

Leland

$$4\sqrt[3]{x^2}\cdot 5\sqrt{x^3} = 4x^{\frac{2}{3}}\cdot 5x^{\frac{3}{2}}$$
$$= 20x^{\frac{6}{6}}$$
$$= 20x$$

Kata

$$12\sqrt[3]{m^2} + 5\sqrt{m^2} - 8\sqrt[3]{m^2} = 9\sqrt{m^2}\,\sqrt[3]{m^2}$$
$$= 9|m|\sqrt[3]{m^2}$$

4. Rewrite $6\sqrt[3]{x^2}(15\sqrt[3]{x} - 13\sqrt[3]{x}) + 8x$ with the fewest terms possible.

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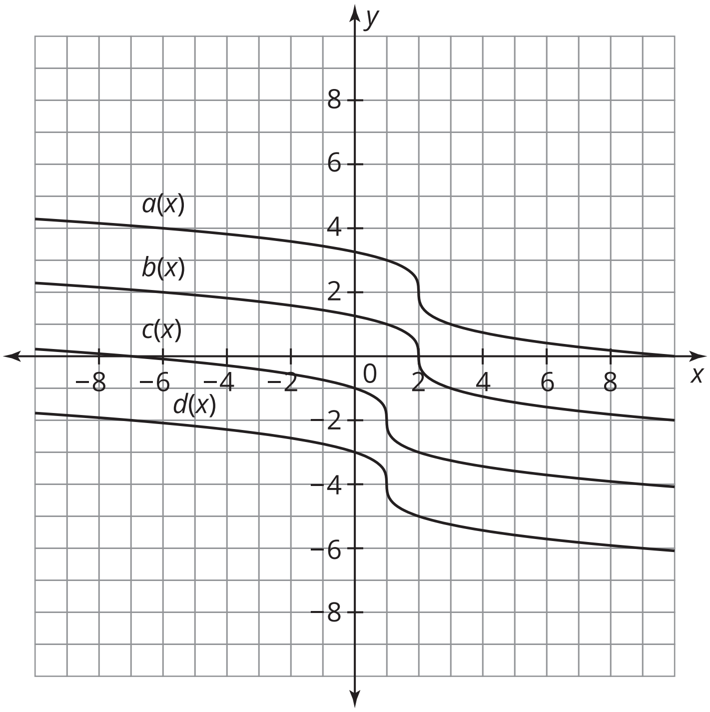
Stretch

Perform each operation, extract all roots, and write your final answers in radical form, without radicals in the denominator.

1. $\frac{8\sqrt{20x^7}}{3\sqrt[3]{8x^5}}$, given $x > 0$
2. $\frac{-2\sqrt[3]{24a^4}}{15\sqrt{27a^3}}$, given $a > 0$
3. $\frac{4\sqrt{28a^2b^2}}{5\sqrt{12ab^3}}$

Review

1. Each of the four curves shown on the graph are transformations of $f(x) = \sqrt[3]{x}$ with a restricted domain.



   a. Describe each transformation of $f(x) = \sqrt[3]{x}$ that was used to create the four functions in the design.

   b. Write each function used in the design. For each function, write the domain as an inequality.

2. Perform each operation. List any restrictions on the variables.

   a. $\frac{5}{x^3 - x^2 - 2x} - \frac{8}{x^2 + 2x + 1}$

   b. $\frac{x^2 - 16}{4(x + 1)} \cdot \frac{x^2 - 1}{x - 4}$

3. Describe the end behavior of the function $f(x) = -x^5 + 4x^3 - 2x - 2$.

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5

Into the Unknown

Solving Radical Equations

Warm Up

Determine the value of x, if possible.

1. $\sqrt{x} = 2$
2. $\sqrt[3]{x} = -2$
3. $\sqrt{x} = -2$

Learning Goals

- Use Properties of Equality to solve radical equations.
- Identify extraneous roots when solving radical equations.

Key Term

- extraneous solutions

You have used the Properties of Equality to isolate the variable in many types of equations in order to solve for an unknown quantity. How can you use these same properties to solve for an unknown quantity when the variable is under the radicand?

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GETTING STARTED

Step To It

Strategies for solving equations using the Properties of Equality to isolate the term containing the unknown are applicable when solving radical equations.

Let's compare the algebraic solution of a two-step quadratic equation to a two-step radical equation.

Worked Example

Solution Steps for a Quadratic Equation	**Solution Steps for a Radical Equation**
$2x^2 - 5 = 13$	$2\sqrt{x} - 5 = 13$
$2x^2 = 18$	$2\sqrt{x} = 18$
$x^2 = 9$	$\sqrt{x} = 9$
$\sqrt{x^2} = \sqrt{9}$	$(\sqrt{x})^2 = (9)^2$
$x = \pm 3$	$x = 81$

1. **Analyze the examples.**

   a. **Describe the similarities in the first two steps of each solution.**

   b. **Describe the differences in the remaining steps of each solution.**

2. **How would the strategy shown in the worked example change for cube and cube root equations? Provide an example to explain your reasoning.**

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ACTIVITY **5.1**

Solving Radical Equations

Let's look at different strategies for solving radical equations.
You can solve a radical equation with a single variable term under the radicand or a radical equation with more than one term under the radicand.

Worked Example

Solve $3\sqrt{x} + 7 = 25$.

$$\begin{aligned} 3\sqrt{x} + 7 &= 25 \\ 3\sqrt{x} &= 18 \\ \sqrt{x} &= 6 \\ (\sqrt{x})^2 &= (6)^2 \\ x &= 36 \end{aligned}$$

Check:

$$\begin{aligned} 3\sqrt{36} + 7 &\stackrel{?}{=} 25 \\ 3(6) + 7 &\stackrel{?}{=} 25 \\ 25 &= 25 ✔ \end{aligned}$$

Solve $\sqrt{x + 1} = 8$.

$$\begin{aligned} \sqrt{x + 1} &= 8 \\ (\sqrt{x + 1})^2 &= (8)^2 \\ x + 1 &= 64 \\ x &= 63 \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{63 + 1} &\stackrel{?}{=} 8 \\ \sqrt{64} &\stackrel{?}{=} 8 \\ 8 &= 8 ✔ \end{aligned}$$

You should always check your answers when solving equations. But with radical equations, it's extra important to check your answers. You'll soon learn why.

1. Analyze the worked examples.

a. How does the form of the equation $3\sqrt{x} + 7 = 25$ compare to the form of the equation $\sqrt{x + 1} = 8$?

b. How does the form of each equation relate to the solution strategy shown?

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Raising both sides of an equation to a power may introduce an *extraneous solution*, so it is important to check your answers.

Extraneous solutions are solutions that result from the process of solving an equation but are not valid solutions to the equation.

2. Solve and check each equation.

a. $\sqrt{2x} = 3$

b. $\sqrt[3]{2x - 3} = 2$

c. $4\sqrt{x - 6} = 8$

d. $\sqrt{2x + 1} = 5$

e. $2\sqrt[3]{x} + 16 = 0$

f. $\sqrt{3x - 1} + 9 = 8$

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3. **Marteiz solved the equation $x - \sqrt{x} = 2$. Explain Marteiz's error. Give the correct solution in your explanation.**

Marteiz

$$x - \sqrt{x} = 2$$
$$-\sqrt{x} = -x + 2$$
$$\sqrt{x} = x - 2$$
$$(\sqrt{x})^2 = (x - 2)^2$$
$$x = x^2 - 4x + 4$$
$$0 = x^2 - 5x + 4$$
$$0 = (x - 4)(x - 1)$$
$$x = 4 \text{ or } x = 1$$

The solution is $x = 4$ or $x = 1$.

4. **Solve the equation $x - 1 = \sqrt{x + 1}$.**

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ACTIVITY

5.2

Radical Equations in Context

The Beaufort scale is a system that measures wind speed and describes conditions at sea and on land. The scale's range is from 0 to 12. A zero on the Beaufort scale means that the wind speed is less than 1 mile per hour and the conditions at sea and on land are calm. A twelve on the Beaufort scale represents hurricane conditions with wind speeds greater than 74 miles per hour, resulting in greater than 50-foot waves at sea and severe damage to structures and landscape.

1. Consider the equation $V = 1.837B^{\frac{3}{2}}$ that models the relationship between wind speed in miles per hour *V* and the Beaufort numbers *B*.

a. Solve the equation for *B*.

b. Determine the Beaufort number for a wind speed of 20 miles per hour.

In medicine, Body Surface Area *BSA* is used to help determine proper dosage for medications.

2. The equation $BSA = \frac{\sqrt{W \cdot H}}{60}$ models the relationship between *BSA* in square meters, the patient's weight *W* in kilograms, and the patient's height *H* in centimeters.

a. Solve the equation for *H*.

b. Determine the height of a patient who weighs 90 kilograms and has a *BSA* of 2.1.

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Big Ben is the nickname of the bell in a well-known clock tower in London, England, that stands 316 feet tall. The clock is driven by a 660-pound pendulum in the tower that continually swings back and forth.

3. The relationship between the length of the pendulum L in feet and the time it takes for a pendulum to swing back and forth one time, or its period T, is modeled by the equation $T = 2\pi\sqrt{\frac{L}{32}}$.

a. Solve the equation for L.

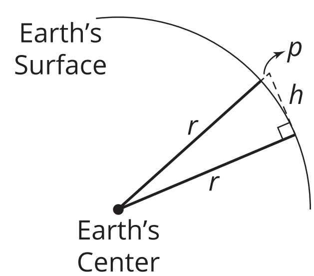
b. If the pendulum's period is 4 seconds, determine the pendulum's length.

A pilot is flying a plane high above the earth. She has clear vision to the horizon ahead.

The Earth is not actually a perfect sphere, but it is very close. The solution will give you a very good estimate of the distance from the pilot to the horizon.

4. Consider the diagram shown. The variable r represents the Earth's radius (miles), p represents the plane's height above the earth, or altitude (miles), and h represents the distance from the pilot to the horizon (miles).

a. Derive an equation to show the relationship between the three sides of the triangle. Then, solve the equation for the distance from the pilot to the horizon, h.



b. Use your equation from part (a) to calculate the plane's altitude if the distance from the pilot to the horizon is 225 miles. The earth's radius is 3959 miles.

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NOTES

TALK the TALK

Balances and Checks

1. Solve each equation for the unknown.

a. $\sqrt{2x - 1} = 7$

b. $3\sqrt[3]{x} + 15 = 0$

c. $\sqrt[4]{2x + 2} + 5 = 3$

d. $\sqrt[3]{4x + 1} = 3$

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Assignment

Write

Explain in your own words why it is important to check all solutions when solving radical equations.

Remember

Strategies to solve equations using the Properties of Equality to isolate the term containing the unknown are applicable when solving radical equations. Increasing the power of the variable during the solution process may introduce extraneous solutions.

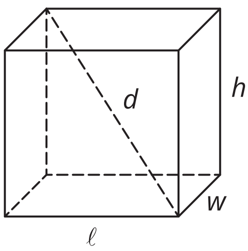
Practice

1. Analyze Jackson's incorrect work. Identify the error(s) and correctly solve the radical equation.

Jackson

$$\begin{aligned} \sqrt[3]{2x - 5} &= 5 \\ \left(\sqrt[3]{2x - 5}\right)^3 &= (5)^3 \\ 2x^3 - 5^3 &= 125 \\ 2x^3 - 125 &= 125 \\ 2^3 &= 250 \\ x^3 &= 125 \\ x &= 5 \end{aligned}$$

2. Solve $x + \sqrt{x} - 9 = -3$ and check your solution.
3. Solve $9.7x^{\frac{2}{3}} = 38.8$ and check your solution.
4. The length, d, of the diagonal in a right rectangular prism can be determined using the equation $d = \sqrt{\ell^2 + w^2 + h^2}$, where ℓ represents the length, w represents the width, and h represents the height. Determine the height of a right rectangular prism with a length of 8 inches, a width of 4 inches, and a diagonal length of 12 inches. Check your solution.



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Stretch

1. Solve $\sqrt{3x + 1} = 4 - \sqrt{5 - x}$ and check your solution.
2. Consider the functions $f(x) = x^{\frac{1}{2}}$ and $g(x) = \left(\frac{1}{2}\right)^x$ for $x \geq 0$.
   a. Complete the table of values for the functions.
   b. Sketch the graph of both functions.
   c. Determine the domain and range for each graph. Explain your reasoning.

x	$f(x) = x^{\frac{1}{2}}$	$g(x) = \left(\frac{1}{2}\right)^x$
−2		
−1		
0		
1		
2		
4		

Review

1. Rewrite each expression by extracting all possible roots.

   a. $\sqrt[3]{54x^5y^{12}}$　　　　b. $\sqrt{44a^3b^7}$

2. Rewrite $-2\sqrt{y^2} + 10\sqrt[3]{y^2} - 12\sqrt[3]{y^2} + 8\sqrt{y^2}$ with the fewest terms possible.

3. Yoon is going to sell T-shirts at a concert. Company A tells her it will cost \$300 up front and \$4.50 per shirt. Company B tells her it will cost \$250 up front and \$4.00 per shirt.

   a. Write a function to represent the average cost per T-shirt for the two companies.

   b. Which company will have a lower average cost per T-shirt if she plans on getting 200 T-shirts made? Show all of your work and explain your reasoning.

4. Identify the number of complex zeros for the polynomial equation $x^5 + x^4 + x^3 + x^2 - 12x - 12 = 0$. Explain your reasoning

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Radical Functions Summary

KEY TERMS

- inverse of a function
- invertible function
- Horizontal Line Test
- square root function
- cube root function
- radical function
- composition of functions

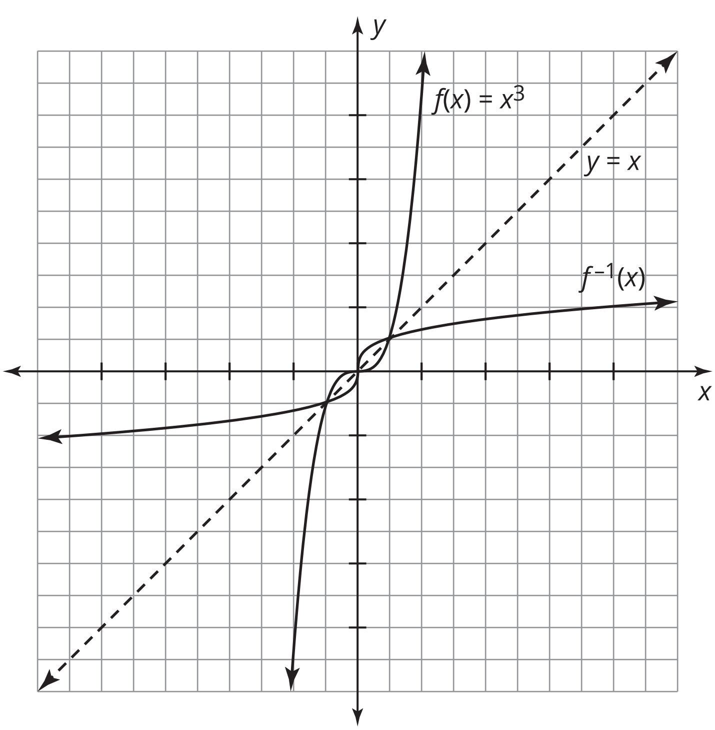
LESSON 1

Strike That, Invert It

Recall that a power function is a polynomial function of the form $P(x) = ax^n$, where n is a non-negative integer.

A function f is the set of all ordered pairs (x, y), or $(x, f(x))$, where for every value of x there is one and only one value of y, or $f(x)$. The **inverse of a function** is the set of all ordered pairs (y, x), or $(f(x), x)$.

For example, the graph shows $f(x) = x^3$ and its inverse, $f^{-1}(x)$.



If the inverse of a function f is also a function, then f is an **invertible function**, and its inverse is written as $f^{-1}(x)$.

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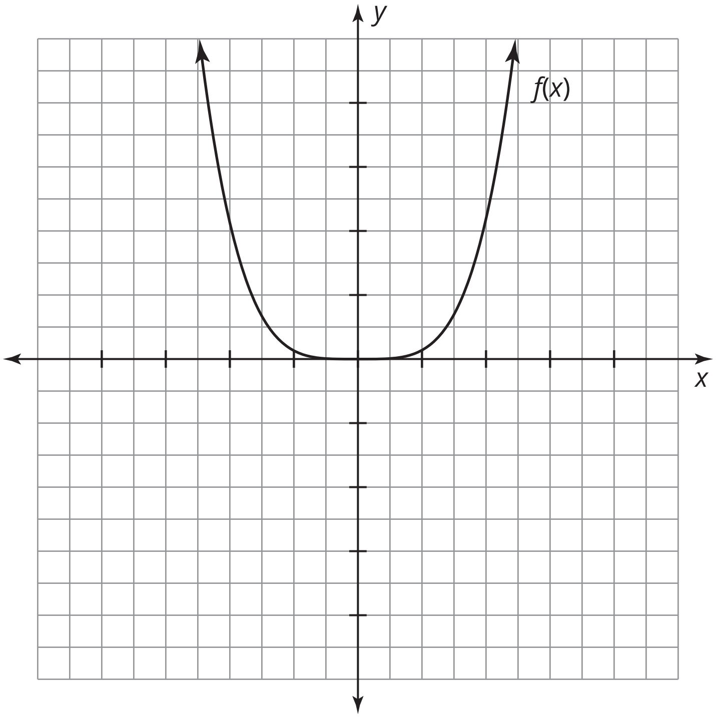
The Vertical Line Test is used to determine whether or not the inverse of a power function is also a function. The **Horizontal Line Test** is a visual method to determine whether a function has an inverse that is also a function. If any of the horizontal lines intersect the graph of the function at more than one point, then the inverse of the function is not a function.

For example, $f(x) = \frac{x^4}{56}$ is not invertible because it fails the Horizontal Line Test.



LESSON 2 Such a Rad Lesson

To determine the equation of the inverse of a power function, transpose x and y in the equation and solve for y.

For example, consider the function $y = x^5$. Determine the inverse of the function.

$$y = x^5$$
$$x = y^5$$
$$\sqrt[5]{x} = y$$

The equation for the inverse of the function $y = x^5$ is $y = \sqrt[5]{x}$.

The function $f(x) = x^2$ fails the Horizontal Line Test. Therefore, it is not an invertible function. But there are conditions that would allow $f(x) = x^2$ to pass the Horizontal Line Test. When you restrict the domain of the power function $f(x) = x^2$ to values greater than or equal to 0, the inverse of the function is called the **square root function** and is written as:

$$f^{-1}(x) = \sqrt{x}\text{, for } x \geq 0$$

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When determining the inverse of a function, you transpose x and y which are the inputs and outputs of a function. Thus, you are also switching the domain and range of the function. The domain of the function becomes the range of the inverse and the range of the function becomes the domain of the inverse. Restricting the domain of the function will restrict the range of the inverse.

Transformations of a function will also transform the inverse.

Transformation of Quadratic Function, $f(x)$	**Transformation of Inverse Function, $f^{-1}(x)$**
translation up D units	translation right D units
translation down D units	translation left D units
translation right C units	translation up C units
translation left C units	translation down C units

The **cube root function** is the inverse of the power function $f(x) = x^3$ and can be written as $f^{-1}(x) = \sqrt[3]{x}$. Because $f(x) = x^3$ is an invertible function the domain does not have to be restricted.

The inverses of power functions with exponents greater than or equal to 2, such as the square root function and the cube root function, are called **radical functions**.

The process of evaluating one function inside of another function is called the **composition of functions**. For two functions f and g, the composition of functions uses the output of $g(x)$ as the input of $f(x)$. It is notated as $f(g(x))$ or $g(f(x))$.

For example, to write a composition of the functions $g(x) = x^2$ and $f(x) = \sqrt{x}$ when the domain of $g(x)$ is restricted to $x \geq 0$, substitute the value of one of the functions for the argument, x, of the other function. This can be done in two ways.

$$\begin{aligned} f(g(x)) &= f(x^2) \\ &= \sqrt{x^2} \\ &= x \\ f(g(x)) &= x, \text{ for } x \geq 0 \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(\sqrt{x}) \\ &= (\sqrt{x})^2 \\ &= x \\ g(f(x)) &= x, \text{ for } x \geq 0 \end{aligned}$$

If $(g(x)) = g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverse functions.

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LESSON 3

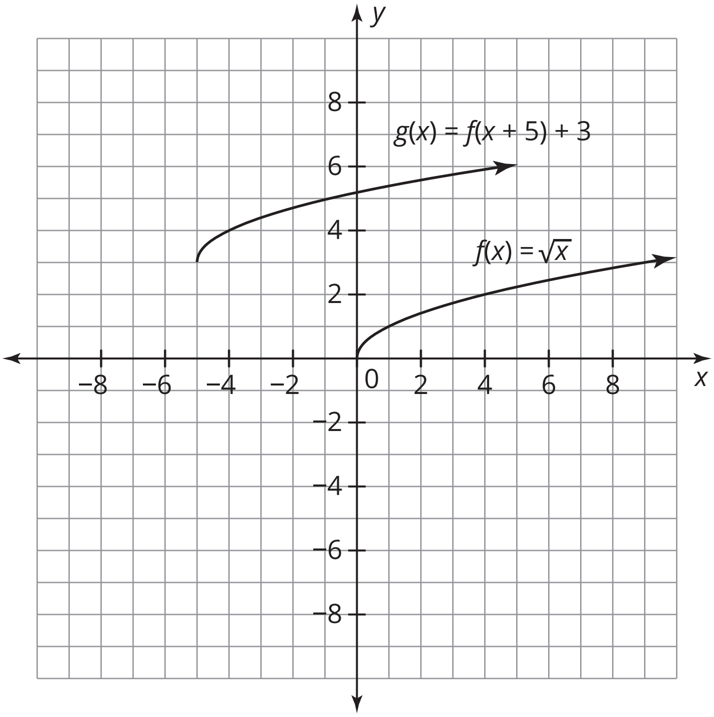
Making Waves

For the square root function and cube root function respectively, the transformation function form can be written as $s(x) = A\sqrt{B(x - C)} + D$ or $c(x) = A\sqrt[3]{B(x - C)} + D$. Transformations that take place inside the radical shift the function left or right. Transformations that take place outside the radical shift the function up or down.

For example, consider the graph of $f(x) = \sqrt{x}$ and its transformation represented by the equation $g(x) = f(x + 5) + 3$.

The domain of $f(x)$ is $[0, \infty)$.
The range of $f(x)$ is $[0, \infty)$.

The range of $g(x)$ is $[3, \infty)$.
The domain of $g(x)$ is $[-5, \infty)$.



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LESSON 4

Keepin' it Real

To rewrite a radical expression, extract the roots by using the rational exponents and the Properties of Powers. To extract a variable from a radical, the expression $\sqrt[n]{x^n}$ can be written as $|x|$ when n is even, and x when n is odd.

For example, you can rewrite the expression $\sqrt[4]{625x^8y^5z}$ as shown.

$$\begin{aligned}\sqrt[4]{625x^8y^5z} &= \sqrt[4]{625 \cdot x^8 \cdot y^5 \cdot z}\\ &= \sqrt[4]{625} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{y^5} \cdot \sqrt[4]{z}\\ &= 5x|y|\sqrt[4]{yz}\end{aligned}$$

You can rewrite a radical as a power with a rational exponent, and rewrite a power with a rational exponent as a radical. If n is an integer greater than 1, and, $x \geq 0$, then $\sqrt[n]{x} = x^{\frac{1}{n}}$.

For example, $\sqrt[4]{b} = b^{\frac{1}{4}}$ and $6^{\frac{1}{5}} = \sqrt[5]{6}$.

You can extract roots to rewrite radicals, using radicals or powers.

For example, you can extract the roots from $\sqrt[3]{8x^6}$ to rewrite the expression using radicals and using powers.

Using Radicals

$$\begin{aligned}\sqrt[3]{8x^6} &= \sqrt[3]{2^3 \cdot x^6}\\ &= \sqrt[3]{2^3 \cdot (x^2)^3}\\ &= \sqrt[3]{2^3} \cdot \sqrt[3]{(x^2)^3}\\ &= 2x^2\end{aligned}$$

Using Powers

$$\begin{aligned}\sqrt[3]{8x^6} &= (8x^6)^{\frac{1}{3}}\\ &= (2^3 \cdot x^6)^{\frac{1}{3}}\\ &= 2^{\frac{3}{3}} \cdot x^{\frac{6}{3}}\\ &= 2^1 \cdot x^2\\ &= 2x^2\end{aligned}$$

The root of a product is equal to the product of its roots, $\sqrt[p]{a^m b^n} = \sqrt[p]{a^m} \cdot \sqrt[p]{b^n}$

For example, you can extract roots to rewrite the product $\sqrt{18a^2} \cdot 4\sqrt{3a^2}$.

$$\begin{aligned}\sqrt{18a^2} \cdot \sqrt[4]{3a^2} &= 4\sqrt{54a^4}\\ &= 4\sqrt{9 \cdot 6 \cdot a^4}\\ &= 4 \cdot 3 \cdot a^2\sqrt{6}\\ &= 12a^2\sqrt{6}\end{aligned}$$

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The root of a quotient is equal to the quotient of its roots, $\sqrt[p]{\frac{a^m}{b^n}} = \frac{\sqrt[p]{a^m}}{\sqrt[p]{b^n}}$.

For example, you can extract the roots to rewrite the quotient $\frac{\sqrt{18a^2}}{4\sqrt{3a^2}}$.

$$\begin{aligned}\frac{\sqrt{18a^2}}{4\sqrt{3a^2}} &= \frac{\sqrt{6 \cdot 3a^2}}{4\sqrt{3a^2}} \\ &= \frac{\sqrt{6}}{4}\end{aligned}$$

In some cases, you can rewrite the sum or difference of two terms as one term. When adding or subtracting radicals, you can combine like terms and write the result using fewer terms.

For example, the two terms, $3\sqrt{x}$ and $\sqrt{x}$ are like terms because their radicand and index, $\sqrt{x}$, are the same. The coefficients do not have to be the same. On the other hand, the terms $-8\sqrt[3]{x^4}$ and $7\sqrt[3]{x}$ are not like terms. The indices are the same, 3, but the radicands are different, x^4 and x^3.

To determine the sum or difference of like radicals, add or subtract the coefficients.

$$3\sqrt{x} + \sqrt{x} = 4\sqrt{x}, \text{ given } x \geq 0$$

You can also write an equivalent expression using powers.

$$3x^{\frac{1}{2}} + x^{\frac{1}{2}} = 4x^{\frac{1}{2}}, \text{ given } x \geq 0$$

LESSON 5 Into the Unknown

To solve a radical equation, isolate the radical term if possible. Then, raise the entire equation to the power that will eliminate the radical. Finally, follow the steps necessary to solve the equation and check for extraneous solutions.

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For example, consider the solution steps to solve $\sqrt{x+2} + 10 = x$.

$$\begin{aligned} \sqrt{x+2} + 10 &= x \\ \sqrt{x+2} &= x - 10 \\ (\sqrt{x+2})^2 &= (x-10)^2 \\ x + 2 &= x^2 - 20x + 100 \\ 0 &= x^2 - 21x + 98 \\ 0 &= (x-14)(x-7) \\ x &= 14 \text{ or } x = 7 \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{14+2} + 10 &\stackrel{?}{=} 14 \\ \sqrt{16} + 10 &\stackrel{?}{=} 14 \\ 14 &= 14 \checkmark \end{aligned}$$

Check:

$$\begin{aligned} \sqrt{7+2} + 10 &\stackrel{?}{=} 14 \\ \sqrt{9} + 10 &\stackrel{?}{=} 14 \\ 13 &\neq 14 \end{aligned}$$

Extraneous solution

There is one solution, $x = 14$.

To solve a problem with radical equations, identify what the problem is asking. Then, determine how to use the given equation to solve the problem. Finally, follow the process for solving radical equations.

For example, consider the following problem situation.

Big Ben is the nickname of a well-known clock tower in London, England, that stands 316 feet tall. The clock is driven by a 660-pound pendulum in the tower that continually swings back and forth. The relationship between the length of the pendulum, L, in feet and the time it takes for a pendulum to swing back and forth one time, or its period T, is modeled by the equation

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$T = 2\pi\sqrt{\frac{L}{32}}$. If the pendulum's period is 4 seconds, solve for L.

$$\begin{aligned} 4 &= 2\pi\sqrt{\frac{L}{32}} \\ \frac{4}{2\pi} &= \sqrt{\frac{L}{32}} \\ \left(\frac{2}{\pi}\right)^2 &= \left(\sqrt{\frac{L}{32}}\right)^2 \\ \frac{4}{\pi^2} &= \frac{L}{32} \\ 4 \cdot 32 &= \pi^2 \cdot L \\ L &= \frac{128}{\pi^2} \approx 12.97 \end{aligned}$$

The length of the pendulum is approximately 12.97 feet.

TOPIC 2

Exponential and Logarithmic Functions

In his 1798 book An Essay on the Principle of Population*, Robert Malthus warned that food supply increases linearly while populations increase exponentially, leading to inevitable famine. Fortunately he was not correct; globally speaking, human population growth is very close to linear.*

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Module 3: Inverting Functions

TOPIC 2: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Students begin this topic by creating exponential graphs using their prior knowledge of geometric sequences. They write exponential growth and decay functions given specified characteristics. The irrational number *e*, or natural base *e*, is then introduced through the development of continuous compound interest. Logarithmic functions are introduced as the inverse of exponential functions. Students explore the key characteristics of the logarithmic function and transformations of logarithmic functions and state restrictions on the variables for any logarithmic equation.

Where have we been?

In previous courses, students have analyzed exponential functions, including their key characteristics and transformations. They have also investigated inverse functions in the previous topic and in earlier courses. Although they may not be familiar with the constant *e*, students have experience with irrational numbers, including some square roots and π.

Where are we going?

Students will use the intuitions they gain by studying the graphs of logarithmic functions in the next topic to analyze logarithmic equations and apply logarithmic functions to situations. As will be shown in the situations in these topics, logarithmic functions have a number of applications in astronomy, medicine, mechanics, physics, and seismology.

The Triangle of Power

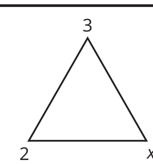
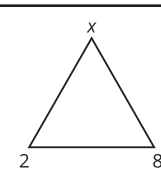
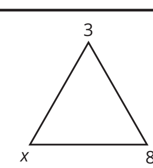
The logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. You can use the "Triangle of Power" to help you make sense of logarithms.

logarithm, or exponent

argument

$$3 = \log_2 8$$

base

	$\log_2 x = 3$ $x = 8$
	$\log_2 8 = x$ $x = 3$
	$\log_x 8 = 3$ $x = 2$

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Earthquakes

The Richter scale is used to rate the amount of energy an earthquake releases. This is calculated using information gathered by a seismograph.

The Richter scale is logarithmic, meaning that whole-number jumps in the rating indicate a tenfold increase in the wave amplitude of the earthquake. For example, the wave amplitude in a Level 4 earthquake is ten times greater than the amplitude of a Level 5 earthquake, and the amplitude increases 100 times between a Level 6 earthquake and a Level 8 earthquake.

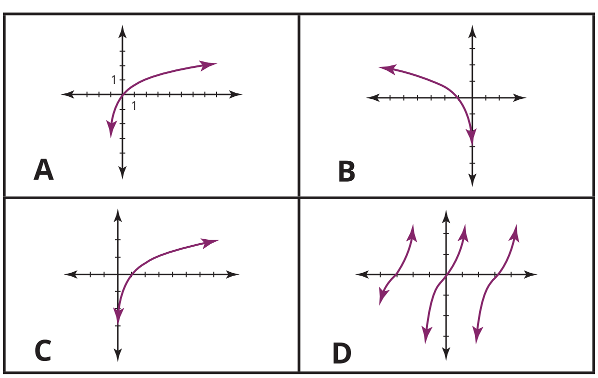
Only a tiny portion, 15 or so, of the 1.4 million quakes that register above 2 each year register at 7 or above, which is the threshold for a quake to be considered major.

Talking Points

Logarithmic functions can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Identify the graph of $f(x) = \ln(-x)$.



The function $f(x) = \ln(-x)$ is a reflection of the function $f(x) = \ln(x)$ across the y-axis.
Choice B is correct.

Key Terms

natural base *e*

The natural base *e* is a mathematical constant approximately equal to 2.71828.

logarithm

The logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number.

natural logarithm

A natural logarithm is a logarithm with base *e*, and is usually written as ln.

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1

Half-Life

Comparing Linear and Exponential Functions

Warm Up

Define each term and provide an example.

1. Arithmetic sequence

2. Geometric sequence

3. Product Rule of Powers

Learning Goals

- Construct an exponential function from a geometric sequence.
- Classify functions as exponential growth or decay.
- Compare tables, graphs, and equations of exponential functions.
- Compare linear and exponential functions.

Key Term

- half-life

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You have studied exponential functions in previous courses. How can you use exponential functions to model real-world situations?

GETTING STARTED

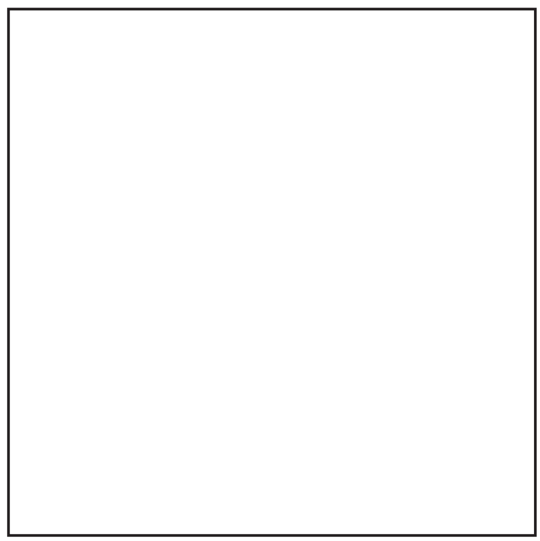
Asymptotic, Don't You Think?

When you work with sequences, you often use something called *repeated reasoning*. This means that you notice when a pattern repeats and think about how a pattern will extend when repeated.

Ask yourself:

What is repeating? What happens when the repeating pattern is extended?

1. A square is shown. Use a ruler to draw a line segment that divides the square in half. Then draw a second line segment in just one of the two halves you created to divide that section in half. Continue like this until the divisions become too small to make.



Remember:

The explicit formula for a geometric sequence is given by $a_n = a_1 \cdot r^{(n-1)}$, where a_n represents the nth term of the sequence, a_1 represents the first term of the sequence, and r represents the common ratio, or constant multiplier.

2. Compare your drawing with your classmates' drawings. What is the area of the smallest section shown in your square if *A* is the area of the original square.

3. Write the explicit formula you could use to determine the area of the smallest section. Explain your reasoning.

4. What would happen to the area of the smallest section if you could keep dividing the square forever?

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ACTIVITY 1.1

Arithmetic and Geometric Sequences

Allison and Beth each receive $10 per week for doing chores for their neighbor. One day, Allison decides to increase her income using her knowledge of exponential growth. She proposes that her payment change to a penny for the first week, and then double each week thereafter.

1. Complete the table to represent the amount that Allison and Beth will earn each week.

Week	Allison's Income (dollars)	Beth's Income (dollars)
1	0.01	10.00
2		
3		
4		
5		
6		
7		
8		

2. How does Allison's income change as the number of weeks increases?

3. Does Allison's income represent an arithmetic or geometric sequence? Explain your reasoning and state the general formula.

4. Write an equation to represent Allison's income, g_n, after n weeks.

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5. **What is the value of g_n for $n = 0$? Does this value make sense in problem situation?**

6. **If the pattern were to continue, how many weeks would it take for Allison to have a greater weekly income than Beth?**

Remember:

Some geometric sequences, when written in function notation, are exponential functions. The function gets its name from the variable in the exponent.

Worked Example

You can write the explicit formula for the geometric sequence $g_n = 0.01 \cdot 2^{(n-1)}$ as a function in the form $f(x) = ab^x$, using the properties of powers.

Statement	**Reason**
$g_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for the geometric sequence
$f(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation
$f(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product Rule
$f(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$f(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative Property of Multiplication
$f(n) = 0.005 \cdot 2^n$	Perform multiplication

So, $g_n = 0.01 \cdot 2^{(n-1)}$ can be written as $f(n) = 0.005 \cdot 2^n$.

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7. Calculate the income that Allison would earn per week in each week given:

a. 15th week

b. 20th week

c. 24th week

8. Predict the shape and characteristics of the graph that models Allison's income as a function of the number of weeks.

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ACTIVITY
1.2

Linear and Exponential Models

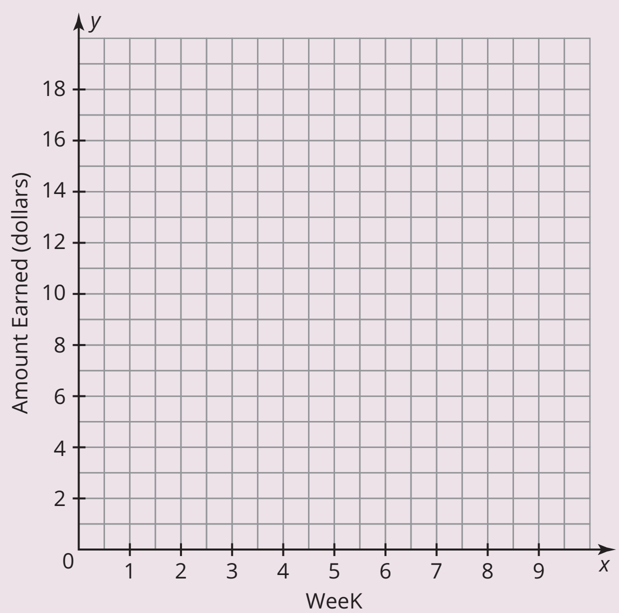
Beth is amazed at how quickly Allison was able to make a lot of money and decides that she wants in on the action. She asks her two friends, Quinton and Alisha, to help her come up with a plan.

1. Consider Quinton's and Alishia's plans. Whose plan should Beth choose? Complete the table and graph to justify your reasoning. Round to the nearest hundredth.

Quinton
You could start with a dollar and ask for 50% more each week.

Alisha
You could start with a dollar and add another dollar each week.

Week	Quinton's Plan (dollars)	Alisha's Plan (dollars)
1		
2		
3		
4		
5		
6		
7		



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2. Write functions to represent Quinton's plan, $q(x)$, and Alisha's plan, $a(x)$.

3. **Use your choice from Question 1 to determine how much Beth will earn in Week 10.**

4. **If Beth and Allison both start using their exponential model to earn income at the same time, who will earn a higher income in Week 12?**

5. **Use technology to determine when Allison's and Beth's incomes will be equal. Does this make sense in the problem situation? Explain your reasoning.**

6. **Compare Allison's and Beth's function models.**

a. **As the number of weeks continues to increase, whose model will generate more per week?**

b. **Consider the *a*- and *b*-values of the exponential functions. How do they further support your claim?**

Remember:

The general form of an exponential equation is $y = a \cdot b^x$.

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ACTIVITY
1.3

Modeling Situations with Exponential Functions

Simeon is studying for a big test and is trying to stay awake. He drank a 12-ounce can of Big Buzz Energy Drink that contains 80 milligrams of caffeine. He is wondering how long the caffeine will stay in his system if the caffeine has a *half-life* of 5 hours.

The term *half-life* does not just apply to radioactive material. It is also used in medicine to describe how long a chemical exists in the human body.

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

1. How much caffeine remains in Simeon's system after 5 hours? After 10 hours? Explain your reasoning.

2. Complete the table to determine the amount of caffeine in Simeon's system at each time interval.

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)					
Number of Half-Life Cycles					

3. What is the initial amount of caffeine in Simeon's system? What is the rate of decay?

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4. **Emily, Tyler, and Renee were asked to write an exponential function $A(t)$ to represent the amount of caffeine remaining in Simeon's system after t hours.**

Emily

$A(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{5}} t$

The variable represents the number of hours, and the half-life occurs in 5 hour cycles, so I divided my exponent by 5.

Tyler

$A(t) = 80\left(\frac{1}{2}\right)^{-\frac{t}{5}}$

The variable t represents the number of hours, and since it's a decay function, I made my exponent negative.

Renee

$A(t) = 80\left(\frac{1}{2}\right)^{5t}$

The variable t represents the number of hours and I multiplied it by 5 to represent the half-life cycle of 5 hours.

a. **Why is Tyler's reasoning incorrect?**

b. **Why is Renee's reasoning incorrect?**

Think about:

It may be helpful to substitute the values from the table to check each student's function.

5. **How much caffeine remains in Simeon's system after 2 hours?**

6. **Kendra suggests that she can calculate the amount of caffeine remaining by rewriting the equation as $A(t) = 40^{\frac{t}{5}}$. Is Kendra correct? Explain your reasoning.**

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7. Use technology to predict when the caffeine will be completely out of Simeon's system. Does this make sense, given what you know about exponential functions? Explain your reasoning.

8. Approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

9. Use the properties of exponents to rewrite your function so that only the variable t is in the exponent. What percentage of caffeine remains after each hour?

10. Suppose Simeon is taking an antibiotic that extends the half-life of caffeine to 8 hours. Write a function $B(t)$ that models the amount of caffeine remaining under these new conditions.

11. Complete the table for the new half-life. Round to the nearest hundredth.

Time Elapsed (hours)	0	5	10	15	20
Caffeine in System (mg)					
Number of Half-Life Cycles					

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12. How does the half-life of 8 hours rather than 5 hours affect the amount of caffeine remaining in Simeon's system?

13. Under these new conditions, approximately when will the amount of caffeine remaining in Simeon's system be less than 1 milligram?

14. What generalization can you make about the effect of greater or lesser half-lives of substances?

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NOTES

TALK the TALK

Exponentials and Dating

Carbon-14 is a radioactive isotope used to date the age of plants and animals. It has a half-life of about 5730 years. While a plant or animal is alive, the amount of carbon-14 present is constant, but when the plant or animal dies, the amount of carbon-14 begins to decay. To determine the length of time since the plant or animal died, you measure the percent of carbon-14 remaining, and apply the decay function.

1. **Write the decay function for carbon-14, using the value 100 as the initial amount present. The unit for 100 is not important because you will express the amount of remaining carbon-14 as a percent.**

2. **Use your function to calculate the percentage of carbon-14 remaining after:**

    a. **100 years.**

    b. **1000 years.**

    c. **10,000 years.**

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Assignment

Write

Define the term *half-life* in your own words.

Remember

The general form of an exponential equation is $y = a \cdot b^x$ where $b > 0$ and $b \neq 1$.

Practice

1. Wildlife biologists are studying the coyote populations on 2 wildlife preserves to better understand the role climate plays in population change. The table displays the coyote populations on both preserves for each year of the study. The Year 0 corresponds to the date of the biologists' initial observations.

Year	Alaska Preserve	Tennessee Preserve
0	200	80
1	220	100
2	242	125
3		
4		

   a. Can the population be represented by an arithmetic or a geometric sequence? Explain your reasoning.
   b. For each wildlife preserve, write a sequence to represent the coyote population in a given year.
   c. For each wildlife preserve, write a function to represent the coyote population as a function of the year of the study.
   d. Use the functions you wrote in part (c) to complete the table. Round all answers to the nearest whole number.
   e. Graph and label the functions you wrote in part (c).
   f. Do the populations represent examples of exponential decay, exponential growth, or neither? Explain your reasoning.
   g. Will the coyote population on the Tennessee preserve ever exceed the coyote population on the Alaska preserve? If so, when will this occur?
   h. Make a hypothesis about the role the climate plays on coyote populations based on the results of the study, assuming all other population growth factors are equal. Explain your reasoning.

Stretch

A sheet of paper is approximately 0.1 mm thick. Suppose you could fold the paper in half as many times as you wished, doubling the thickness each time, without tearing the paper. How many times would you need to fold the paper in order for its thickness to reach the Moon?

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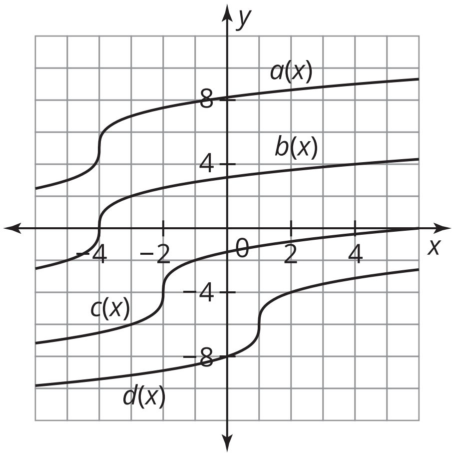
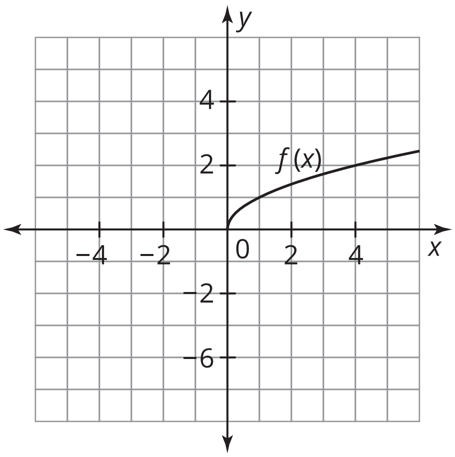
Review

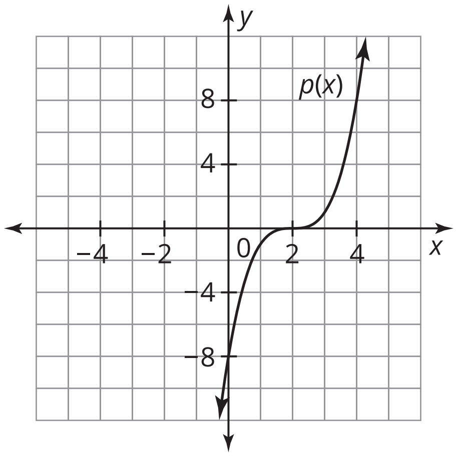
1. Each of the four curves shown on the graph are transformations of $f(x) = \sqrt[3]{x}$ with a restricted domain.
   a. Describe each of the four transformations of $f(x) = \sqrt[3]{x}$ that were used to create the four functions in the design.
   b. Write each function used in the design. For each function, write the domain as an inequality.



2. The graph of $f(x) = \sqrt{x}$ is shown with the restricted domain of $0 \le x \le 6$.
   a. Sketch a function $g(x)$ to represent the reflection of $f(x)$ across the x-axis. Write the function $g(x)$.
   b. Sketch the functions $f'(x)$ and $g'(x)$ to represent reflections of the $f(x)$ and $g(x)$ across the x-axis. Write the functions $f'(x)$ and $g'(x)$.



3. Identify the extrema, zeros, and intercepts of the graph of $p(x)$.



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2

Pert and Nert

Properties of Exponential Graphs

Warm Up

Juno is opening her first savings account and is depositing \$100. The bank offers 3% annual interest to be calculated at the end of each year.

1. Write a function $A(t)$ to model the amount of money in Juno's savings account after t years.

2. Calculate the amount of money in Juno's savings account at the end of 1 year.

3. Calculate the amount of money in Juno's savings account at the end of 5 years.

Learning Goals

- Identify the domain and range of exponential functions.
- Investigate graphs of exponential functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Explore the irrational number e.

Key Term

- natural base e

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You have calculated interest compounded annually using an exponential equation in the form $A = P \cdot (1 + r)^t$. How does this equation change when interest is compounded at more frequent intervals? Can interest be compounded continuously?

GETTING STARTED

You Have the Power

1. **Cut out the exponential graphs and equations located at the end of the lesson. Match each equation to its graph. Sort them into growth or decay functions, and attach them onto the graphic organizer located at the end of the lesson. Then, complete each table.**

2. **Analyze the exponential growth and decay functions.**

   a. **What point do the graphs have in common? Why?**

   b. **Compare the equations of the six functions you just sorted. What differentiates an exponential growth function from an exponential decay function?**

Think about:

How does the base affect the graph of an exponential function?

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ACTIVITY 2.1

Key Characteristics of Exponential Functions

Consider the equations of the six functions you sorted in the Getting Started.

1. Sara and Scott's teacher asked them to each write a rule that determines whether a function represents exponential growth or decay, based on its equation.

Sara
For exponential growth functions, b is a value greater than 1, but for exponential decay functions, b is a fraction or decimal between 0 and 1.

Scott
For exponential growth functions, b is greater than 1. For exponential decay functions, b is less than 1.

Why is Scott's reasoning incorrect? Provide a counterexample that would disprove his claim and explain your reasoning.

2. Which *b*-values for the function $f(x) = b^x$ produce neither growth nor decay? Provide examples to support your answer.

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Ask yourself:

Do you have enough information to write a unique exponential function?

3. **Write an exponential function that satisfies the given characteristics.**

   a. **Increasing over $(-\infty, \infty)$**
   Reference point (1, 6)

   b. **Decreasing over $(-\infty, \infty)$**
   Reference point (−1, 4)

   c. **End behavior:** **As $x \to -\infty$, $f(x) \to 0$**
   As $x \to \infty$, $f(x) \to \infty$
   Point (2, 6.25)

4. **Summarize the characteristics for the basic exponential growth and exponential decay functions.**

	Basic Exponential Growth $f(x) = b^x$, $b > 1$	Basic Exponential Decay $f(x) = b^x$, $0 < b < 1$
Domain		
Range		
Asymptote		
Intercepts		
End Behavior		
Intervals of Increase or Decrease		

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ACTIVITY 2.2

The Natural Base *e*

Helen is opening her first savings account and is depositing $500. Suppose she decides on a bank that offers 6% annual interest to be calculated at the end of each year.

Remember:

The formula for compound interest is $A = P \cdot \left(1 + \frac{r}{k}\right)^{kt}$, where P represents the initial amount, r is the interest rate, k is the number of compounding periods per year, and t represents time in years.

1. Write a function $A(t)$ to model the amount of money in Helen's savings account after t years and determine the amount in the account at the end of 1 year and 5 years.

2. Suppose that the bank decides to start compounding interest at the end of every 6 months. If they still want to offer 6% per year, how much interest would they offer per 6-month period?

3. John, Betty Jo, and Lizzie were each asked to calculate the amount of money Helen would have in her savings at the end of the year if interest was compounded twice a year. Who's correct? Explain your reasoning.

John

1st 6 months:

$A(t) = 500(1 + 0.03)$

$A(t) = 515$

2nd 6 months:

$A(t) = 515(1 + 0.03)$

$A(t) = 530.45$

Betty Jo

$A(t) = 500\left(1 + \frac{0.06}{2}\right)^2$

$A(t) = 530.45$

Lizzie

$A(t) = 2(500(1 + 0.03))$

$A(t) = 1030$

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4. **Write a function to model the amount of money in Helen's savings account at the end of *t* years, compounded *k* times during the year.**

5. **Determine the amount of money in Helen's account at the end of 3 years if it is compounded:**

   a. **twice a year.**

   b. **monthly.**

   c. **daily.**

6. **What effect does the frequency of compounding have on the amount of money in her savings account?**

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Recall that in Question 4 the variable *k* represented the number of compounding periods per year. Let's examine what happens as the interest is compounded more frequently.

7. Imagine that Helen finds a different bank that offers her 100% interest. Complete the table to calculate how much Helen would accrue in 1 year for each period of compounding if she starts with $1.

Period of Compounding	$k =$	Formula	Amount
Yearly	1	$1\left(1+\frac{1}{1}\right)^{1\cdot 1}$	2.00
Semi-Annually	2	$1\left(1+\frac{1}{2}\right)^{2\cdot 1}$	2.25
Quarterly	4	$1\left(1+\frac{1}{4}\right)^{4\cdot 1}$	
Monthly	12		
Weekly			
Daily			
Hourly			
Every Minute			
Every Second			

8. Make an observation about the frequency of compounding and the amount that Helen earns. What amount is it approaching?

Think about:

Even though you are working with money, to see the pattern, it is helpful to compute the compound interest to at least six decimal places.

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The amount that Helen's earnings approach is actually an irrational number called e.

$$e \approx 2.7182818284590 45...$$

It is often referred to as the **natural base *e***.

You have worked with π, an irrational number that was approximated as 3.14159265. . . Pi is an incredibly important part of many geometric formulas and occurs so frequently that, rather than write out "3.14159265 . . ." each time, you use the symbol π.

The symbol e is used to represent the constant 2.718281. . . It is often used in models of population changes as well as radioactive decay of substances, and it is vital in physics and calculus.

The symbol for the natural base e was first used by Swiss mathematician Leonhard Euler in 1727 as part of a research manuscript he wrote at age 21. In fact, he used it so much, e became known as Euler's number. The constant e represents continuous growth and has many other mathematical properties that make it unique, which you will study further in calculus.

In Question 7, you showed that the expression $\left(1 + \frac{1}{k}\right)^k$ approaches e as k approaches infinity. You can use this fact to derive the formula for compound interest with continuous compounding.

Worked Example

For compounded interest, the amount in an account, A, can be represented with the equation $A = P \cdot \left(1 + \frac{r}{k}\right)^{kt}$, where P represents the principal, or starting amount, r represents the interest rate, k represents the number of compounding periods per year, and t represents time in years. You can rewrite the compound interest formula using algebraic reasoning.

		Reasoning
$P \cdot \left(1 + \frac{r}{k}\right)^{kt} = P \cdot \left(\left(1 + \frac{r}{k}\right)^{k}\right)^{t}$	$\longleftarrow$	$a^{mn} = (a^m)^n$
$= P \cdot \left(\left(1 + \frac{r}{k}\right)^{\frac{k}{r}}\right)^{rt}$	$\longleftarrow$	$(a^m)^n = (a^{\frac{m}{b}})^{bn}$
$= P \cdot \left(\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}\right)^{rt}$	$\longleftarrow$	$\frac{a}{b} = \frac{1}{\frac{b}{a}}$

Consider the expression $\left(1 + \frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}$. If $\frac{k}{r} = n$, then the expression can be written as $\left(1 + \frac{1}{n}\right)^n$, which approaches e as n approaches infinity.

Thus, the formula for compound interest with continuous compounding can be written as:

$$A = Pe^{rt}$$

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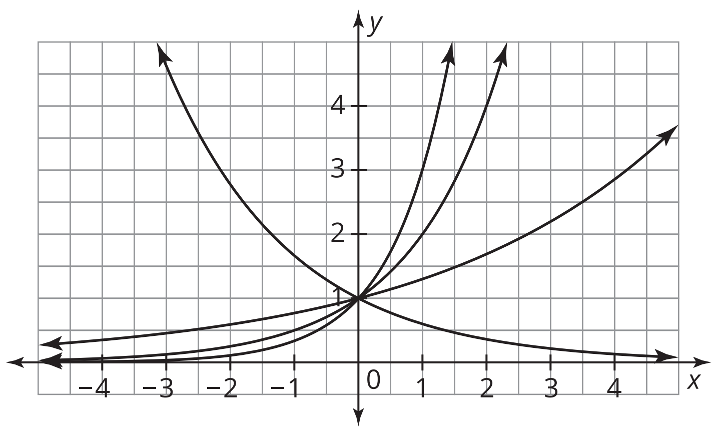
9. **Suppose Helen's investment of \$500 in her savings account is compounded continuously at a rate of 6%. How much would be in her account at the end of 1 year and 5 years?**

10. **The given functions are sketched on the coordinate plane shown.**

$f(x) = 2^x$ $\quad g(x) = 3^x$ $\quad j(x) = \left(\frac{3}{5}\right)^x$ $\quad k(x) = 1.3^x$

a. **Label each function.**



b. **Consider the function $m(x) = e^x$. Use your knowledge of the approximate value of e to sketch its graph. Explain your reasoning.**

c. **Using the functions $f(x) = 2^x$, $g(x) = 3^x$, and $m(x) = e^x$, approximate the values of $f(2)$, $g(2)$, and $m(2)$ on the number line. Explain your reasoning.**

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ACTIVITY 2.3 Modeling Population Change With *e*

In the previous activity, you explored the continuous growth of a principal amount using the value *e*. How can *e* be used to model population changes?

1. The formula for population growth is $N(t) = N_0e^{rt}$. Describe the contextual meaning of each quantity.

Quantity	Contextual Meaning
N_0	
r	
t	
$N(t)$	

2. Why is *e* used as the base?

3. How could this formula be used to represent a decline in population?

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4. **The population of the city of Fredericksburg, Virginia, was approximately 19,360 in 2000 and has been continuously growing at a rate of 2.9% each year.**

   a. **Use the formula for population growth to write a function to model this growth.**

   b. **Use your function model to predict the population of Fredericksburg in 2020.**

   c. **What value does your function model give for the population of Fredericksburg in the year 1980?**

5. **Use technology to estimate the number of years it would take Fredericksburg to grow to 40,000 people, assuming that the population trend continues.**

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NOTES

TALK the TALK

Easy *E*

In this lesson, you learned about the natural base *e* and derived the formula for interest compounded continuously. You also looked at key characteristics of graphs of exponential functions.

1. Jan invested \$5000 in a savings account for 6 years. The bank pays 2% interest, compounded monthly. How much more would Jan's investment be worth at the end of 6 years if the interest were compounded continuously? Show your work.

2. Describe the key characteristics of the basic exponential function with natural base *e*. Include domain, range, asymptote, intercepts, end behavior, and intervals of increase and decrease.

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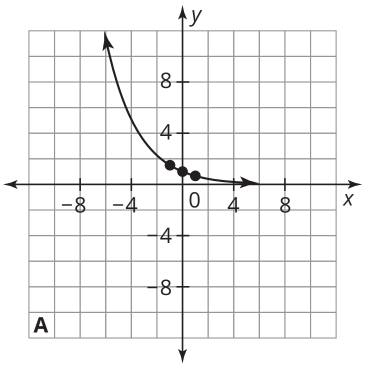
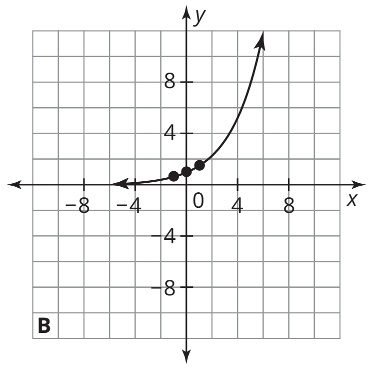
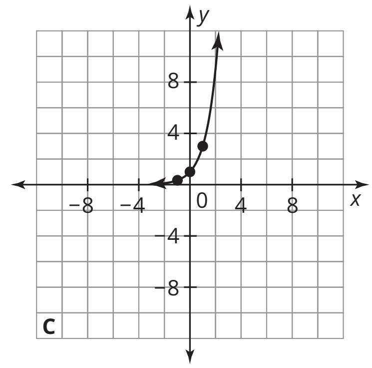
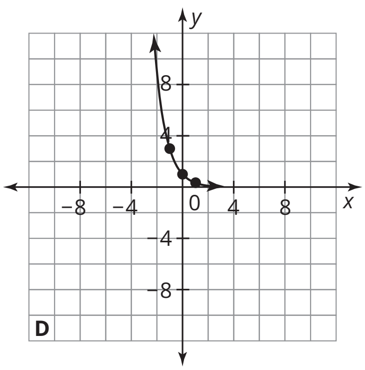
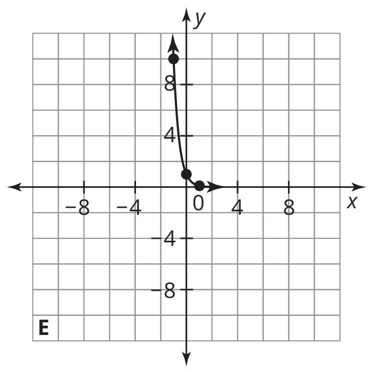
Exponential Graphs and Equations Cutouts

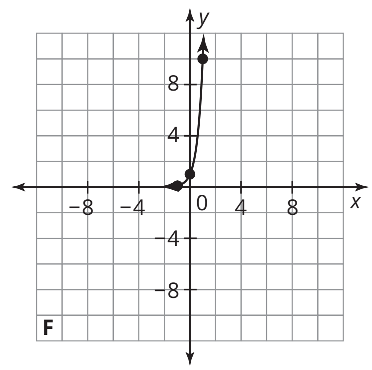












$f(x) = \left(\frac{2}{3}\right)^x$	$f(x) = 10^x$	$f(x) = 1.5^x$
$f(x) = 0.1^x$	$f(x) = 3^x$	$f(x) = \left(\frac{1}{3}\right)^x$

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Exponential Graphs and Equations Graphic Organizer

Growth

x	$f(x)$
−1	
0	
1	

Decay

x	$f(x)$
−1	
0	
1	

Growth

x	$f(x)$
−1	
0	
1	

Decay

x	$f(x)$
−1	
0	
1	

Growth

x	$f(x)$
−1	
0	
1	

Decay

x	$f(x)$
−1	
0	
1	

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Assignment

Write

Explain how the natural base e is similar to and different from π.

Remember

The natural base $e \approx 2.7182818$ is an irrational number that represents continuous growth and is used to model population changes as well as continuously compounded interest.

For continuous compounding, the compound interest formula is $A = Pe^{rt}$.

Practice

1. Caleb wants to invest \$1000 in a savings account. Lincoln Federal Bank is offering 6% interest compounded yearly. Washington National Bank is offering 5.5% interest compounded daily.
   a. For each bank, write an exponential function to represent the amount of money Caleb would have in the account after t years.
   b. Determine which bank Caleb should choose if he plans to invest his money for 5 years. If Caleb decides to leave the money in the bank for a longer period of time, will the other bank be a better deal in the long run? Explain your reasoning.
   c. Discuss the domain, range, asymptotes, intercepts, end behavior, and intervals of increase and decrease for each function as they relate to this problem situation.
2. In 2010, Bolivia had a population of 10.5 million people and an annual growth rate of 1.6%.
   a. Write a function to model Bolivia's population with respect to t, the number of years since 2010. Write your function in the form $N(t) = N_0e^{rt}$.
   b. Use your model to predict what Bolivia's population will be in the year 2030.
   c. Use your model to estimate Bolivia's population in the years 1990 and 1970.
   d. Use technology to estimate when Bolivia's population will reach 20 million people.
   e. Discuss the domain, range, asymptotes, intercepts, end behavior, and intervals of increase and decrease for your population model as they relate to this problem situation.

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Stretch

It has been called the most beautiful equation in all of mathematics. This equation links together four of the most important constants in math, including e and the imaginary number i:

$$e^{i\pi} + 1 = ?$$

Three of these constants are shown. The right-hand side of the equation is the fourth constant. What is it? What does the expression $e^{i\pi}$ tell us? Research this question.

Review

1. Nadine bought a house in 1995 for $155,000. The table displays the value of the house over several years. The year 0 corresponds to the date of the year Nadine bought the house.
   a. Does the value of the house represent an example of exponential decay, exponential growth, or neither? Explain your reasoning.
   b. Write a function to represent the value of the house, V, as a function of the year, n.
   c. What was Nadine's house worth in 2010, assuming the value of the house increased each year at the same rate?

Year	Value of House ($)
0	155,000
1	162,750
2	170,887.50
3	179,431.875

2. The population of a small town in 2015 was 8,455. The table displays the population of the town for several years. The year 0 corresponds to the the year 2015.
   a. Does the population represent an example of exponential decay, exponential growth, or neither? Explain your reasoning.
   b. Write a function to represent the population of the town, P, as a function of the year, t.
   c. Use the function you wrote in part (a) to complete the table. Round all answers to the nearest whole number.

Year	Population
0	8455
1	6764
2	5411
3	
4	
5	

3. Solve each equation and check your solution.
   a. $\sqrt[3]{4x + 3} = -2$
   b. $\sqrt{x + 2} + 4 = x$
   c. $\frac{m - 4}{4} + \frac{m}{3} = 6$

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3

Return of the Inverse

Logarithmic Functions

Warm Up

For each function, determine the domain, range, and end behavior. Describe the function as exponential growth or decay when $x > 0$.

1. $f(x) = e^x$

2. $g(x) = e^{-x}$

Learning Goals

- Graph the inverses of exponential functions with bases of 2, 10, and e.
- Recognize the inverse of an exponential function as a logarithm.
- Identify the domain and range of logarithmic functions.
- Investigate graphs of logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.

Key Terms

- logarithm
- logarithmic function
- common logarithm
- natural logarithm

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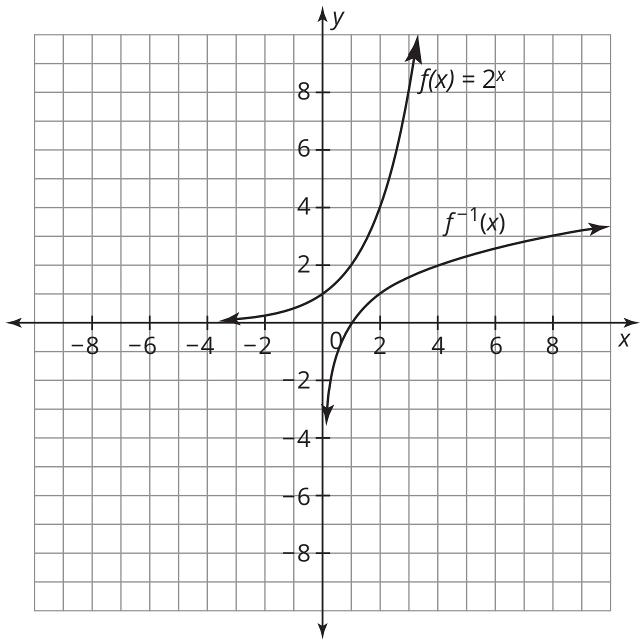
You have investigated the inverses of linear, quadratic, and power functions. What characteristics can you identify from the inverse of exponential functions—logarithmic functions?

GETTING STARTED

Across $y = x$

Consider the table and graph for the basic exponential function $f(x) = 2^x$ and its inverse.

x	$f(x) = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



x	$f^{-1}(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

Remember:

For any function f with ordered pairs (x, y), or $(x, f(x))$, the inverse of the function f is the set of all ordered pairs (y, x), or $(f(x), x)$.

1. Complete the table to identify the key characteristics of each function. Compare the key characteristics of the function and its inverse.

	$f(x)$	$f^{-1}(x)$
Domain		
Range		
***y*-Intercept**		
Asymptote		
Intervals of Increase and Decrease		
End Behavior		

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ACTIVITY 3.1 Logarithms Are Exponents

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number is the exponent to which the given base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $\log_b y = x$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

For example, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

argument

$$\log_2 8 = 3$$

logarithm, or exponent

base

Worked Example

You can write any exponential equation as a logarithmic equation and vice versa.

Example	Exponential Form	⇔	Logarithmic Form
A	$y = b^x$	⇔	$\log_b y = x$
B	$16 = 4^2$	⇔	$\log_4 16 = 2$
C	$1000 = 10^3$	⇔	$\log_{10} 1000 = 3$
D	$32 = 16^{1.25}$	⇔	$\log_{16} 32 = 1.25$
E	$a = b^c$	⇔	$\log_b a = c$

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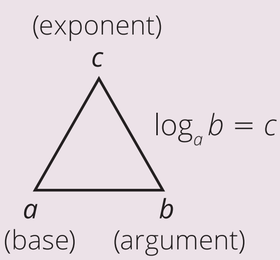
1. Analyze the exponential equation $y = b^x$ and its related logarithmic equation, $\log_b y = x$. State the restrictions, if any, on the variables. Explain your reasoning.

$y = b^x \Leftrightarrow \log_b y = x$		
Variable	**Restrictions**	**Explanation**
x		
b		
y		

Triangle of Power



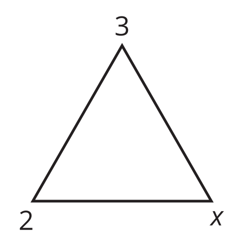
You can use a "Triangle of Power" to help you think about logarithms.

Worked Example

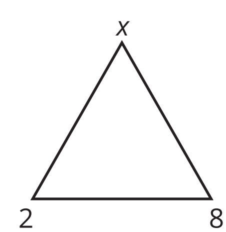
Consider the equation $2^3 = 8$.

Logarithmic Equation | **Triangle of Power**

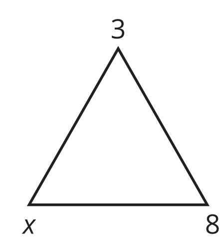
$$\log_2 x = 3$$
$$x = 8$$



$$\log_2 8 = x$$
$$x = 3$$



$$\log_x 8 = 3$$
$$x = 2$$



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2. **Draw Triangles of Power to illustrate each equation. Then rewrite in exponential form.**

   a. $\log_{10} 1000 = x$ b. $\log_5 x = 5$ c. $\log_x 9 = 1$

ACTIVITY
3.2

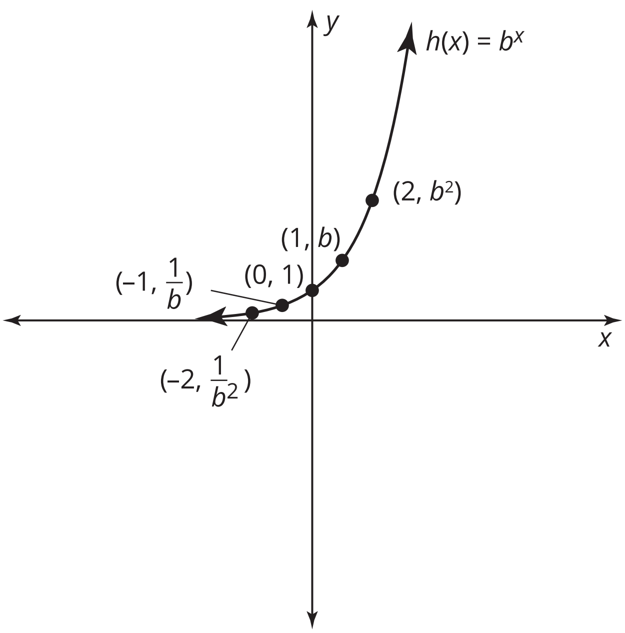
Graphing Logarithmic Functions

Recall that the logarithm of a number is the exponent to which the given base must be raised in order to produce the number. If $y = b^x$, then the logarithm is written as $\log_b y = x$, where $b > 0$, $b \neq 1$, and $y > 0$. A **logarithmic function** is a function involving a logarithm. What do the graphs of logarithmic functions look like?

Think about:

What are some methods you can use to sketch the graph of the inverse?

1. **The graph of $h(x) = b^x$ is shown. Sketch the graph of the inverse of $h(x)$ on the same coordinate plane. Label coordinates of points on the inverse of $h(x)$.**



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2. Write the equation for the inverse of $h(x) = b^x$. Label the graph.

3. Do you think all exponential functions are invertible? If so, explain your reasoning. If not, provide a counterexample.

Logarithms were first conceived by a Swiss clockmaker and amateur mathematician Joost Bürgi, but became more widely known and used after the publication of a book by Scottish mathematician John Napier in 1614. Tables of logarithms were originally used to make complex computations in astronomy, surveying, and other sciences easier and more accurate. With the invention of calculators and computers, the use of logarithm tables as a tool for calculation has decreased. However, many real-world situations can be modeled using logarithmic functions.

Two frequently used logarithms are logarithms with base 10 and base *e*. A **common logarithm** is a logarithm with base 10 and is usually written without a specified base.

$$c(x) = \log_{10} x \iff c(x) = \log x$$

A **natural logarithm** is a logarithm with base *e*, and is usually written as *ln*.

$$n(x) = \log_e x \iff n(x) = \ln x$$

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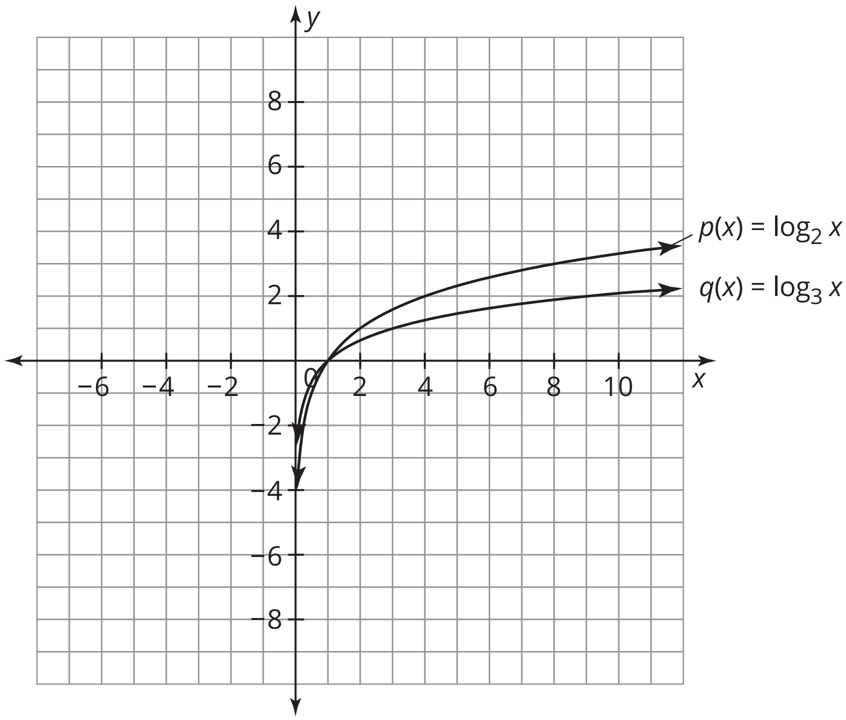
4. **The functions $p(x) = \log_2 x$ and $q(x) = \log_3 x$ have been graphed for you.**

   a. **Sketch and label the functions $c(x) = \log x$ and $n(x) = \ln x$.**



   b. **Explain how you determined the graphs of $c(x)$ and $n(x)$.**

   c. **Analyze the key characteristics of $p(x)$, $q(x)$, $c(x)$, and $n(x)$. Describe the similarities and differences.**

   d. **What is the inverse of the logarithmic function $c(x) = \log x$?**

   e. **What is the inverse of the logarithmic function $n(x) = \ln x$?**

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ACTIVITY
3.3

Applying Logarithms

The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M, is determined using the equation, $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake or "threshold quake" whose seismograph amplitude is 10^{-4} cm.

1. Write an equation to model the magnitude of a standard earthquake.

2. An earthquake in California measured 6.8 on the Richter scale, while an earthquake in Japan measured 7.2. Use technology to determine how many times more intense the Japanese earthquake was than the Californian earthquake.

3. Early in the century, an earthquake in Indonesia registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in Chile that was 4 times stronger. Use technology to determine the magnitude of the Chilean earthquake.

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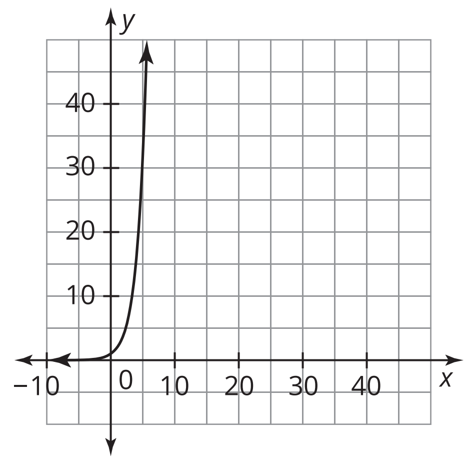
In the sciences, people often make use of logarithmic scales on graphs to make interpretations of exponential growth or decay simpler.

What happens to the graph of an exponential function when a logarithmic scale is used on the *y*-axis?

4. The graph shows $f(x) = 2^x$. Convert the *y*-axis to a logarithmic scale by interpreting each value on the *y*-axis as an exponent of a power with a base of 2.

Draw the new graph of $f(x) = 2^x$ with the re-interpreted *y*-values.



5. How does the logarithmic scale affect the graph? Will this happen with every exponential function? Explain your reasoning.

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TALK the TALK

Getting Into a Logarithm

1. Identify the key characteristics for exponential and logarithmic functions to complete the table.

	Basic Exponential Function	Basic Logarithmic Function
Function		
Graph		
Domain		
Range		
Intercepts		
Asymptotes		
End behavior		

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Assignment

Write

Write the term that best completes each sentence.

1. The ___________ of a number for a given base is the exponent to which the base must be raised in order to produce that number.
2. A ___________ is a logarithm with base e, and is usually written as ln.
3. A ___________ is a function involving a logarithm.
4. A ___________ is a logarithm with a base 10 and is usually written without a base specified.

Remember

All exponential functions are invertible.

A logarithm is an exponent. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b y$.

Key characteristics of the basic logarithmic function include a domain of positive numbers, a range of all real numbers, and a vertical asymptote at $x = 0$.

Practice

1. Given $f(x) = 3^x$.
   a. Write the function $f^{-1}(x)$, the inverse of $f(x) = 3^x$.
   b. Graph and label the functions $f(x)$ and $f^{-1}(x)$ on the same coordinate plane.
   c. Describe how to calculate $f^{-1}(3)$ without a calculator. Then, calculate $f^{-1}(3)$, $f^{-1}(9)$, and $f^{-1}(27)$.
   d. Determine the domain, range, asymptotes, intercepts, end behavior, and intervals of increase and decrease for $f^{-1}(x)$.
2. The loudness of sounds is measured in decibels (dB). The loudness, L, of a sound is a function of its intensity, I, and can be determined using the function $L(I) = 10 \log\left(\frac{I}{I_0}\right)$, where both I and I_0 are measured in watts per square meter (W/m^2). In the function, I_0 represents a barely audible sound or "threshold sound" and is equal to 10^{-12} W/m^2.
   a. The Guinness World Record for the Loudest Crowd Roar at an Outdoor Stadium was set during an NFL game in Seattle. The roar measured 136.6 dB. During an NFL game in Kansas City, the roar of the crowd was measured at 133.1 dB. How many times more intense was the roar at the Seattle game?
   b. The fans in Kansas City attempted to break the Guinness World Record for the Loudest Crowd Roar. Their goal was to create a roar that was 2 times as intense as the Seattle roar. In order for the fans in Kansas City to be successful, how many decibels did their roar need to be?
   c. In fact, the fans in Kansas City successfully recorded a new record of 140 dB. Determine the intensity of the roar.

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Stretch

Evaluate each logarithm.

1. $\log_{49} 7 - \log_{8} 64$
2. $\log_{100} 10 + \log_{81} 3$

Review

1. Drew wants to invest \$5000 in a savings account. Sun Bank is offering 4.5% interest compounded monthly. Brightside Bank is offering 4.75% interest compounded quarterly.
   a. For each bank, write an exponential function to represent the amount of money Drew would have in the account after t years.
   b. Determine which bank Drew should choose if she plans to invest her money for 5 years. If Drew decides to leave the money in the bank for a longer period of time, will the other bank be a better deal in the long run? Explain your reasoning.
2. In the year 2012, Graceville had a population of 1.4 million people and an annual growth rate of 1.35%.
   a. Write a function in the form $N(t) = N_0e^{rt}$ to model Graceville's population with respect to t, the number of years since 2012.
   b. Use your model to predict what Graceville's population will be in the year 2025.
   c. Use your model to estimate Graceville's population in the year 1989.
   d. Use technology to estimate when Graceville's population will reach 2 million people.
3. Multiply $8\sqrt[3]{a^2}\left(-2\sqrt{a} + 5\sqrt[3]{a^4}\right)$, given $a \geq 0$. Extract all roots and write your answer in radical form.
4. Determine the domain of $f(x)$. Explain your reasoning. $f(x) = \frac{x^2 + 2x + 1}{x^2 - 1}$

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4

I Like to Move It

Transformations of Exponential and Logarithmic Functions

Warm Up

Determine the inverse of each function.

1. $f(x) = 2^x$

2. $g(x) = \log_4 x$

3. $h(x) = 7^x$

4. $k(x) = \log x$

Learning Goals

- Dilate, reflect, and translate exponential and logarithmic functions using reference points and transformation function form.
- Investigate graphs of exponential and logarithmic functions through intercepts, asymptotes, intervals of increase and decrease, and end behavior.
- Describe how transformations of exponential and logarithmic functions affect their key characteristics.

You have explored transformations with functions in many function families. How are the transformations of exponential and logarithmic functions related?

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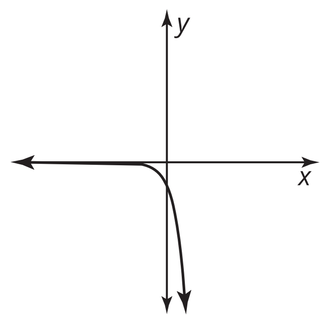
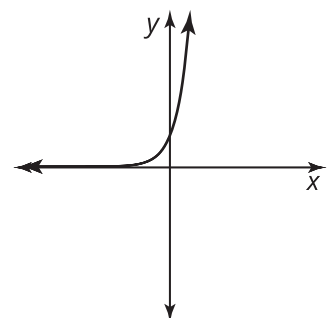
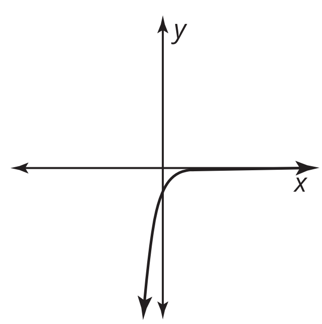
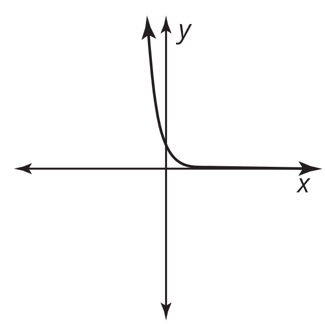
Don't Flip Out

Consider the two tables that show four exponential functions represented as equations and as graphs.

Exponential Equations	
$f(x) = 10^x$	$g(x) = 10^{-x}$
$h(x) = -10^x$	$j(x) = -10^{-x}$

Exponential Graphs	
A. 	**B.** 
C. 	**D.** 

Remember:

All transformations can be written in transformation function form in terms of the original function.

1. Analyze the graphs.

a. Match each exponential equation to its corresponding graph, and write the equation under the graph it represents.

b. Write an equation for each function $g(x)$, $h(x)$, and $j(x)$ in terms of $f(x)$. Describe each transformation on $f(x)$.

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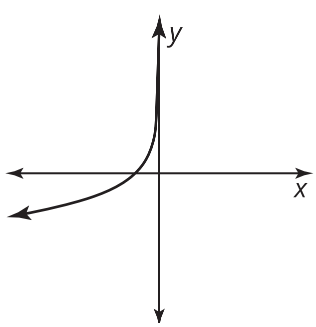
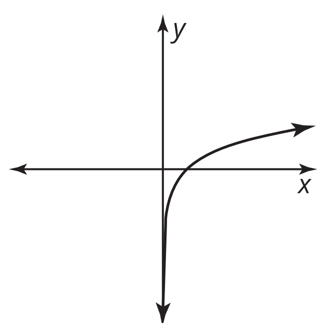
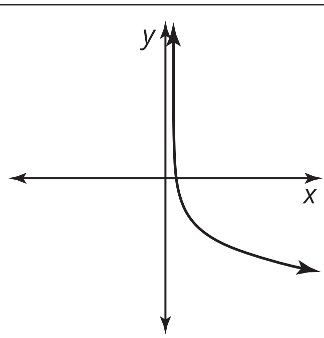
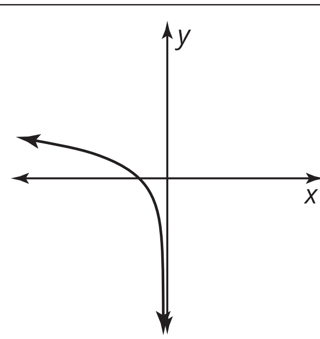
2. Analyze the key characteristics of each exponential function. What similarities exist among the four functions?

Key characteristics include asymptotes, intervals of increase or decrease, and end behavior.

3. What generalizations can you make about the effects of these transformations on the domain and range of an exponential function?

Consider the two tables that show four logarithmic functions represented as equations and as graphs.

Logarithmic Graph	
A. 	**B.** 
C. 	**D.** 

Logarithmic Equations	
$f(x) = \log_2 x$	$g(x) = -\log_2 x$
$h(x) = \log_2(-x)$	$j(x) = -\log_2(-x)$

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4. Analyze the graphs.

a. Match the logarithmic equation to its corresponding graph, and write the equation under the graph it represents.

b. Write an equation for each function $g(x)$, $h(x)$, and $j(x)$ in terms of $f(x)$. Describe each transformation on $f(x)$.

5. Analyze the key characteristics of each logarithmic function. What similarities exist among the four functions?

6. What generalizations can you make about the effects of these transformations on the domain and range of a logarithmic function?

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ACTIVITY **4.1**

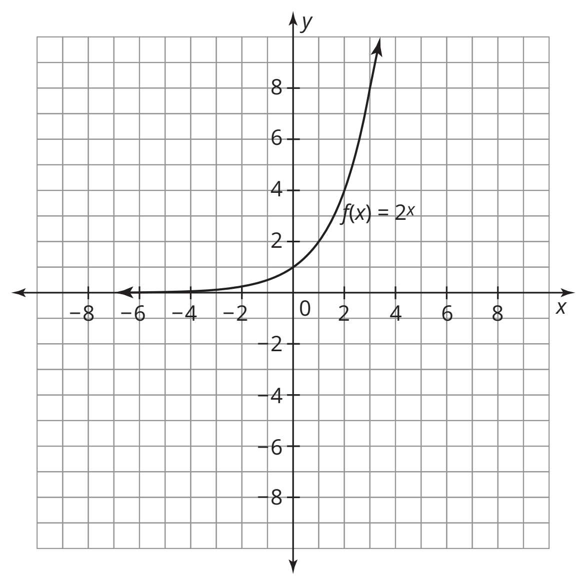
Graphing Transformations

In the Getting Started, you analyzed transformations of exponential and logarithmic functions that were a reflection across either axis, or reflections across both axes.

You know that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformation function form.

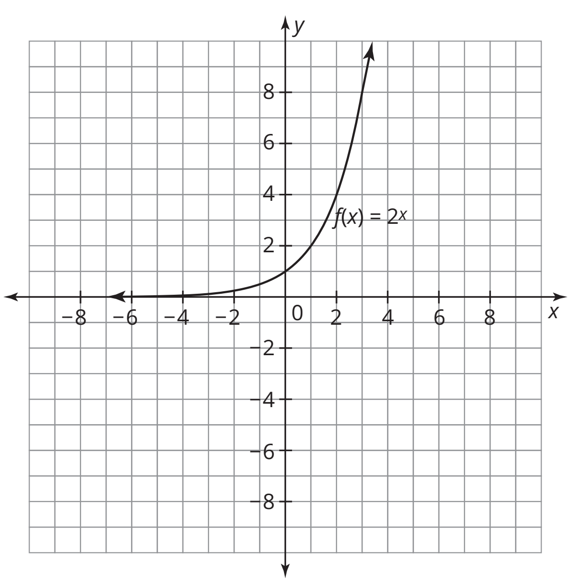
$$g(x) = Af(Bx - C) + D$$

1. Consider each transformation to $f(x) = 2^x$. Describe the effects of each transformation, including effects on the key attributes of the graph. Then, sketch the transformed graph.

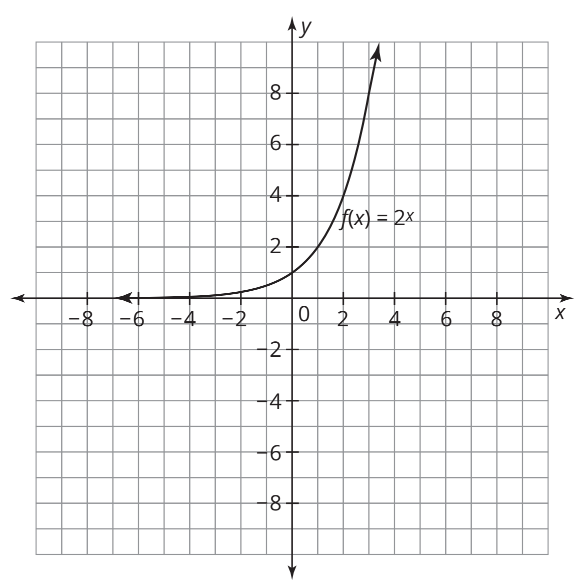


a. $g(x) = 5 \cdot 2^x$

b. $p(x) = 2^{x-4}$



c. $k(x) = 2^x - 3$



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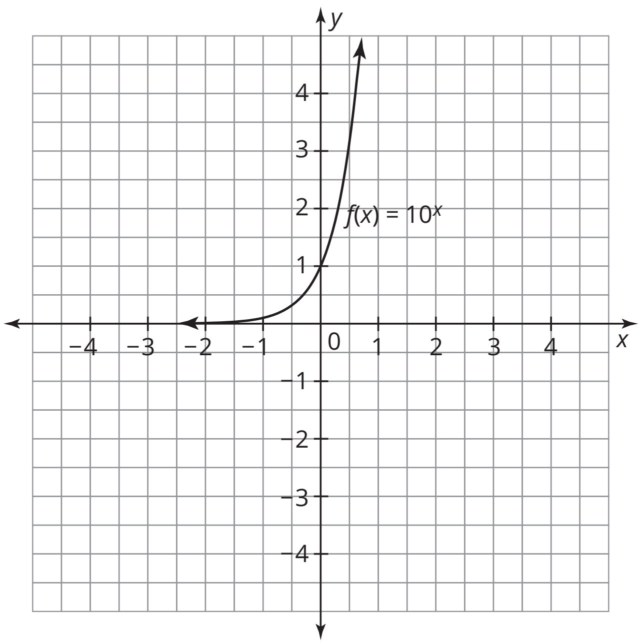
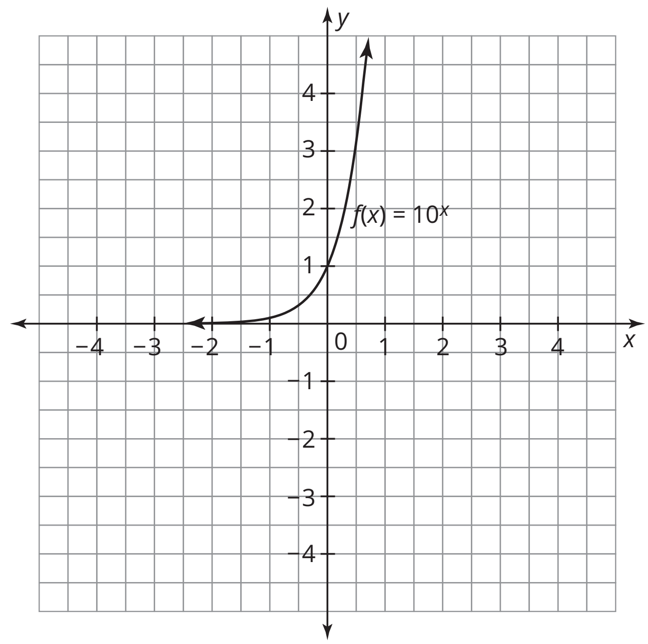
Now let's consider multiple transformations of an exponential function.

2. Consider the transformations to $f(x) = 10^x$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.

a. $s(x) = -3 \cdot 10^{x+2} - 1$



b. $t(x) = 2 \cdot 10^{x-1} + 3$



3. Analyze the transformations performed on $f(x)$ in Questions 4 and 5.

a. Which, if any, of these transformations affected the domain, range, and asymptotes?

b. What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of exponential functions?

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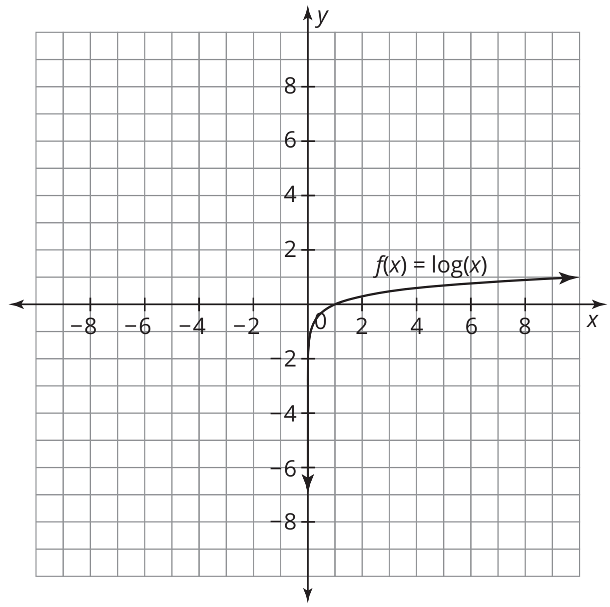
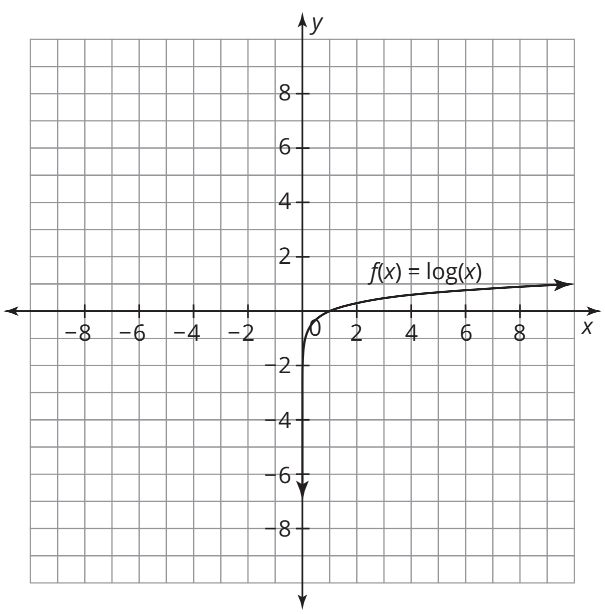
Let's consider single transformations of a logarithmic function.

4. Consider each transformation to $f(x) = \log(x)$. Describe the effects of each transformation, including effects on the key attributes of the graph. Then, sketch the transformed graph.

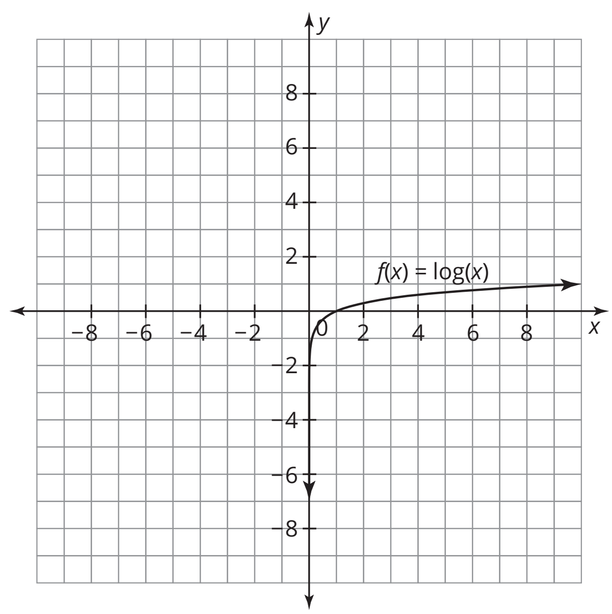
a. $g(x) = 3\log(x)$



b. $p(x) = \log(x - 3)$



c. $k(x) = \log(x) - 3$



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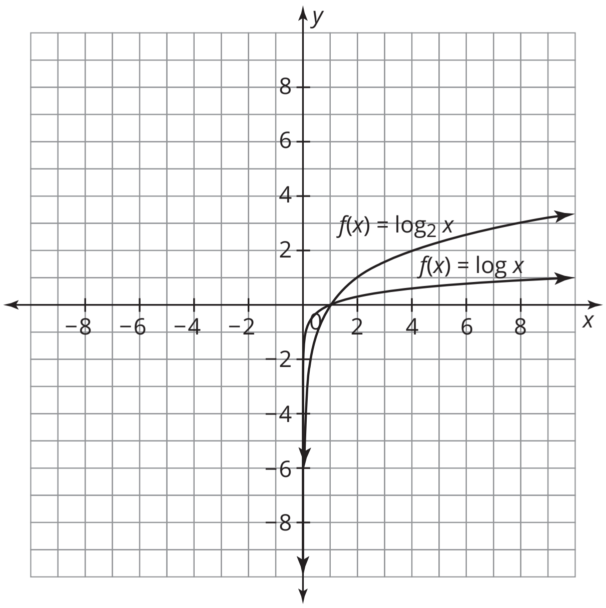
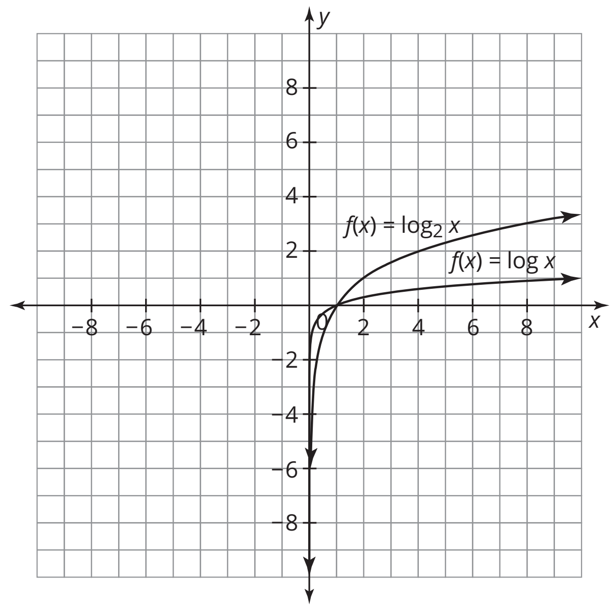
Now let's consider multiple transformations of a logarithmic function.

5. Consider the transformations to $f(x) = \log_2(x)$. Describe the effects of the multiple transformations, including effects on the key attributes of the graph. Then, sketch the transformed graph.

a. $s(x) = 4\log_2(x - 1) + 3$



b. $t(x) = -\log_2(x + 3) - 4$



6. Analyze the transformations performed on $f(x)$ in Questions 4 and 5.

a. Which, if any, of these transformations affected the domain, range, and asymptotes?

b. What generalizations can you make about the effects of transformations on the domain, range, and asymptotes of logarithmic functions?

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ACTIVITY

4.2 Describing Transformations

You can describe a transformation in more than one way—in terms of the original function, as an equation in transformation function form, using key characteristics, or words. Is it possible to describe the same transformation using different notation?

1. Inaz and Topher each described the effects of transforming the graph of $f(x) = 3^x$, such that $p(x) = 3f(x)$.

Inaz

$p(x) = 3f(x)$

The A-value is 3 so the graph is stretched vertically by a scale factor of 3.

Topher

$p(x) = 3f(x)$

$p(x) = 3 \cdot 3^x$

$p(x) = 3^{1+x}$

$p(x) = f(x + 1)$

The C-value is −1 so the graph is horizontally translated 1 unit to the left.

Who is correct? Explain your reasoning.

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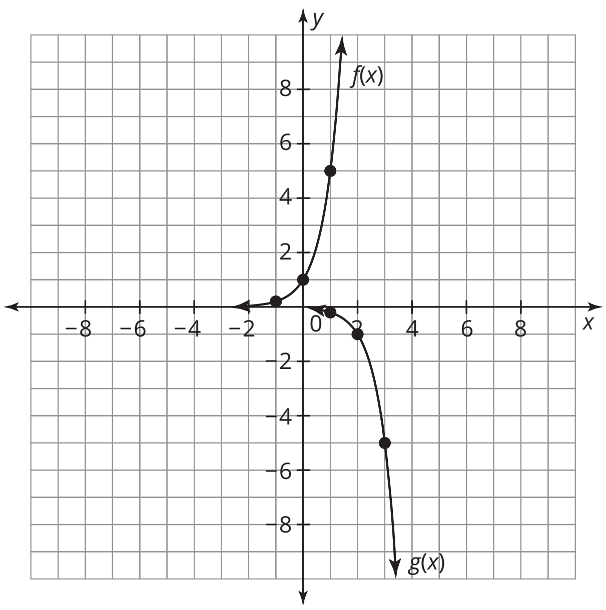
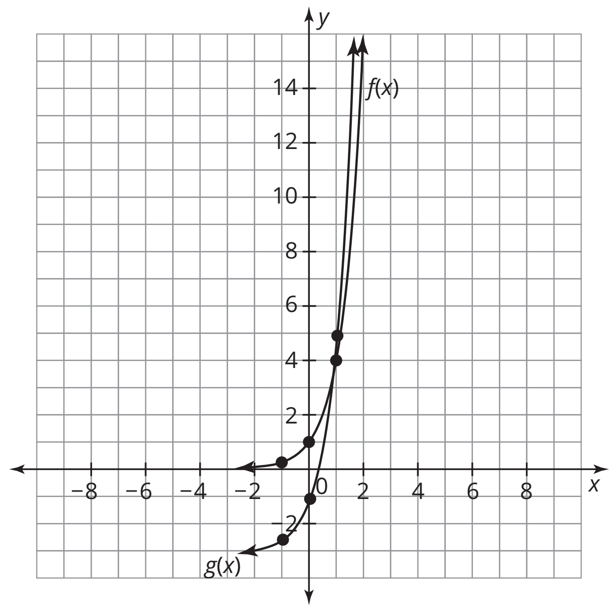
You can describe the transformation(s) performed on a function by comparing the graphs or equations of the original function and the transformed function.

2. Analyze the graphs of $f(x)$ and $g(x)$. Describe the transformations performed on $f(x)$ to create $g(x)$. Then, write an equation for $g(x)$ in terms of $f(x)$. For each set of points shown on $f(x)$, the corresponding points are shown on $g(x)$.

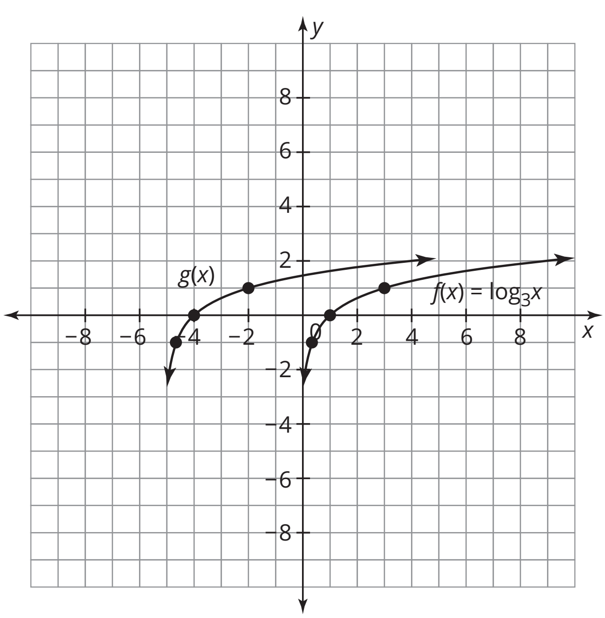
a. $g(x) =$ ____________



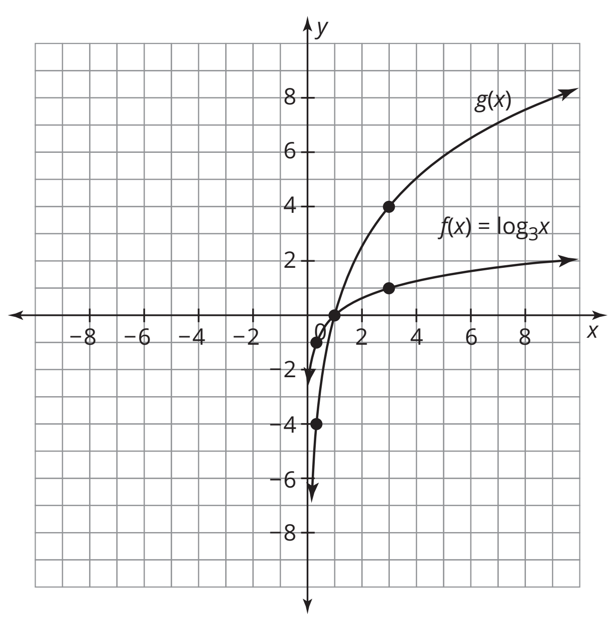
b. $g(x) =$ ____________



c. $g(x) =$ ____________



d. $g(x) =$ ____________



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3. **The equation for an exponential function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write an exponential equation for $t(x)$.**

   a. $m(x) = 2^x$
      $t(x) = 0.5m(x + 3)$

   b. $m(x) = e^x$
      $t(x) = -m(x) - 1$

   c. $m(x) = 6^x$
      $t(x) = 2m(-x)$

4. **The equation for a logarithmic function $m(x)$ is given. The equation for the transformed function $t(x)$ in terms of $m(x)$ is also given. Describe the graphical transformation(s) on $m(x)$ that produce(s) $t(x)$. Then, write a logarithmic equation for $t(x)$.**

   a. $m(x) = \log x$
      $t(x) = 3m(x) + 4$

   b. $m(x) = \ln x$
      $t(x) = m(-x) - 1$

   c. $m(x) = \log_2 x$
      $t(x) = 0.2m(x - 3)$

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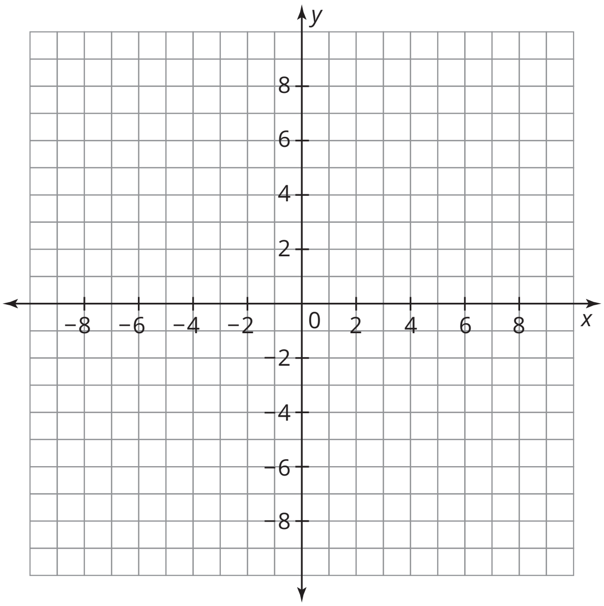
You can describe the transformation of a function given the original function and a characteristic, or characteristics, of the transformed function.

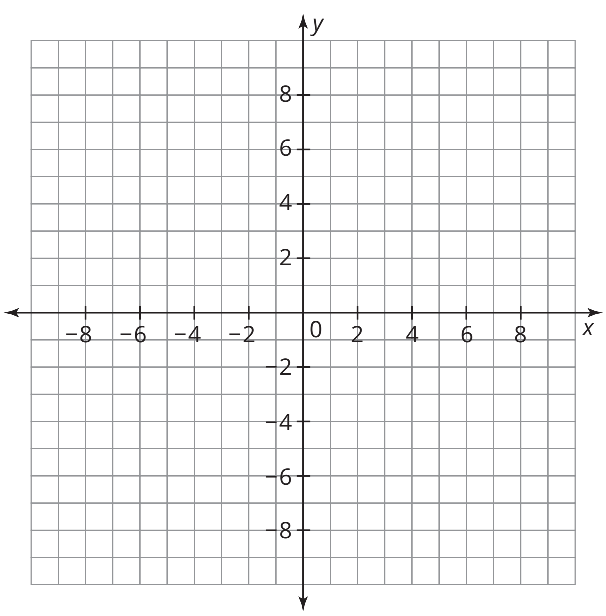
5. Write a transformed logarithmic function in terms of $f(x)$ with the characteristic(s) given. Then, graph the transformed function.



a. $f(x) = \log_2 x$

vertical asymptote at $x = -3$

$m(x) =$ ____________



b. $f(x) = \log_3 x$

Reference Points on $f(x)$	▯	Corresponding Points on $h(x)$
$\left(\frac{1}{3}, -1\right)$	▯	$\left(\frac{1}{3}, -\frac{1}{2}\right)$
(1, 0)	▯	(1, 0)
(3, 1)	▯	$\left(3, \frac{1}{2}\right)$

$h(x) =$ ____________

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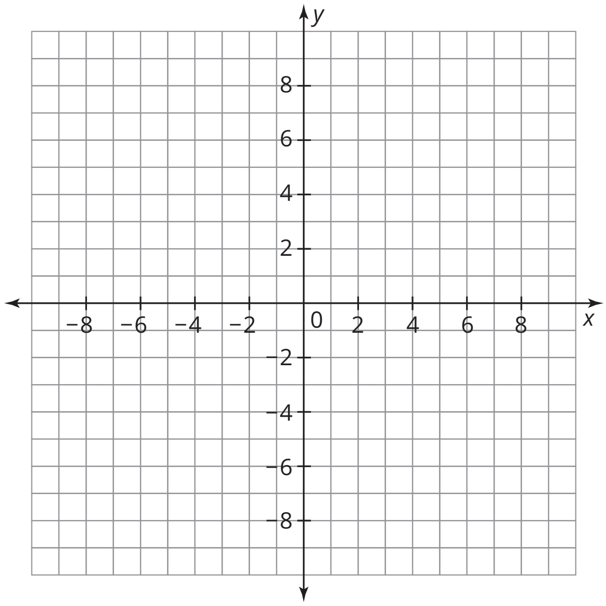
c. $f(x) = \log_5 x$
Domain: $(-\infty, 2)$



$k(x) =$ ____________

ACTIVITY
4.3

Relating Inverses of Transformations

Recall that the inverse of an exponential function is a logarithmic function. Let's explore how the transformation of an exponential function and its inverse are related.

Remember:

You can fold the cutouts to determine the inverse graph.

1. Consider the functions $f(x) = 2^x$ and $g(x) = f^{-1}(x)$, or $\log_2 x$. The graphs of $f(x)$ and $g(x)$ are provided.

- Graph the transformations on $f(x)$.
- Describe the transformation(s) performed on each.
- Then graph the inverse of the transformed function.
- Finally, describe and label the graph of the inverse of the transformed function as a transformation on $g(x)$.

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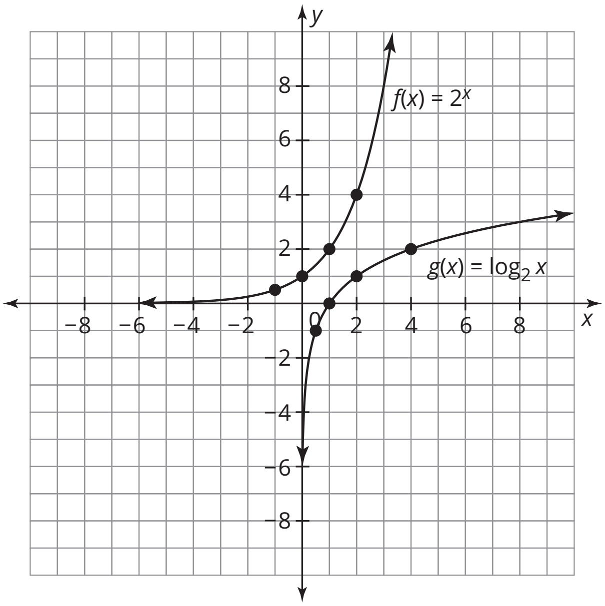
a. $f(x + 4)$

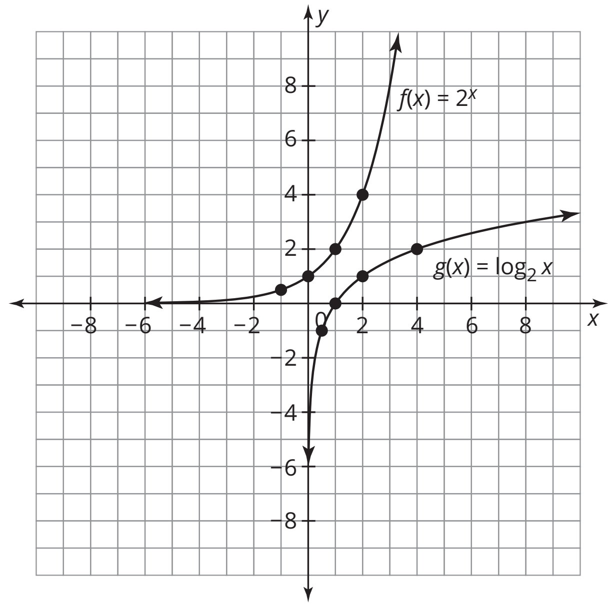
Transformation(s) on $f(x)$**:**



Transformation(s) on $g(x)$**:**

b. $f(x) + 5$

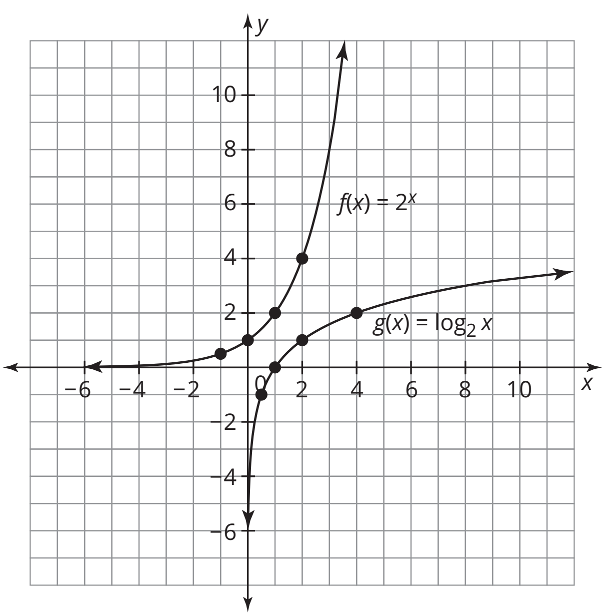
Transformation(s) on $f(x)$**:**



Transformation(s) on $g(x)$**:**

c. $f(x - 3) + 6$

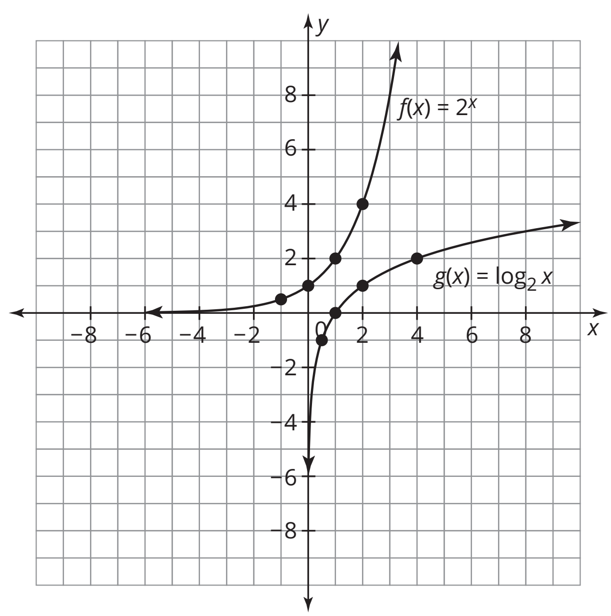
Transformation(s) on $f(x)$**:**



Transformation(s) on $g(x)$**:**

d. $f(x + 1) + 1$

Transformation(s) on $f(x)$**:**



Transformation(s) on $g(x)$**:**

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2. **Generalize the effects of transformations on $f(x)$ and its inverse function, $f^{-1}(x)$. Complete the table to organize your results.**

	Transformation on $f(x)$	Effect of Transformation on $f^{-1}(x)$
$f(x + C)$		
$f(x - C)$		
$f(x) + D$		
$f(x) - D$		

3. **Consider the function $y = f(x)$ and the transformed function $g(x)$. Write an equation for $g^{-1}(x)$ in terms of $f^{-1}(x)$.**

   a. $g(x) = f(x - 1)$ $\quad$ $g^{-1}(x) =$ ____________

   b. $g(x) = f(x) - 2$ $\quad$ $g^{-1}(x) =$ ____________

   c. $g(x) = f(x + 5)$ $\quad$ $g^{-1}(x) =$ ____________

   d. $g(x) = f(x - 4) + 3$ $\quad$ $g^{-1}(x) =$ ____________

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The function $f(x) = 2^x$ and its inverse function $f^{-1}(x) = \log_2 x$ are shown in the table.

$f(x)$	$f^{-1}(x)$
$\left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -1\right)$
(0, 1)	(1, 0)
(1, 2)	(2, 1)

4. Complete each table. Write the inverse of the transformed function in terms of $f^{-1}(x)$ and identify the effect of the transformation on the inverse.

a.

$3f(x)$	____________

Transformation on $f(x)$:

Effect on the inverse:

b.

$\frac{1}{2}f(x)$	____________

Transformation on $f(x)$:

Effect on the inverse:

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c.

$f\left(\frac{x}{4}\right)$	______

Transformation on $f(x)$:

Effect on the inverse:

d.

$f(2x)$	______

Transformation on $f(x)$:

Effect on the inverse:

5. How does a vertical dilation on a function affect its inverse?

6. How does a horizontal dilation on a function affect its inverse?

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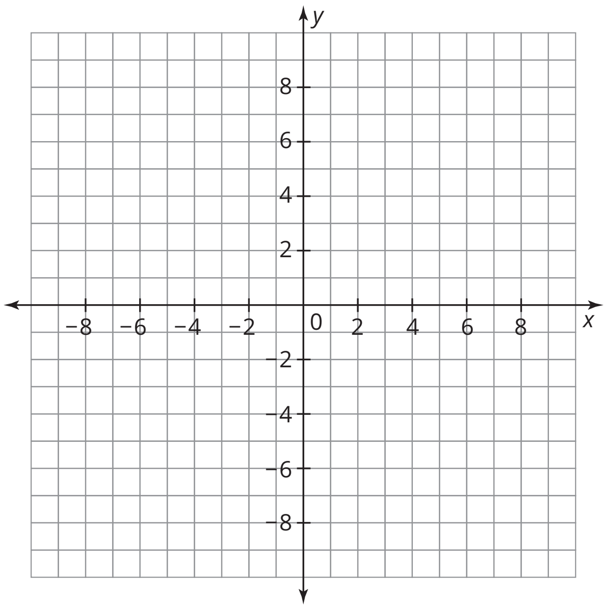
7. **Given the function $f(x)$, write the equation for the inverse function, $f^{-1}(x)$. Then graph the inverse function.**

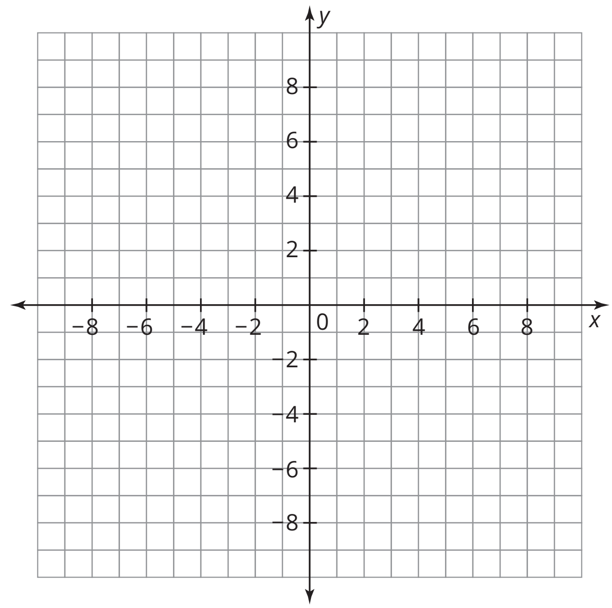
a. $f(x) = 2 \cdot 3^x$

$f^{-1}(x) =$ ____________



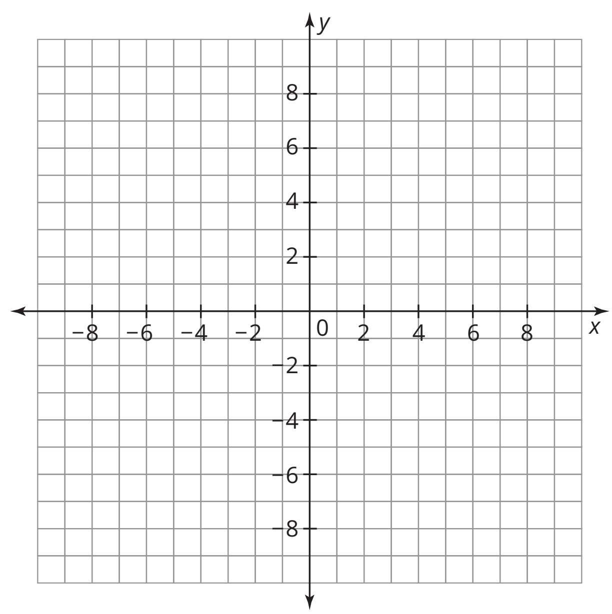
b. $f(x) = \log_4\left(\frac{x}{2}\right)$

$f^{-1}(x) =$ ____________



c. $f(x) = -5^x$

$f^{-1}(x) =$ ____________



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NOTES

TALK the TALK

Moving Right Along

In this lesson, you graphed and analyzed transformations of exponential and logarithmic functions. You also explored how the transformation of an exponential function is related to the transformation of its inverse function, a logarithmic function.

1. **Determine whether each statement is true or false. If the statement is false, rewrite the statement so that it is true.**

   a. **The range of logarithmic functions is not affected by translations or dilations.**

   b. **Vertical translations do not affect the range and the horizontal asymptote of exponential functions.**

   c. **Horizontal translations do not affect the domain and vertical asymptote of logarithmic functions.**

   d. **The domain of exponential functions is not affected by translations or dilations.**

   e. **Vertical dilations affect the range and the vertical asymptote of logarithmic functions.**

   f. **Horizontal dilations do not affect the domain of exponential functions.**

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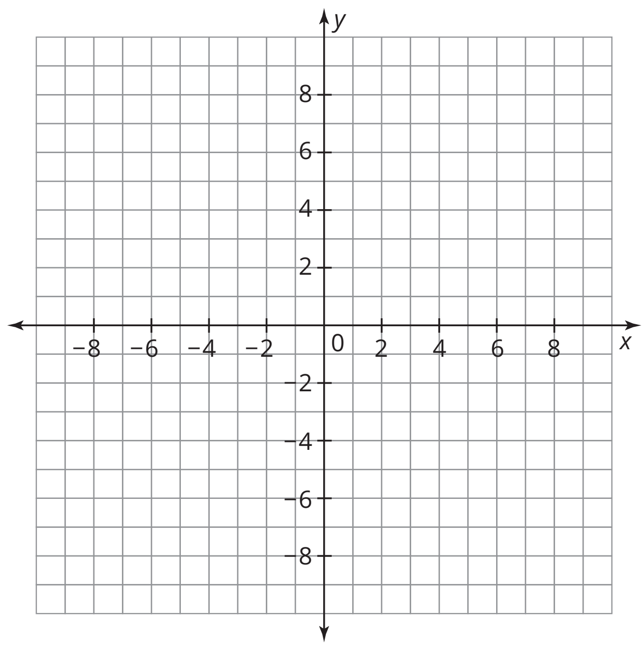
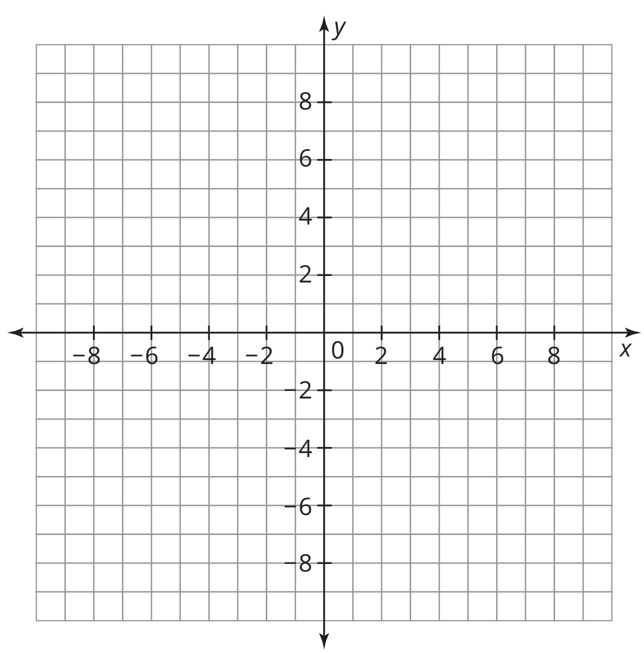
NOTES

2. **Given the function $f(x)$:**
   - **Graph the function.**
   - **Write the equation for the inverse function, $f^{-1}(x)$.**
   - **Graph the inverse function.**

a. $f(x) = -6^x$



b. $f(x) = \log_5\left(\frac{x}{3}\right)$



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Assignment

Write

Match the transformation of $f(x)$ with its effect on $f^{-1}(x)$.

1. $f(x)$ translates up D units	a. $f^{-1}(x)$ translates up C units
2. $f(x)$ translates right C units	b. $f^{-1}(x)$ translates left D units
3. $f(x)$ translates down D units	c. $f^{-1}(x)$ translates right D units
4. $f(x)$ translates left C units	d. $f^{-1}(x)$ translates down C units

Remember

The transformation function form $g(x) = Af(B(x - C)) + D$ can be applied to exponential and logarithmic functions.

A horizontal translation on a function produces a vertical translation on its inverse, while a vertical translation on a function produces a horizontal translation on its inverse. A vertical dilation on a function produces a horizontal dilation by the same factor on its inverse, while a horizontal dilation on a function produces a vertical dilation by the same factor on its inverse.

Practice

1. Given $p(x) = 5^x$ and $t(x) = p(2x) - 4$.
   a. Describe the transformation of $p(x)$ that produces $t(x)$.
   b. Write $t(x)$ as an exponential function.
2. Given $m(x) = 1.5^x$ and $k(x) = \frac{1}{3}m(-x)$.
   a. Describe the transformation of $m(x)$ that produces $k(x)$.
   b. Write $k(x)$ as an exponential function.
3. Consider the function $g(x)$, which is formed by translating the function $f(x) = \log_2 x$ left 3 units and up 4 units.
   a. Write $g(x)$ in terms of $f(x)$.
   b. Complete the table by determining the corresponding point on $g(x)$ for each reference point on $f(x)$.
   c. Graph and label $f(x)$ and $g(x)$ on the same coordinate plane.
   d. Write $g(x)$ as a logarithmic function.
   e. List the domain, range, and any asymptotes of the logarithmic function $g(x)$.

Reference Point on $f(x)$	Corresponding Point on $g(x)$
(0.5, −1)	
(1, 0)	
(2, 1)	
(4, 2)	

4. Consider $p(x) = 2^{\frac{x}{3}}$, which is a transformation of the function $f(x) = 2^x$.
   a. Describe the transformation(s) on the graph of $f(x)$ to produce $p(x)$.
   b. Write the equations of the inverse functions $f^{-1}(x)$ and $p^{-1}(x)$.
   c. Describe the transformation(s) on the graph of $f^{-1}(x)$ to produce $p^{-1}(x)$.
   d. Graph and label the inverse function $p^{-1}(x)$.

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Stretch

1. Consider the function $f(x) = 2^x$. The table shows the corresponding point on $g(x)$ for each reference point on $f(x)$ from the transformation of $f(x)$. Determine the function $g(x)$.

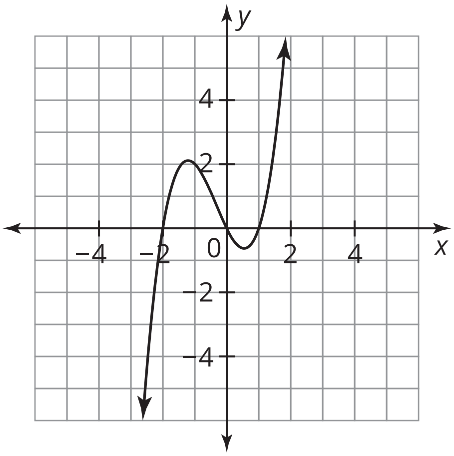
Reference Point on $f(x)$	Corresponding Point on $g(x)$
(−2, 0.25)	(−2, −2)
(−1, 0.5)	(−1, 0)
(0, 1)	(0, 1)
(1, 2)	(1, 1.5)
(2, 4)	(2, 1.75)

2. Solve for x in each equation.
   a. $25 = 5^{x+1}$
   b. $64 = 2^{3x+10}$
   c. $\frac{8}{125} = \left(\frac{2}{5}\right)^{\frac{x}{3}}$

Review

1. Consider the power function, $g(x) = x^6$.
   a. Sketch the graph of $g(x)$.
   b. Determine whether $g(x)$ invertible? Explain your reasoning.

2. Determine whether the inverse of the graphed function is a function. Explain your reasoning.



3. Consider the function $f(x) = 16x^3$.
   a. Determine the domain and range of $f(x)$.
   b. Write the inverse function $f^{-1}(x)$.
   c. Determine the domain and range of $f^{-1}(x)$.

4. Algebraically determine whether $f(x) = \frac{3}{4}x + 2$ and $g(x) = \frac{4}{3}x - \frac{1}{2}$ are inverses. Show your work.

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Exponential and Logarithmic Functions Summary

KEY TERMS

- half-life
- natural base e
- logarithm
- logarithmic function
- common logarithm
- natural logarithm

LESSON 1

Half-Life

The explicit formula for a geometric sequence is given by $a_n = a_1 \cdot r^{(n-1)}$, where a_n represents the nth term of the sequence, a_1 represents the first term of the sequence, and r represents the common ratio.

Allison's income is shown. Her income represents a geometric sequence.

Week	Allison's Income ($)
1	0.01
2	0.02
3	0.04
4	0.08
5	0.16
6	0.32
7	0.64
8	1.28
9	2.56
10	5.12
11	10.24

The explicit formula that represents Allison's income, $g_n = 0.01 \cdot 2^{(n-1)}$, can be written in function notation using the properties of powers.

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Statement	Reason
$g_n = 0.01 \cdot 2^{(n-1)}$	Explicit formula for a geometric sequence
$g(n) = 0.01 \cdot 2^{(n-1)}$	Rewrite in function notation.
$g(n) = 0.01 \cdot 2^n \cdot 2^{-1}$	Product Rule
$g(n) = 0.01 \cdot 2^n \cdot \frac{1}{2}$	Definition of negative exponents
$g(n) = 0.01 \cdot \frac{1}{2} \cdot 2^n$	Commutative Property of Multiplication
$g(n) = 0.005 \cdot 2^n$	Associative Property of Multiplication

So $g_n = 0.01 \cdot 2^{(n-1)}$ written in function notation is $f(n) = 0.005 \cdot 2^n$. A geometric sequence, when written in function notation, is called an *exponential function*. The function gets its name from the variable in the exponent.

A **half-life** is the amount of time it takes a substance to decay to half of its original amount.

The table shows the amount of caffeine in a person's system over time, starting with 80 milligrams of caffeine. The caffeine has a half-life of 5 hours.

Time Elapsed (hours)	0	5	10	15	20	25
Caffeine in System (mg)	80	40	20	10	5	2.5
Number of Half-Life Cycles	0	1	2	3	4	5

An equation to represent the half-life of the caffeine would be, $g(n) = 80\left(\frac{1}{2}\right)^{\left(\frac{n}{5}\right)}$, where n is the number of hours.

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LESSON 2 Pert and Nert

Basic exponential functions, whether increasing or decreasing, have the same domain, range, asymptote, and intercepts.

	Basic Exponential Growth $f(x) = b^x$, $b > 1$	Basic Exponential Decay $f(x) = b^x$, $0 < b < 1$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$(0, \infty)$	$(0, \infty)$
Asymptote	$y = 0$	$y = 0$
Intercept	$(0, 1)$	$(0, 1)$
End Behavior	As $x \to -\infty$, $f(x) \to 0$ As $x \to \infty$, $f(x) \to \infty$	As $x \to -\infty$, $f(x) \to \infty$ As $x \to \infty$, $f(x) \to 0$
Intervals of Increase or Decrease	Increasing over $(-\infty, \infty)$	Decreasing over $(-\infty, \infty)$

For compounded interest, the amount in an account, A, can be represented with the equation $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$, where P represents the principal, or starting amount, r represents the interest rate, and n represents the number of times the interest is compounded in a given amount of time.

The table shows the amount of compound interest earned with an initial investment of \$1 over year at various compounding intervals.

Period of Compounding	$n =$	Formula	Amount
Yearly	1	$1\left(1 + \frac{1}{1}\right)^{1 \cdot 1}$	2.00
Semi-Annually	2	$1\left(1 + \frac{1}{2}\right)^{2 \cdot 1}$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^{4 \cdot 1}$	2.44141
Monthly	12	$1\left(1 + \frac{1}{12}\right)^{12 \cdot 1}$	2.61303
Weekly	52	$1\left(1 + \frac{1}{52}\right)^{52 \cdot 1}$	2.692597
Daily	365	$1\left(1 + \frac{1}{365}\right)^{365 \cdot 1}$	2.714567
Hourly	8760	$1\left(1 + \frac{1}{8760}\right)^{8760 \cdot 1}$	2.718127
Every Minute	525600	$1\left(1 + \frac{1}{525600}\right)^{525600 \cdot 1}$	2.718279
Every Second	31536000	$1\left(1 + \frac{1}{31536000}\right)^{31536000 \cdot 1}$	2.718282

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The amount that the interest earnings approaches as the compounding frequency increases is an irrational number called *e*.

$$e \approx 2.718281828459045\ldots$$

It is often referred to as the **natural base *e***.

As the frequency of compounding, *n*, increases, the value $\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}$ approaches *e*. Thus, the formula for compound interest, with continuous compounding, can be written as:

$$A = Pe^{rt}$$

LESSON 3

Return of the Inverse

It is necessary to define a new function in order to write the equation for the inverse of an exponential function. The **logarithm** of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$, then x is the logarithm and can be written as $x = \log_b y$. The value of the base of a logarithm is the same as the base in the exponential expression b^x.

For example, the number 3 is the logarithm to which base 2 must be raised to produce the argument 8. The base is written as the subscript 2. The logarithm, or exponent, is the output 3. The argument of the logarithm is 8.

exponent argument

$$3 = \log_2 8$$

base

You can write any exponential equation as a logarithmic equation and vice versa.

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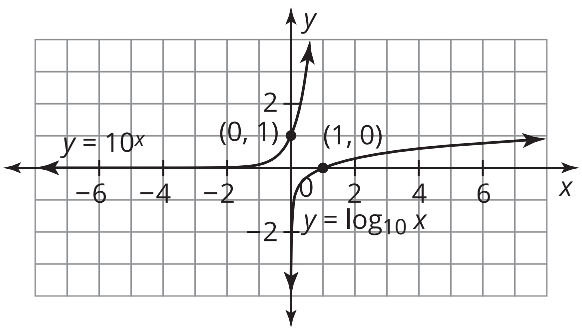
Exponential Form	Logarithmic Form
$y = b^x$	$x = \log_b y$
$16 = 4^2$	$2 = \log_4 16$
$1000 = 10^3$	$3 = \log_{10} 1000$
$a = b^c$	$c = \log_b a$

Recall that the logarithm of a number for a given base is the exponent to which the base must be raised in order to produce the number. If $y = b^x$ then the logarithm is written as $x = \log_b y$, where $b > 0$, $b \neq 1$, and $y > 0$. A **logarithmic function** is a function involving a logarithm.

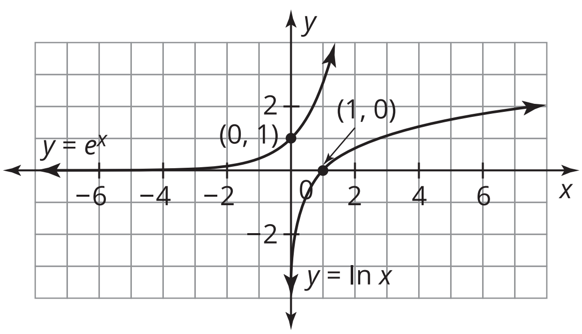
Two frequently used logarithms are logarithms with base 10 and base e. A **common logarithm** is a logarithm with base 10 and is usually written as *log* without a base specified.

$$c(x) = \log_{10} x \Leftrightarrow c(x) = \log x$$



A **natural logarithm** is a logarithm with base e, and is usually written as *ln*.

$$n(x) = \log_e x \Leftrightarrow n(x) = \ln x$$



One example of the use of logarithms in real-world applications is the Richter scale. The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M, is determined using the equation, $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake or "threshold quake" whose seismograph amplitude is 10^{-4} cm.

Since the Richter Scale is a logarithmic scale it is based on powers of ten. This can be used to determine the difference in intensity of earthquakes. For example, consider an earthquake in California that measured 6.8 on the Richter scale, and an earthquake in Japan that measured 7.2. The difference between them is $7.2 - 6.8 = 0.4$, thus the earthquake in Japan is $10^{0.4} \approx 2.511$ times more intense than the one in California.

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LESSON 4

I Like to Move It

The transformation function form of a logarithmic function is $g(x) = A \cdot \log(B(x - C)) + D$.

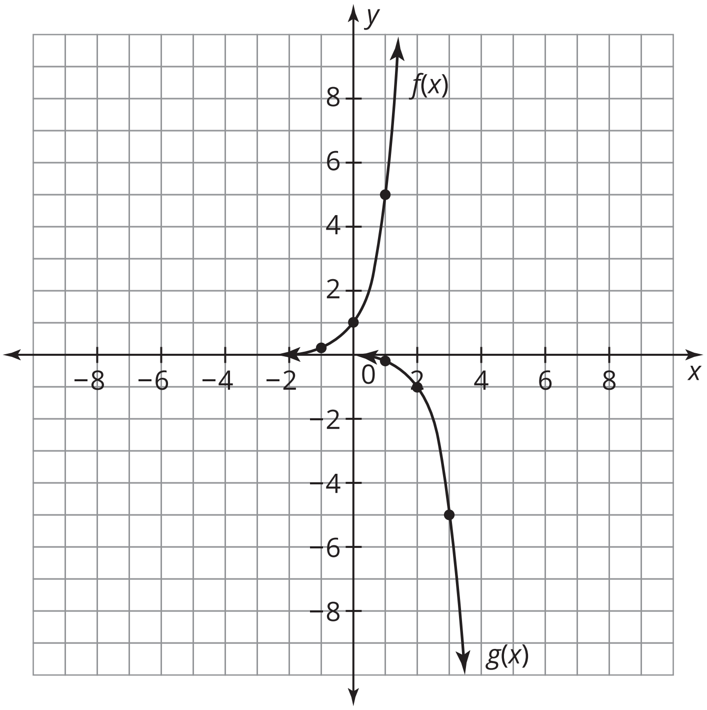
The domain of an exponential function and the range of a logarithmic function are not affected by translations or dilations, either vertical or horizontal. Vertical translations affect the range and the horizontal asymptote of exponential functions, while horizontal translations affect the domain and the vertical asymptote of logarithmic functions. Horizontal translations and dilations—either vertical or horizontal—do not affect the range and the horizontal asymptote of exponential functions. A vertical dilation on a function will produce a horizontal dilation the same number of units on its inverse, and a horizontal dilation on a function will produce a vertical dilation the same number of units on its inverse

You can describe a transformation in more than one way—in terms of the original function, as an equation in transformation function form, using key characteristics, or words.

You can describe the transformation(s) performed on a function by comparing the graphs or equations of the original function and the transformed function.

To create $g(x)$, the graph of $f(x)$ is horizontally translated right 2 units and reflected across the x-axis.

$$g(x) = -f(x - 2)$$



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You can describe the transformation of a function given the original function and a characteristic, or characteristics, of the transformed function. Consider the following function, $f(x) = \log_2(x)$. Create a transformed function that has a vertical asymptote at $x = -3$.

$$m(x) = f(x + 3)$$

Recall that the inverse of an exponential function is a logarithmic function.

	Transformation on $f(x)$	**Effect of Transformation on $f^{-1}(x)$**
$f(x + C)$	Translate horizontally left C units	Translate vertically down C units
$f(x - C)$	Translate horizontally right C units	Translate vertically up C units
$f(x) + D$	Translate vertically up D units	Translate horizontally right D units
$f(x) - D$	Translate vertically down D units	Translate horizontally left D units

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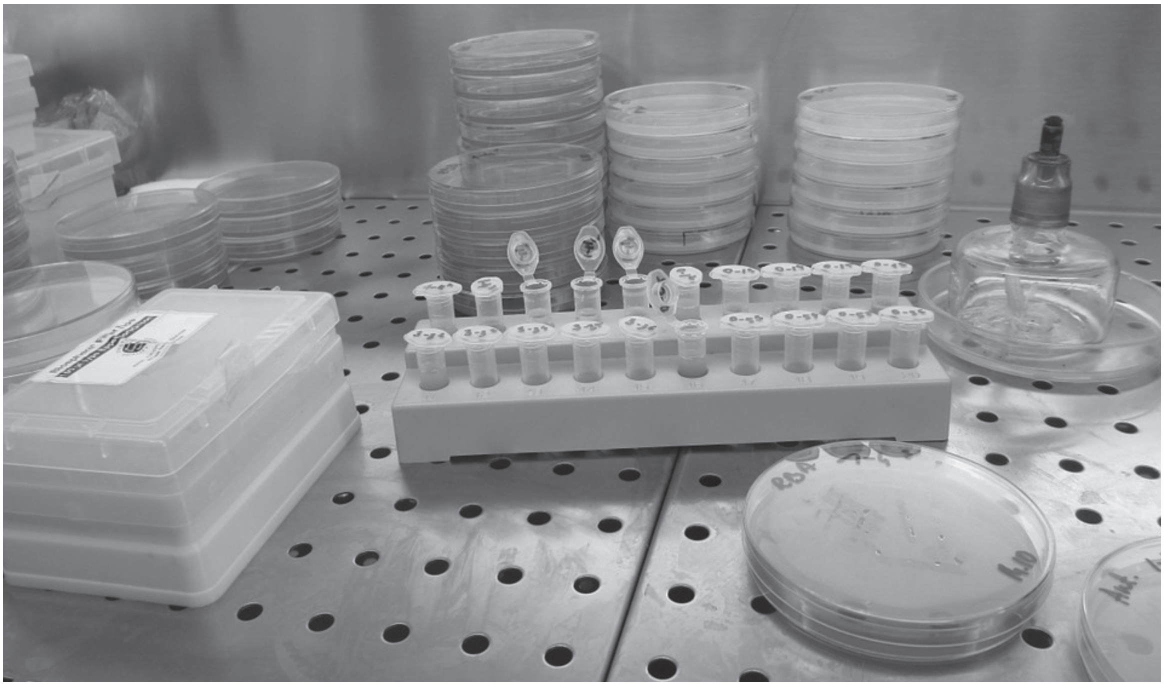
TOPIC 3

Exponential and Logarithmic Equations



Since bacteria typically grow by splitting, a bacterial population in a Petri dish is an excellent example of exponential growth.

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Module 3: Inverting Functions

TOPIC 3: EXPONENTIAL AND LOGARITHMIC EQUATIONS

In this topic, students first convert between exponential and logarithmic forms of an equation, and then use this relationship to solve for an unknown base, exponent, or argument in a logarithmic equation. They then develop rules and properties of logarithms based on their prior knowledge of various exponent rules and properties. Students derive the Change of Base Formula and solve logarithmic equations for the base, argument, or exponent using the formula or by rewriting them as exponential equations.

Where have we been?

Students oriented their thinking around exponential and logarithmic functions in the previous topic. From their prior work with quadratic functions, students have experience with manipulating equations of function inverses.

Where are we going?

In this topic, students use exponential and logarithmic equations that model real-world situations to solve problems. They use technology to determine logarithmic regressions. As evidenced by the scenarios in this topic, logarithmic equations are used to represent a large variety of complex real-world situations.

Properties of Logarithms

The properties of powers that you know can be applied to derive properties of logarithm operations. For example, you can use the Product Rule of Powers to derive a similar property for logarithms called the Product Rule of Logarithms.

Given: $b^m \cdot b^n = b^{m+n}$

Let $x = b^m$. Let $y = b^n$. → Use substitution to rewrite the given: $xy = b^{m+n}$

Rewrite each exponential equation as a logarithmic equation.

$\log_b x = m$ $\log_b y = n$ $\log_b (xy) = m + n$

Finally, use substitution to write the property in terms of x and y: $\log_b (xy) = \log_b x + \log_b y$

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Newton's Law of Cooling

Crime investigators use logarithmic equations to estimate a body's time of death based on two temperature readings of the body. Specifically, investigators use what's known as Newton's Law of Cooling, which states that an object cools down at a rate that is proportional to the temperature difference between the object and the environment.

Coroners—government officials who are responsible for verifying deaths—often use a rule of thumb to estimate the time of death: subtract 2 degrees from normal body temperature for the first hour after death and then 1 degree for each hour after that.

Talking Points

Logarithmic equations can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Solve $5^x = 432$ for x. Write your answer to the nearest hundredth.

If you take the log of both sides, you get $\log 5^x = \log 432$. Using the Power Rule of Logarithms, you can rewrite $\log 5^x$ as $x \cdot \log 5$.

So, $x = \frac{\log 432}{\log 5} \approx 3.77$.

Key Terms

Zero Property of Logarithms

The Zero Property of Logarithms states that $\log_b 0 = 1$.

Power Rule of Logarithms

The Power Rule of Logarithms states that $\log_b a^m = m \cdot \log_b a$.

Change of Base Formula

The Change of Base Formula allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base: $\log_b c = \frac{\log_a c}{\log_a b}$, where $a, b, c > 0$ and $a, b \neq 1$.

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1

All the Pieces of the Puzzle

Logarithmic Expressions

Warm Up

Convert each logarithmic equation to an exponential equation. Then solve for the unknown.

1. $\log_{10} 0.1 = x$

2. $\log_2 16 = x$

3. $\log_{10} 10^9 = x$

4. $\log_2 \frac{1}{4} = x$

Learning Goals

- Convert exponential equations into logarithmic equations.
- Convert logarithmic equations into exponential equations.
- Solve exponential and simple logarithmic equations.
- Estimate the values of logarithms on a number line.
- Evaluate logarithmic expressions.

Key Term

- logarithmic expression

You know that logarithmic functions are inverses of exponential functions. How can you use inverses to solve exponential and logarithmic equations?

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GETTING STARTED

Two-Way Street

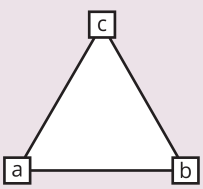
Recall that a logarithmic function is the inverse of an exponential function.

1. Write the equivalent form of the given exponential or logarithmic equation.

Exponential Form $y = b^x$	⇔	Logarithmic Form $x = \log_b y$
$12^2 = 144$	⇔	
	⇔	$\log_{16} 4 = \frac{1}{2}$
$10^5 = 100{,}000$	⇔	
	⇔	$\ln 20.086 \approx 3$
$\left(\frac{2}{3}\right)^3 = \frac{8}{27}$	⇔	
	⇔	$\log_9 27 = \frac{3}{2}$
	⇔	$\log_2 x = 8$
$6^x = 36$	⇔	
$n^5 = 243$	⇔	$\log_n 243 = 5$

Remember:

Use the Triangle of Power.



$a^c = b \Leftrightarrow \log_a b = c$

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ACTIVITY 1.1 Solving Logarithmic Equations

When you evaluate a logarithmic expression (logarithm), you are determining the value of the exponent in the corresponding exponential expression.

$$\text{base}^{\text{exponent}} = \text{argument} \Leftrightarrow \log_{\text{base}}(\text{argument}) = \text{exponent}$$

The variables of the logarithmic equation have the same restrictions as the corresponding variables of the exponential equation. The base, *b*, must be greater than 0 but not equal to 1; the argument must be greater than 0; and the value of the exponent has no restrictions.

It is important to become familiar with how the base, argument, and exponent fit into a *logarithmic equation*. A **logarithmic equation** is an equation that contains a logarithm.

To write a logarithmic equation, sometimes it is helpful to consider the exponential form first and then convert it to logarithmic form.

1. Arrange the given terms to create a true logarithmic equation.

a. 49, 2, 7

b. $-3, 6, \frac{1}{216}$

c. 4, 4, 1

d. 256, 4, 4

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Let's consider the relationship between the base, argument, and exponent. You can use that relationship to solve for any unknown in a logarithmic equation.

It is important to note that you can convert a logarithmic equation to an exponential equation regardless of which term is unknown.

Worked Example

To solve for any unknown in a simple logarithmic equation, begin by converting it to an exponential equation.

Argument Is Unknown	Exponent Is Unknown	Base Is Unknown
$\log_4 y = 3$	$\log_4 64 = x$	$\log_b 64 = 3$
$4^3 = y$	$4^x = 64$	$b^3 = 64$
$64 = y$	$4^x = 4^3$	$b^3 = 4^3$
	$x = 3$	$b = 4$

2. Justify the last step of each case in the worked example.

a. If $4^3 = y$, why does $y = 64$?

b. If $4^x = 4^3$, why does $x = 3$?

c. If $b^3 = 4^3$, why does $b = 4$?

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3. **Solve for the unknown in each logarithmic equation.**

a. $\log_8 64 = n$

b. $\log_n \frac{1}{16} = -2$

c. $\log_{\frac{1}{2}} 64 = n$

d. $\log n = -3$

e. $\log_n \sqrt[3]{49} = \frac{2}{3}$

f. $\log_9 27 = n$

4. **Write three logarithmic expressions that are equivalent to each given expression. Explain your strategy.**

a. $\log_5 625$

b. $\log_7 \frac{1}{7}$

c. $\log_{64} 8$

d. $\log_2 -2$

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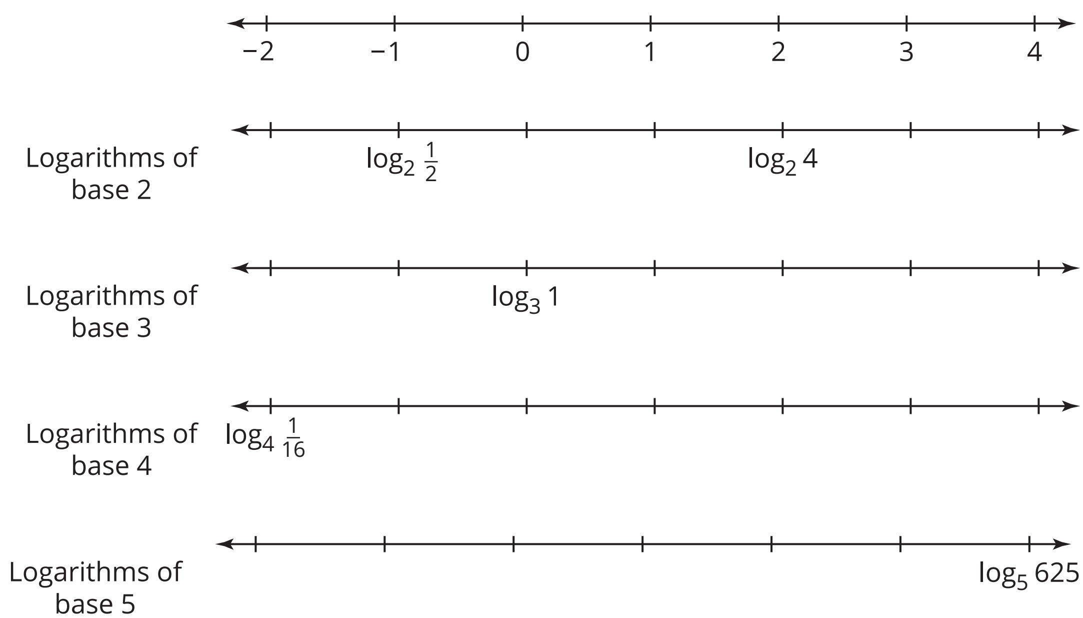
ACTIVITY **1.2**

Estimating with Logarithms

A logarithm is an exponent, so it can be any real number, even an irrational number.

1. Label each number line using logarithmic expressions with the indicated base.



Describe the restrictions on the variables when appropriate.

2. Compare the logarithms on the number lines.

a. Analyze all the logarithms that are equivalent to 0. Write a general statement using the base *b* to represent this relationship.

b. Analyze all the logarithms that are equivalent to 1. Write a general statement using the base *b* to represent this relationship.

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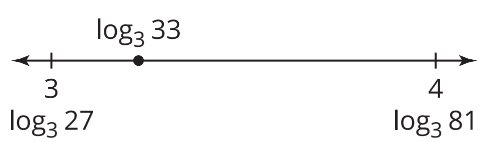
c. **Rewrite the general statements from parts (a) and (b) in exponential form. Use exponent rules to verify that each statement is true.**

You can estimate the value of a logarithm that is not an integer using a number line as a guide.

Worked Example

Estimate the value of $\log_3 33$.

To estimate $\log_3 33$ to the tenths place, identify the closest integer logarithm whose argument is less than 33 and the closest integer logarithm whose argument is greater than 33 on a number line that represents base 3.



Closest integer logarithm whose argument is less than 33:	Logarithm you are estimating:	Closest integer logarithm whose argument is greater than 33:
$\log_3 27$	$\log_3 33$	$\log_3 81$

You know that $\log_3 27 = 3$ and $\log_3 81 = 4$. This means the estimate of $\log_3 33$ is between 3 and 4.

$$\begin{array}{ccccc} \log_3 27 & < & \log_3 33 & < & \log_3 81 \\ 3 & < & x & < & 4 \end{array}$$

Next, estimate the decimal digit.

Because 33 is closer to 27 than to 81, the value of $\log_3 33$ is closer to 3 than to 4.

In this case, 3.2 is a good estimate for $\log_3 33$.

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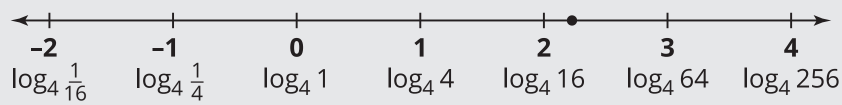
3. **Sutton and Silas were each asked to estimate $\log_4 28$.**

Sutton

I estimated $\log_4 28$ using the number line.



$$\log_4 16 < \log_4 28 < \log_4 64$$
$$2 < x < 3$$
$$\log_4 28 \approx 2.3$$

Silas

I estimated $\log_4 28$ by converting the log into exponential form and estimating based on powers of 4.

$$\log_4 28 = x$$
$$4^x = 28$$

I know that $4^2 = 16$ and $4^3 = 64$ so the estimate of $\log_4 28$ must be between 2 and 3.

$$\log_4 28 \approx 2.4$$

Silas did not use the number line, but his estimate was about the same as Sutton's. Will Silas's method always work?

4. **Estimate each logarithm to the tenths place and explain your reasoning.**

a. $\log_2 10$

b. $\log_5 4$

c. $\log_4 300$

d. $\log 2500$

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5. **Mark and Scotty were each asked to determine which base was used to estimate the value of $\log_b 58 = 2.9$.**

Mark

The log of 58 falls between 2 and 3 when the base is 4.

$\log_4 16 < \log_b 58 < \log_4 64$

$2 < 2.9 < 3$

So, $b \approx 4$.

Scotty

The log of 58 falls between 2 and 3 when the base is 5.

$\log_5 25 < \log_b 58 < \log_5 125$

$2 < 2.9 < 3$

So, $b \approx 5$.

Who is correct? Explain your reasoning.

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6. Use the number lines from Question 1 to determine the appropriate base of each logarithm.

a. $\log_b 108 = 2.9$

b. $\log_b 0.4 = -1.3$

c. $\log_b 74 = 3.1$

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ACTIVITY
1.3

Estimating with Natural Logarithms

You have estimated with logarithms that have rational number bases. Let's consider how you can estimate the values of natural logarithms—those with a base of *e*.

1. For a fixed base greater than 1, as the value of the argument gets larger, what happens to the value of the logarithm? Provide an example to illustrate your statement.

2. Plot $\log_2 18$, $\log_3 18$, $\log_4 18$, and $\log_5 18$ on the appropriate number lines in Question 1 in the previous activity. Then use the number lines to estimate the numeric value of each logarithm to the tenths place. Verify your answers in exponential form.

3. For a fixed argument, when the value of the base is greater than 1 and increasing, what happens to the value of the logarithm?

4. How could you use the number lines to predict the value of ln 18?

5. Make a prediction for the value of ln 18.

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NOTES

TALK the TALK

Always, Sometimes, Never

Complete each sentence with *always, sometimes, or never* to make it true. Explain your reasoning.

1. **The value of a logarithm is ______________ equal to the exponent of the corresponding exponential equation.**

2. **The argument of a logarithmic expression is ______________ a negative number.**

3. **The value of a logarithm is ______________ equal to a negative number.**

4. **The base of a logarithm is ______________ a negative number.**

5. **A logarithm is ______________ a value that is not an integer.**

6. **For a base greater than 1, if $b > c$ then the value of $\log_a b$ is ______________ greater than $\log_a c$.**

7. **If $a > b$, then the value of $\log_a 1$ is ______________ less than $\log_b 1$.**

8. **The base of a logarithm is ______________ equal to 1.**

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Assignment

Write

Describe how to estimate the value of a logarithm.

Remember

The value of a logarithmic expression is equal to the value of the exponent in the corresponding exponential expression.

For a fixed base greater than 1, as the value of the argument increases, the value of the logarithm increases as well.

For a fixed argument, when the value of the base is greater than 1 and increasing, the value of the logarithm is decreasing.

Practice

1. Solve for the unknown in each logarithmic equation.

   a. $\log 1000 = n$

   b. $\log_n \frac{1}{27} = -3$

   c. $\log_{\frac{1}{3}} 81 = n$

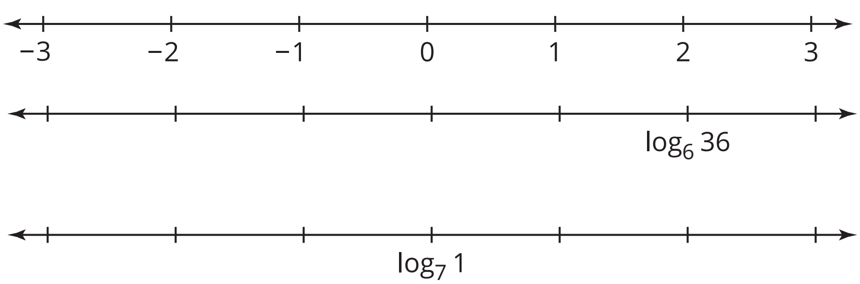
   d. $\log_8 16 = n$

2. Consider base 6 and base 7 logarithms.

   a. Label each number line using logarithmic expressions with the indicated base to match the given number line.

Logarithms of base 6

Logarithms of base 7



   b. Estimate $\log_6 40$.

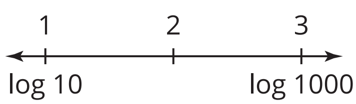
   c. Estimate $\log_7 40$.

   d. Estimate $\log_6 200$.

   e. Estimate $\log_7 100$.

3. Estimate $\log_2 15$ to the nearest tenth. Explain your reasoning.

4. Two students are trying to finish labeling the number line with the base 10 logarithmic expression that equals 2. Dylan says the missing logarithm should be log 505 because 505 is halfway between 10 and 1000, just like 2 is halfway between 1 and 3. Jakob disagrees. He says the missing logarithm should be log 100.



   a. Which student is correct? Explain your reasoning.

   b. For a logarithm with a base greater than 1, how does the argument change for every increase of 1 in the value of the logarithm?

   c. Estimate log 55. Explain your reasoning.

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Stretch

Solve the equation $4 = \log_2(x) + \log_2(x - 6)$.

Review

1. Given: $f(x) = 1.5^x$ and $g(x) = -\frac{1}{2}f(4x)$.
   a. Describe the transformation of $f(x)$ that produces $g(x)$.
   b. Write $g(x)$ as an exponential function.
2. Consider $s(x) = 3^{x-1}$, which is a transformation of the function $f(x) = 3^x$.
   a. Describe the transformation(s) of $f(x)$ to produce $s(x)$.
   b. Write the equations of the inverse functions $f^{-1}(x)$ and $s^{-1}(x)$.
   c. Describe the transformation(s) on the graph of $f^{-1}(x)$ to produce $s^{-1}(x)$.
3. Given $p(x) = 2^x$ and $t(x) = 3p(x + 1) + 7$.
   a. Describe the transformation of $p(x)$ that produces $t(x)$.
   b. Write $t(x)$ as an exponential function.
4. Consider the function $h(x)$, which is formed by translating the function $g(x) = \log_3 x$ right 2 units and down 1 unit.
   a. Write $h(x)$ in terms of $g(x)$.
   b. Complete the table by determining the corresponding point on $h(x)$ for each reference point on $g(x)$.

Reference Point on $g(x)$	**Corresponding Point on $h(x)$**
$(\frac{1}{3}, -1)$	
(1, 0)	
(3, 1)	
(9, 2)	

   c. Write $h(x)$ as a logarithmic function.
   d. List the domain, range, and any asymptotes of the logarithmic function $h(x)$.
5. Determine the quadratic equation that goes through the points (−1, 10), (2, 4), and (3, −6).

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2

Mad Props

Properties of Logarithms

Warm Up

Identify the property of powers associated with each example.

1. $2^5 \cdot 2^2 = 2^7$

2. $2^1 = 2$

3. $2^0 = 1$

4. $\frac{2^{14}}{2^5} = 2^9$

Learning Goals

- Derive the properties of logarithms.
- Expand logarithmic expressions using properties of logarithms.
- Rewrite multiple logarithmic expressions as a single logarithmic expression.

Key Terms

- Zero Property of Logarithms
- Logarithm with Same Base and Argument
- Product Rule of Logarithms
- Quotient Rule of Logarithms
- Power Rule of Logarithms

You have studied properties of powers. How can you use the properties of powers to understand the properties of logarithms?

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GETTING STARTED

Show What You Know

You have studied the properties of powers in previous lessons and in previous courses. Let's recall some of these properties.

1. Rewrite each expression. Select the properties you used to justify each step in your process.

Product Rule of Powers	Power to a Power Rule	Quotient Rule of Powers
Zero Power Rule	Negative Exponent Rule	

a. $4x^5 \cdot 6x^2y^6 \cdot xy$

b. $42m^5n^3 \cdot \frac{m^4n^2}{6m^6n^5}$

c. $(-3x^7y^3)^5$

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ACTIVITY **2.1**

Properties of Logarithms

Now let's consider how the logarithmic rules and properties correspond to various exponential rules and properties you already know.

Logarithms by definition are exponents, so they have properties that are similar to those of exponents and powers.

1. **Let's consider the Zero Property of Powers to develop a corresponding logarithmic property.**

   a. **Write a sentence to summarize the Zero Property of Powers, $b^0 = 1$.**

   b. **Write the Zero Property of Powers in logarithmic form. This is a corresponding logarithmic property called the *Zero Property of Logarithms*.**

   c. **State the Zero Property of Logarithms in words.**

2. **Let's consider the Base Raised to a Power of One Rule that says that any number raised to a power of one is equal to the base.**

   a. **Write an exponential equation to represent this rule. Use b as the base.**

   b. **Write your exponential equation from part (a) in logarithmic form.**

   c. **State this logarithmic relationship in words.**

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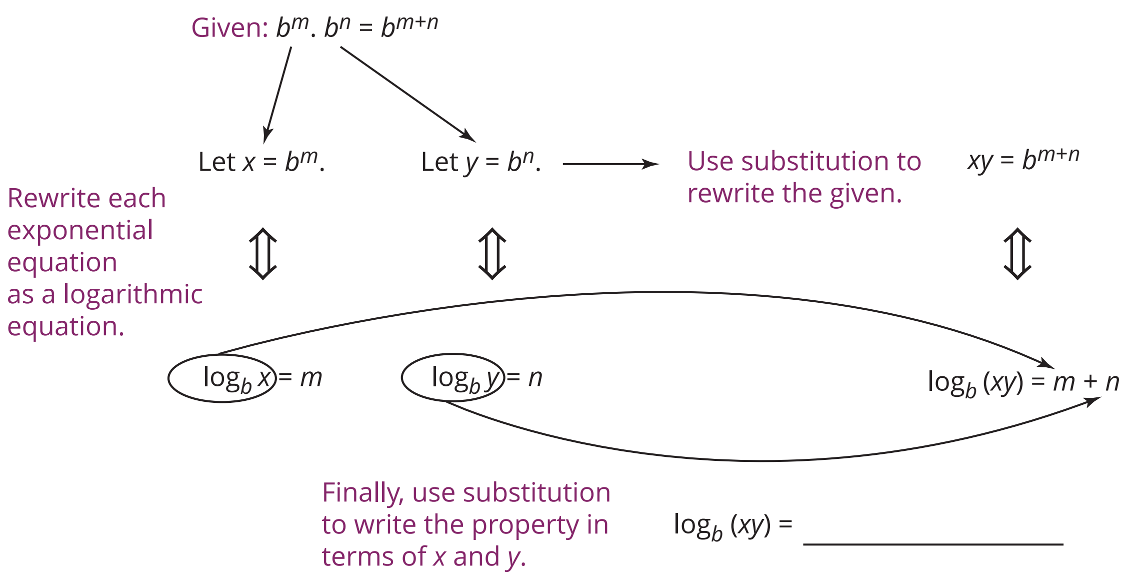
3. **Let's consider the Product Rule of Powers to derive a corresponding logarithmic property.**

   a. **Write a sentence to summarize the Product Rule of Powers, $b^m \cdot b^n = b^{m+n}$.**

   b. **Analyze the steps that begin with the Product Rule of Powers to derive a corresponding logarithmic property called the *Product Rule of Logarithms*. Complete the last line of the diagram.**



   c. **State the Product Rule of Logarithms in words.**

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4. Let's consider the Quotient Rule of Powers to derive a corresponding logarithmic property.

a. Write a sentence to summarize the Quotient Rule of Powers, $\frac{b^m}{b^n} = b^{m-n}$, if $b \neq 0$.

b. Analyze the steps that begin with the Quotient Rule of Powers to derive a corresponding logarithmic property called the *Quotient Rule of Logarithms*.

Given: $\frac{b^m}{b^n} = b^{m-n}$, if $b \neq 0$

Let $x = b^m$. Let $y = b^n$. $\rightarrow$ Use substitution to rewrite the given. ______________

Rewrite each exponential equation as a logarithmic equation.

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$\log_b x = m$ $\log_b y = n$ ______________

Finally, use substitution to write the property in terms of x and y.

$\log_b\left(\frac{x}{y}\right) =$ ____________________

c. Summarize the Quotient Rule of Logarithms in words.

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5. Let's consider the Power to a Power Rule to derive a corresponding logarithmic property.

a. Write a sentence to summarize the Power to a Power Rule, $(b^m)^n = b^{mn}$.

b. Complete the steps that begin with the Power to a Power Rule to derive a corresponding logarithmic property called the *Power Rule of Logarithms*.

Given: $(b^m)^n = b^{mn}$

Let $x = b^m$. → Use substitution to rewrite the given. ______________

Rewrite each exponential equation as a logarithmic equation.

$\Updownarrow$ $\Updownarrow$

______________ ______________

Finally, use substitution to write the property in terms of x.

$\log_b x^n =$ ______________

c. State the Power Rule of Logarithms in words.

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ACTIVITY
2.2

Using Properties of Logarithms to Rewrite Expressions

You can use the properties of logarithms to rewrite and combine logarithmic expressions.

The logarithm properties apply to both natural logarithms and common logarithms.

1. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.

a. $\log_4 (6x^5)$

b. $\log_7 \left(\frac{3y^4}{x^3}\right)$

c. $\ln (3xy^3)$

2. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.

a. $\log_2 10 + 3 \log_2 x$

b. $4 \log 12 - 4 \log 2$

c. $3(\ln 3 - \ln x) + (\ln x - \ln 9)$

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3. **Suppose $\log_a 5 = p$, $\log_a 3 = q$, and $\log_a 2 = r$. Write an algebraic expression for each logarithmic expression.**

   a. $\log_a 50$

   b. $\log_a 0.3$

   c. $\log_a \frac{1}{27}$

4. **Identify the property of logarithms associated with each example.**

   a. $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^7$

   b. $4^1 = 4$

   c. $0.25^0 = 1$

   d. $\frac{2^{14}}{2^5} = 2^9$

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NOTES

TALK the TALK

Organize and Mathematize

You have derived different properties of logarithms.

1. Complete the tables to define each exponential and logarithmic property verbally and symbolically. Provide examples for each property.

Exponential Property	Logarithmic Property
Zero Property of Powers	**Zero Property of Logarithms**
Verbal:	Verbal:
Symbolic:	Symbolic:
Examples:	Examples:
Base Raised to a Power of One	**Logarithm with Same Base and Argument**
Verbal:	Verbal:
Symbolic:	Symbolic:
Examples:	Examples:

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NOTES

Exponential Property	Logarithmic Property
Product Rule of Powers	**Product Rule of Logarithms**
Verbal:	Verbal:
Symbolic:	Symbolic:
Examples:	Examples:
Quotient Rule of Powers	**Quotient Rule of Logarithms**
Verbal:	Verbal:
Symbolic:	Symbolic:
Examples:	Examples:
Power to a Power Rule	**Power Rule of Logarithms**
Verbal:	Verbal:
Symbolic:	Symbolic:
Examples:	Examples:

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Assignment

Write

Define each term in your own words.

1. Zero Property of Logarithms
2. Logarithm with Same Base and Argument
3. Power Rule of Logarithms

Remember

The Product Rule of Logarithms states: "The logarithm of a product is equal to the sum of the logarithms of the factors."

The Quotient Rule of Logarithms states: "The logarithm of a quotient is equal to the difference of the logarithms of the dividend and divisor."

Practice

1. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.
   a. $\log_3 (ab^2c^3)$
   b. $\log \left(\frac{x^3}{5y^2}\right)$
   c. $\log_2 (6mn^4)$
   d. $\ln \left(\frac{2x}{y^{10}}\right)$
2. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.
   a. $2 \log_5 3 - \log_5 y$
   b. $7 \ln x + \ln 8 - 3 \ln y$
   c. $2(\log 5 + \log m) - \log (m^3)$
   d. $8 \log_2 x - 3(\log_2 y + 2 \log_2 x)$
3. Suppose $\log_a 2 = m$, $\log_a 5 = n$, and $\log_a 7 = t$. Write an algebraic expression for each logarithmic expression.
   a. $\log_a 14$
   b. $\log_a 20$
   c. $\log_a \left(\frac{5}{14}\right)$
   d. $\log_a \left(\frac{1}{49}\right)$
   e. $\log_a 100$
   f. $\log_a \left(\frac{10}{7}\right)$
4. An earthquake's magnitude, M, can be determined using the formula $M = \log \left(\frac{I}{10^{-4}}\right)$, where I represents the intensity of the earthquake. Rewrite the logarithmic expression in the formula in expanded form.
5. The loudness, L, of a sound, in decibels, can be determined using the formula $L = 10 \log \left(\frac{I}{10^{-12}}\right)$, where I represents the intensity of the sound. Rewrite the logarithmic expression in the formula in expanded form.

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Stretch

Use properties of logarithms to rewrite $\log_8 \left(\frac{\sqrt{x}}{y^3}\right)$ in expanded form.

Review

1. Solve for the unknown in each logarithmic equation.
   a. $\log 10000 = n$
   b. $\log_n \left(\frac{1}{81}\right) = -2$
   c. $\log_{\frac{1}{2}} 64 = n$
   d. $\log_9 27 = n$
2. Estimate $\log_4 5$ to the nearest tenths place. Explain your reasoning.
3. The Richter scale is used to rate the magnitude of an earthquake, or the amount of energy released. An earthquake's magnitude, M, is determined using the equation, $M = \log \left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake), and I_0 is the intensity of a standard earthquake or "threshold quake" whose seismograph amplitude is 10^{-4} cm.
   a. An earthquake in San Francisco measured 7.9 on the Richter scale, while an earthquake in Chile measured 8.8. How many times more intense was the Chilean earthquake?
   b. An earthquake in Mexico City measured 8.0 on the Richter scale. An earthquake was recorded in Haiti that was three times stronger. What was the magnitude of the Haitian earthquake?
4. Determine the quadratic equation that goes through the points (1, 3), (−2, −3), and (−1, −5).

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3

More Than One Way to Crack an Egg

Solving Exponential Equations

Warm Up

Solve each exponential equation.

1. $2^x = 32$

2. $5^x = 625$

3. $x^3 = 27$

4. $x^7 = 128$

Learning Goals

- Solve exponential equations using the Change of Base Formula.
- Solve exponential equations by taking the log of both sides.
- Analyze different solution strategies to solve exponential equations.

Key Term

- Change of Base Formula

You have previously solved exponential equations using common bases. How can you use logarithms to solve exponential equations?

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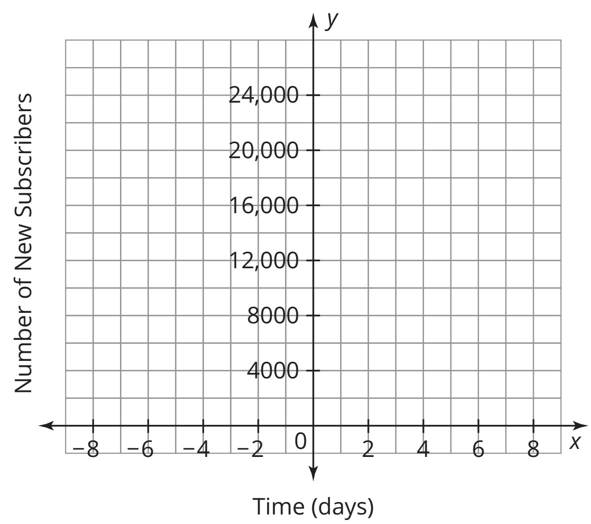
GETTING STARTED

Don't Burst My Bubble

The newest online game is Bubblez Burst, a highly addictive game that runs on social media. Before the game was even released, there were 50 early subscribers. The creators estimate that everyone who subscribes will then send 3 more people invitations to subscribe the next day. So, on the first day of its actual release, there were 150 new subscribers.

A person is considered a new subscriber for one day only, the actual day they first subscribe.

1. Write a function using the creator's estimate to model the number of new subscribers, *P*, who will be playing Bubblez Burst on day *t*. Use technology to graph your function and then sketch it on the coordinate plane.



Ask Yourself:

How can I check that my function makes sense for Day 1?

2. Use your graph to approximate each answer and then verify algebraically. Show all work and explain your reasoning.

a. On what day will Bubblez Burst reach 4050 new subscribers?

b. On what day will Bubblez Burst reach 20,000 new subscribers?

3. How did your methods in Question 2 parts (a) and (b) differ?

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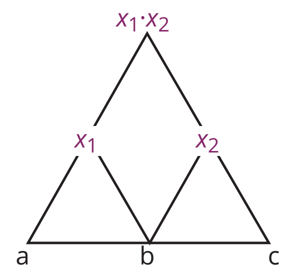
ACTIVITY **3.1**

The Change of Base Formula

So far, you have used estimation to determine the value of logarithms that were not integers. The *Change of Base Formula* allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base. First, you will use the Change of Base Formula and then you will derive it.

The **Change of Base Formula** states:

$\log_b c = \dfrac{\log_a c}{\log_a b}$, where $a, b, c > 0$ and $a, b \neq 1$.



$$\log_b c = \log_a c \div \log_a b$$
$$\uparrow \qquad \uparrow \qquad \uparrow$$
$$x_2 = x_1 \cdot x_2 \div x_1$$

Many calculators can evaluate only common logs and natural logs. So, the Change of Base Formula can be helpful to evaluate logs of other bases.

1. Rewrite the exponential equation you wrote in Question 2, part (b), of the Getting Started as a logarithmic equation.

2. Use the Change of Base Formula to evaluate the logarithmic expression using common logs. Round to the nearest thousandth.

3. Compare your estimate in Question 2, part (b), of the Getting Started with the calculated value in Question 2 by substituting each value back into the original equation. What do you notice?

Ask yourself:

Do a few decimal places really make that much of a difference?

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4. **Use a calculator to determine how many days it will take for Bubblez Burst to reach one million subscribers.**

5. **Tammy was asked to approximate how many days it would take Bubblez Burst to reach 30,000 subscribers. Describe the calculation error Tammy made. Then, use your knowledge of estimation to explain why *t* cannot be equal to 2.3.**

Tammy

$$30{,}000 = 50 \cdot 3^t$$
$$600 = 3^t$$
$$\log_3 (600) = t$$
$$\frac{\log 600}{\log 3} = t$$
$$\log 200 = t$$
$$2.3 \approx t$$

6. **In 2017, there were approximately 326 million people in the United States. In that year, how long would it take for everyone in the country to subscribe to Bubblez Burst?**

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ACTIVITY
3.2

Taking the Logarithm of Both Sides of an Equation

Previously, you solved exponential equations with common bases as well as exponential equations with common exponents.

1. Solve each exponential equation and explain the strategy you used.

a. $2^x = 64$

b. $y^3 = 125$

2. Solve each logarithmic equation and explain the strategy you used.

a. $\log_3 w = \log_3 20$

b. $\log_m 9 = \log_4 9$

You just derived the relationship that if $\log_b a = \log_b c$, then $a = c$. The converse is also true. If $a = c$, then $\log_b a = \log_b c$. You can use this knowledge to now derive the Change of Base Formula.

Taking the logarithm of both sides of an equation keeps the equation balanced.

3. Consider the exponential equation $b^x = c$, where x is the unknown in the exponent and b and c are constants.

a. Solve the exponential equation for x by rewriting it in logarithmic form.

b. Solve the exponential equation for x by taking the logarithm of both sides.

c. How do the results from these two methods demonstrate the Change of Base Formula?

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4. Todd and Danielle each solved the exponential equation $4^{x-1} = 50$.

Todd

$$4^{x-1} = 50$$
$$x - 1 = \log_4 (50)$$
$$x - 1 = \frac{\log 50}{\log 4}$$
$$x - 1 \approx 2.822$$
$$x \approx 3.822$$

Danielle

$$4^{x-1} = 50$$
$$\log (4^{x-1}) = \log 50$$
$$(x - 1) \log 4 = \log 50$$
$$x - 1 = \frac{\log 50}{\log 4}$$
$$x = \frac{\log 50}{\log 4} + 1$$
$$x \approx 3.822$$

Describe how Todd's and Danielle's methods are different.

5. Solve the exponential equation $8^x = 38.96$ using both Todd's and Danielle's methods. Round to the nearest thousandth and check your work.

Remember:

When solving equations, it can be helpful to isolate the term with the variable first before solving.

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ACTIVITY **3.3**

Analyzing Strategies to Solve Exponential Equations

In the previous activity, you explored solving exponential equations by taking the logarithm of both sides. There are multiple strategies for solving exponential equations. Let's consider some different methods used by students to solve the same exponential equation.

1. John, Bobbi, and Randy each solved the equation $9^{x+4} = 27$.

John

$$9^{x+4} = 27$$
$$x + 4 = \log_9 27$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

Bobbi

$$9^{x+4} = 27$$
$$\log (9^{x+4}) = \log 27$$
$$(x + 4)\log 9 = \log 27$$
$$x + 4 = \frac{\log 27}{\log 9}$$
$$x + 4 = 1.5$$
$$x = -2.5$$

Randy

$$9^{x+4} = 27$$
$$(3^2)^{x+4} = 3^3$$
$$3^{2(x+4)} = 3^3$$
$$2(x + 4) = 3$$
$$2x + 8 = 3$$
$$2x = -5$$
$$x = -2.5$$

a. Describe each method used.

b. Will each method work for every logarithmic equation? Describe any limitations of each method.

2. Ameet and Neha each took the logarithm of both sides of the equation to solve $24^x = 5$.

Ameet

$$24^x = 5$$
$$\ln (24^x) = \ln 5$$
$$x \ln 24 = \ln 5$$
$$x = \frac{\ln 5}{\ln 24}$$
$$x \approx 0.506$$

Check:

$$24^x \stackrel{?}{=} 5$$
$$24^{0.506} \stackrel{?}{=} 5$$
$$4.99 \approx 5$$

Neha

$$24^x = 5$$
$$\log (24^x) = \log 5$$
$$x \log 24 = \log 5$$
$$x = \frac{\log 5}{\log 24}$$
$$x \approx 0.506$$

Check:

$$24^x \stackrel{?}{=} 5$$
$$24^{0.506} \stackrel{?}{=} 5$$
$$4.99 \approx 5$$

Describe the similarities and differences in their methods. Explain why each student's method is correct.

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NOTES

TALK the TALK

What's Your Strategy?

1. **Solve each exponential equation. Explain why you chose the method that you used.**

   a. $2^{x-5} + 6 = 30$

   b. $7 \cdot (2)^{3x} = 840$

   c. $4^{x-3} - 5 = 16$

   d. $10 \cdot \left(\frac{3^{2x}}{2}\right) = 360$

   e. $2 \cdot 3^{5x} + 1 = 55$

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Assignment

Write

Write the Change of Base Formula.

Remember

The Change of Base Formula allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base.

If $\log_b a = \log_b c$, then $a = c$.

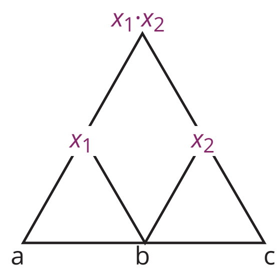
If $a = c$, then $\log_b a = \log_b c$.

Practice

1. Ten volunteers begin recruiting people to be volunteers for a large fundraising event. After 1 week, the total number of volunteers has doubled to 20. Each subsequent week, the total number of volunteers doubles.
   a. Write a function to model the total number of volunteers, V, in the group after t weeks.
   b. How many weeks will it take for the total number of volunteers to reach 1280?
   c. How many weeks will it take for the volunteers to reach their goal of 15,000 total volunteers?
2. A group of citizens established a new political party called the People's Party. The number of members in the party, P, can be modeled by the function $P(m) = 500 \cdot 1.2^m$, where m represents the number of months since the founding of the party.
   a. How many months will it take for the membership of the party to grow to 10,000 members?
   b. How many years will it take for the membership of the party to grow to 200,000 members?
3. Solve each exponential equation. Round your answer to the nearest hundredth.
   a. $9^{x+2} + 3 = 52$
   b. $\frac{7^{3x-2}}{5} = 2$

Stretch

Consider the Triangle of Power diagram. Explain how the diagram shows that the Change of Base Formula works for $a = 10$, $b = 100$, and $c = 1000$. Determine the values of x_1 and x_2.



$$\log_b c = \log_a c \div \log_a b$$

$$x_2 = x_1 \cdot x_2 \div x_1$$

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Review

1. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.
   a. $\log_4 (x^2y^5z)$
   b. $\log \left(\frac{2a^6}{b^3}\right)$
   c. $\log_3 (55r^2st^8)$
   d. $\ln \left(\frac{a^3}{2b^4}\right)$
2. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.
   a. $\log_2 x - 4 \log_2 y$
   b. $10 \ln a - \ln 6 - 2 \ln b$
   c. $10 \log x - 4(\log x + \log 2)$
   d. $8 \log_2 x - 3(\log_2 y + 2 \log_2 x)$
3. Given: $f(x) = 2^x$.
   a. Write the function $f^{-1}(x)$, the inverse of $f(x) = 2^x$.
   b. Graph and label the functions $f(x)$ and $f^{-1}(x)$ on the coordinate plane.
   c. Describe how to calculate $f^{-1}(2)$ without a calculator. Then, calculate $f^{-1}(2)$, $f^{-1}(16)$, and $f^{-1}(128)$.
   d. Determine the domain, range, asymptotes, intercepts, end behavior, and intervals of increase and decrease for $f^{-1}(x)$.

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4

Logging On

Solving Logarithmic Equations

Warm Up

Use the properties of logarithms to rewrite each expression.

1. $\log_3 (8x^4)$

2. $\log_2 \left(\frac{5y^6}{x^4}\right)$

3. $2 \log x - 2 \log y$

Learning Goals

- Solve for the base, argument, and exponent of logarithmic equations.
- Solve logarithmic equations using logarithmic properties.
- Solve logarithmic equations arising from real-world situations.
- Complete a decision tree to determine efficient methods for solving exponential and logarithmic equations.

You have used the properties of logarithms to rewrite and evaluate logarithmic expressions. How can these properties be applied when solving logarithmic equations?

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GETTING STARTED

Sound Off

A decibel is a unit used to measure the loudness of sound. The formula for the loudness of a sound is given by

$$dB = 10 \log \left(\frac{I}{I_0}\right)$$

where dB is the decibel level. The quantity I_0 is the intensity of the threshold sound—a sound that can barely be perceived. The quantity I is the number of times more intense a sound is than the threshold sound.

1. **The sound in a quiet library is about 1000 times as intense as the threshold sound, or $I = 1000I_0$. Calculate the decibel level of a quiet library.**

2. **The sound of traffic on a city street is calculated to be 500 million times as intense as the threshold sound. Calculate the decibel level for city traffic.**

3. **The sound of a crying baby registers at 115 decibels. How many times more intense is this sound than the threshold sound?**

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ACTIVITY

4.1

Solving Logarithmic Equations

While most medications have guidelines for dosage amounts, doctors must also determine the amount of time a medication will remain in a patient's body before it is metabolized when they write prescriptions. The amount of medicine remaining in a patient's body can be predicted by the formula

$$t = \frac{\log\left(\frac{C}{A}\right)}{\log(1 - r)}$$

where t is the time in hours since the medicine was administered, C is the current amount of medicine remaining in the patient's body in milligrams, A is the original dose of the medicine in milligrams, and r is the rate at which the medicine is metabolized.

1. A patient is given 10 milligrams of medicine which is metabolized at the rate of 20% per hour. How long will it take for 2 milligrams of the medicine to metabolize?

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Suppose you know that 4 hours after administering a 12-miligram dose of medicine, 9 milligrams remain in the patient's body. How can you determine the rate at which the medicine metabolizes?

You can solve for an unknown in a logarithmic equation by using the Properties of Equality and by rewriting the logarithmic equation as an exponential equation to isolate the variable.

Worked Example

$4 = \dfrac{\log\left(\frac{9}{12}\right)}{\log(1 - r)}$	Substitute the values for t, C, and A into the formula.
$4\log(1 - r) = \log(0.75)$	Multiply both sides of the equation by $\log(1 - r)$.
$4\log(1 - r) \approx -0.125$	Evaluate $\log(0.75)$.
$\log(1 - r) \approx -0.03125$	Divide both sides of the equation by 4.
$10^{-0.03125} \approx 1 - r$	Rewrite as an exponential equation.
$r \approx 1 - 10^{-0.03125}$	Isolate the variable r.
$r \approx 0.0694$	

The medicine is metabolized at an approximate rate of 6.94% per hour.

2. Six hours after administering a 20-milligram dose of medicine, 5 milligrams remain in a patient's body. At what rate is the medicine metabolized?

3. A patient is undergoing an 8-hour surgery. If 4 milligrams of medicine must remain in the patient's body at the end of surgery, and the medicine is metabolized at the rate of 15% per hour, how much medicine must be administered at the start of surgery?

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ACTIVITY 4.2

Determining Efficient Methods to Solve Logarithmic Equations

You have analyzed different methods for solving exponential equations. There are also multiple methods for solving logarithmic equations.

1. Solve each logarithmic equation using two different methods.

	Example	First rewrite as an exponential equation. Then solve for *x*.	First apply the Change of Base Formula. Then solve for *x*.
Argument Is Unknown	$\log_5 x = 3.1$		
Exponent Is Unknown	$\log_8 145 = x$		
Base Is Unknown	$\log_x 24 = 6.7$		

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2. Consider the position of the unknown for each logarithmic equation in Question 1. Circle your preferred method for solving. Explain your choice.

3. When might it be more efficient to solve a simple logarithmic equation by rewriting it in exponential form?

4. When might it be more efficient to solve a simple logarithmic equation by applying the Change of Base Formula?

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5. Circle the logarithmic equations that can be solved more efficiently when rewritten as exponential equations. Draw a box around the equations that can be solved more efficiently by applying the Change of Base Formula. Explain your choice.

a. $\log_4 (x + 3) = \frac{1}{2}$

b. $\log_{4.5} 9 = x - 1$

c. $\log_{x+2} 7.1 = 3$

d. $\log_3 4.6 = 2 - x$

e. $\ln (x + 4) = 3.8$

f. $\log_{11} 12 = x - 7$

g. $\log_{1-x} 8 = 14.7$

h. $\log (4 - x) = 1.3$

6. Solve each logarithmic equation. Check your work.

a. $\log_2 (x^2 - 6x) = 4$

b. $\log_6 (x^2 + x) = 1$

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ACTIVITY
4.3 Solving Logarithmic Equations with Multiple Logarithms

If a logarithmic equation involves multiple logarithms, you can use the properties of logarithms to rewrite the equation in a form you already know how to solve.

1. Analyze Georgia's, Santiago's, and Lorenzo's work.

Georgia

$$\begin{aligned} \log 5 + \log x &= 2 \\ \log(5 + x) &= 2 \\ 10^2 &= 5 + x \\ 95 &= x \end{aligned}$$

Santiago

$$\begin{aligned} \log 5 + \log x &= 2 \\ 5x &= 2 \\ x &= \frac{2}{5} \end{aligned}$$

Lorenzo

$$\begin{aligned} \log 5 + \log x &= 2 \\ 5 + x &= 2 \\ x &= -3 \end{aligned}$$

a. Explain the error in each student's reasoning.

b. Solve $\log 5 + \log x = 2$. Check your work.

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2. **Solve each logarithmic equation. Check your work.**

   a. $\log_5 45x - \log_5 3 = 1$

   b. $\log_2 8 + 3\log_2 x = 6$

   c. $\ln 18x - \ln 6 = 2$

Remember:

Recall the restrictions on the variables for the logarithmic equation, $y = \log_b x$. The variable y can be any real number, the base b must be greater than 0 and not equal to 1, and the argument x must be greater than 0.

Remember:

The natural log, $\ln x$, is equivalent to $\log_e x$.

3. **Pippa and Kate disagree about the solution to the logarithmic equation $\log_5 x^2 - \log_5 4 = 2$.**

$$\log_5 x^2 - \log_5 4 = 2$$
$$\log_5\left(\frac{x^2}{4}\right) = 2$$
$$5^2 = \frac{x^2}{4}$$
$$25 = \frac{x^2}{4}$$
$$100 = x^2$$
$$x = 10, -10$$

Kate says the solutions are $x = 10$, $x = -10$. Pippa says that the solution $x = -10$ should be rejected because the argument of a logarithm must be greater than zero.

Who is correct? Explain your reasoning.

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4. Solve $\log_3 (x - 4) + \log_3 (x + 2) = 3$. Check your work.

5. Elijah and Zander each solved the logarithmic equation $2 \log 6 = \log x - \log 2$. Explain why each student's method is correct.

Elijah

$$2 \log 6 = \log x - \log 2$$
$$\log (6^2) = \log\left(\frac{x}{2}\right)$$
$$36 = \frac{x}{2}$$
$$72 = x$$

Check:

$$2 \log 6 \stackrel{?}{=} \log 72 - \log 2$$
$$\log (6^2) \stackrel{?}{=} \log\left(\frac{72}{2}\right)$$
$$\log 36 = \log 36$$

Zander

$$2 \log 6 = \log x - \log 2$$
$$\log (6^2) - \log x + \log 2 = 0$$
$$\log\left(\frac{36}{x}\right) + \log 2 = 0$$
$$\log\left(\frac{72}{x}\right) = 0$$
$$\frac{72}{x} = 10^0$$
$$\frac{72}{x} = 1$$
$$72 = x$$

Check:

$$2 \log 6 \stackrel{?}{=} \log 72 - \log 2$$
$$\log (6^2) \stackrel{?}{=} \log\left(\frac{72}{2}\right)$$
$$\log 36 = \log 36$$

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6. Solve each logarithmic equation. Check your work.

a. $2 \log (x + 1) = \log x + \log (x + 3)$

b. $2 \log (x - 3) = \log 4 + \log \left(x - \frac{15}{4}\right)$

c. $\ln (x - 3) + \ln (x - 2) = \ln (2x + 24)$

d. $2 \log_3 x = \log_3 4 + \log_3 16$

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NOTES

TALK the TALK

As Easy As Falling Off a Log

You have solved a variety of exponential and logarithmic equations. When doing so, you have had to consider the structure and characteristics of each equation.

The list shown contains an example of each type of exponential and logarithmic equation you have solved.

- $2^{x+2} = 32$
- $\log_6 x + 7 = \log_6 31$
- $4^{5.1} = x$
- $\log_3 x - \log_3 9 = 5$
- $5^x = 17$
- $\ln x + \ln 2 = \ln 24 - \ln 3$
- $x^{5.5} = 22$
- $\log_8 x = 0.3$

1. Complete the decision tree on the following pages to demonstrate the most efficient strategy to solve each type of equation.

a. For each branch of the decision tree, write the appropriate example equation from the list.

b. Describe the first step you would use to solve each equation.

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Solving Logarithmic Equations

Identify the number and location of the logarithms.

- multiple log = multiple log
  - Example:
  - 1^{st} step:
- log = log
  - Example:
  - 1^{st} step:
- log = constant
  - Example:
  - 1^{st} step:
- multiple log = constant
  - Example:
  - 1^{st} step:

Solving Exponential Equations

Identify the location of the unknown.

- base
  - Example:
  - 1^{st} step:
- exponent
  - Example:
  - 1^{st} step:
  - Example:
  - 1^{st} step:
- value
  - Example:
  - 1^{st} step:

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NOTES

2. **Create an example problem to fit each first step described.**

   a. **Use common bases or common exponents.**

   b. **Convert to exponential form.**

   c. **Evaluate the power.**

   d. **Take the logarithm of both sides.**

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Assignment

Write

State whether applying the Change of Base formula or rewriting in exponential form would be most efficient for solving the logarithmic equation based on its unknown quantity. Explain your reasoning in your own words.

1. argument of the logarithm is unknown
2. value of the logarithm is unknown
3. base of the logarithm is unknown

Remember

To solve for an unknown in a simple logarithmic equation, consider the relationship between the base, argument, and exponent.

Practice

1. In music, the cent, c, is a unit of measure that is used to measure the differences in frequencies between musical notes. The formula $c = 1200 \log_2 \left(\frac{f_1}{f_2}\right)$ can be used to determine the number of cents, c, between a musical note of a higher frequency f_1 and a musical note of a lower frequency f_2.
   a. Determine the number of cents between the notes F4 and C4 if F4 has a frequency of 349.23 Hz (hertz) and C4 has a frequency of 261.63 Hz. Round your answer to the nearest cent.
   b. Determine the frequency (in hertz) of the note G4 if the notes A4 and G4 are separated by 200 cents and A4 has a frequency of 440 Hz. The note A4 has a higher frequency than the note G4. Round your answer to the nearest hertz.
2. The magnitude limit, L, of a telescope refers to the magnitude of the faintest star that is observable with the telescope. Stars with lower magnitudes are brighter than stars with higher magnitudes. The equation $L = \frac{\log_6 d}{0.257} + 2$ can be used to determine the magnitude limit of a telescope with an objective lens diameter, d, in millimeters.
   a. Lorinda purchases a telescope that has an objective lens that is 80 mm in diameter. Determine the magnitude limit of her telescope to the nearest tenth.
   b. After using her telescope, Lorinda realizes she should have purchased a larger telescope. She wants to purchase a telescope with a magnitude limit of 13.5. Determine the minimum diameter of the objective lens to the nearest tenth for such a telescope.
3. Solve each logarithmic equation. Check your work.
   a. $4 = \log_2 x + \log_2 (x - 6)$
   b. $\log_x 216 = 1.5$

Stretch

Solve $2 \log_3 x - \log_3 (x - 4) = 2 + \log_3 2$.

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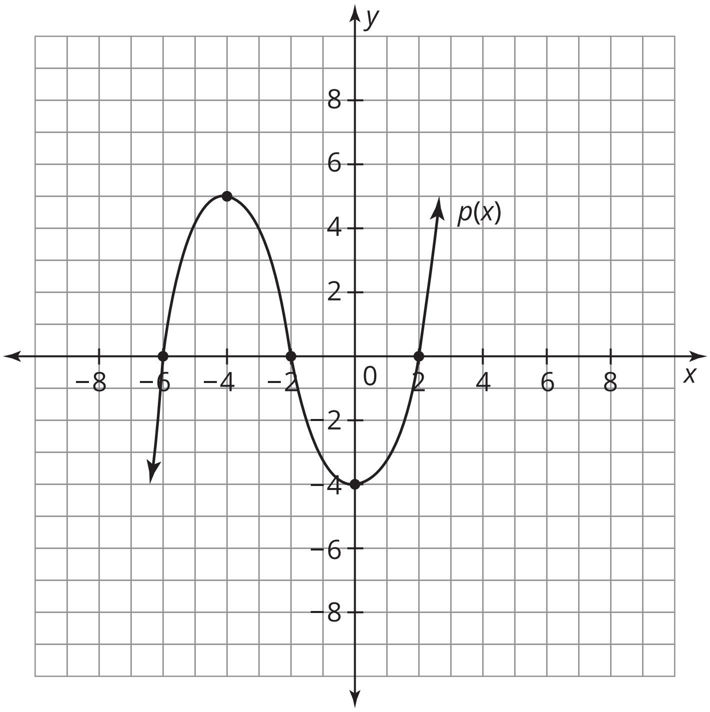
Review

1. A species of bird was recently introduced to an island. The bird population on the island, B, can be modeled by the function $B(t) = 40 \cdot 1.4^t$, where t represents the number of years since the birds were introduced to the island.
   a. How many years will it take for the bird population to grow to 500 birds?
   b. How many years will it take for the bird population to grow to 1,000 birds?
2. Solve each exponential equation. Round your answer to the nearest hundredth.
   a. $5^{x-1} - 8 = 70$
   b. $2^{2x+7} = 15$
3. In the year 2007, Mint Beach had a population of 23 thousand people and an annual growth rate of 1.02%.
   a. Write a function to model the population of Mint Beach with respect to t, the number of years since 2007. Write your function in the form $N(t) = N_0e^{rt}$.
   b. Discuss the domain, range, asymptotes, intercepts, end behavior, and intervals of increase and decrease for your population model as they relate to the problem situation.
4. Identify the extrema, zeros, and intercepts of the graph of $p(x)$.



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5

What's the Use?

Applications of Exponential and Logarithmic Equations

Warm Up

Solve each equation for x.

1. $\log x = 3$

2. $8^{3x + 2} = 512$

3. $600 = e^{0.5x}$

4. $\log_x 200 = 4$

Learning Goals

- Use exponential models to analyze problem situations.
- Use logarithmic models to analyze problem situations.

You have explored multiple methods for solving exponential and logarithmic equations. How can you apply these methods to solve real-world problems modeled by exponential or logarithmic equations?

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GETTING STARTED

Time Flies Like an Arrow

Think back to when you were eight years old and had to wait an entire year before you turned nine years old. Does that year of waiting for your next birthday feel the same length to you now? As you age, your perception of time changes because on each birthday a year represents a smaller and smaller portion of your life.

1. Determine the percent of time that one year adds to the life of:

a. a two-year old.

b. a 10-year-old.

c. a 20-year-old.

d. a 50-year-old.

2. To perceive the same passage of time as a 10-year-old perceives the passage of one year:

a. How many years have to pass for a 30-year-old?

b. How many years have to pass for a 40-year-old?

3. Describe the mathematical relationship between the passage of time and the change in the perceived passage of time. Explain your reasoning.

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ACTIVITY

5.1

Solving Problems with a Logarithmic Model

The pH scale is a scale for measuring the acidity or alkalinity of a substance, which is determined by the concentration of hydrogen ions. The formula for pH is pH $= -\log H^+$, where H^+ is the concentration of hydrogen ions measured in moles per cubic liter. Solutions with a pH value less than 7 are acidic. Solutions with a pH value greater than 7 are alkaline. Solutions with a pH of 7 are neutral. For example, plain water has a pH of 7.

A mole is a counting unit- just a word that stands for a number. Just as *dozen* means 12, a *mole* means 6×10^{23}.

1. **The H^+ concentration in orange juice is 0.000199 mole per cubic liter. Determine the pH level of orange juice, and then state whether it is acidic or alkaline.**

2. **The concentration of hydrogen ions in baking soda is 5.012×10^{-9} mole per cubic liter. Determine the pH level of baking soda, and then state whether it is acidic or alkaline.**

3. **Vinegar has a pH of 2.2. Determine the concentration of hydrogen ions in vinegar.**

4. **Lime water has a pH of 12. Determine the concentration of hydrogen ions in lime water.**

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The pH level of soil is used to determine which plants grow best in an area. Different types of plants and vegetables require varying degrees of soil acidity. Generally, the soil tends to be acidic in moist climates and alkaline in dry climates.

The chart shows the optimal pH soil levels for a variety of fruits and vegetables.

Asparagus 6.5 – 7.5	**Avocados** 6.0 – 7.0	**Beets** 5.6 – 6.6
Carrots 5.0 – 6.0	**Garlic** 5.0 – 6.0	**Lettuce** 6.5 – 7.0
Mushrooms 7.0 – 8.0	**Onions** 6.2 – 6.8	**Peanuts** 5.0 – 6.0
Peppers 6.0 – 8.0	**Potatoes** 5.8 – 6.5	**Raspberries** 6.0 – 6.5
Spinach 5.0 – 7.0	**Sweet Corn** 6.0 – 7.0	**Yams** 6.0 – 8.0

5. **A farmer measures the hydrogen ion concentration of the soil to be 6×10^{-7} mole per cubic liter. List the crops that the farmer can grow in this type of soil.**

6. **Determine the range of the soil's optimal hydrogen ion concentration for growing onions.**

7. **The farmer wants to plant spinach, carrots, and beets. She measures the hydrogen ion concentration of the soil to be 3.16×10^{-6} mole per cubic liter. Is her plan feasible? Explain your reasoning.**

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ACTIVITY 5.2

Solving Problems with an Exponential Model

Gina, a social media manager, uses the model for continuous population growth to project the monthly increase in the number of followers on her clients' social media sites.

1. Recall that the formula for continuous population growth is $N(t) = N_0\, e^{rt}$. State what each quantity represents in terms of this problem situation.

2. Gina claims that when she started working with an up-and-coming boy band, they had 18,450 online followers, and she was able to increase their followers by 26% per month.

a. Use Gina's claim to write a function to represent the number of online followers that the boy band had after *t* months.

b. If Gina started working with the band on May 1st, how many online followers did they have by September 1st?

c. How long did it take for the band to surpass 100,000 online followers?

Don't round your decimals until the very end.

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3. Gina's top client is a pro football quarterback. On her website, Gina claims that since she began managing his account three years ago, his online followers have grown to 105,326.

   a. If he had 4125 online followers when he hired Gina, at what monthly rate did his number of online followers grow?

   b. Write an exponential function to represent the number of online followers the quarterback will have in any given month, assuming the number of followers continues to grow at the same rate.

   c. How long would it take the quarterback to double his current online following of 105,326?

4. Gina also manages a running back on the same team who currently has 62,100 online followers. If he has seen a 14% increase in his followers, how many people were following him when he hired Gina 15 months ago?

5. Will the running back ever have the same number of online followers as the quarterback? If so, when? If not, explain your reasoning.

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6. Gina acquires a talk show host as a new client. The host currently has 5200 online followers.

a. Gina claims that she can more than triple the host's number of online followers in 6 months. Determine the rate per month of increased online following.

b. Josh decides to check Gina's work.

Josh

$N(t) = 5200\,e^{0.18t}$

$N(6) = 5200\,e^{0.18(6)}$

$N(6) \approx 15312.33367$

$15312 \neq 3(5200)$

Gina did not triple the host's followers.

What is the error in Josh's method? Check Gina's work correctly.

c. Assuming that Gina's promised rate of growth is true, project how many followers the talk show host will have in a year.

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ACTIVITY
5.3

Solving Problems with a Natural Logarithmic Model

Notice that the formula assumes a constant room temperature. If the surrounding temperature is not constant, then there is a more involved formula.

A coroner uses a formula derived from Newton's Law of Cooling, a general cooling principle, to calculate the elapsed time since a person has died. The formula is $t = -10 \ln \left(\frac{T - R}{98.6 - R}\right)$, where T is the body's measured temperature in °F, R is the constant room temperature in °F, and t is the elapsed time since death in hours.

1. At 8:30 AM a coroner was called to the home of a person who had died. The constant temperature of the room where the body was found is 70°F.

a. At 9:00 AM the body's measured temperature was 85.5°F. Use this body temperature to estimate the time of death.

A more accurate estimate of the time of death is found by taking two or more readings and averaging the calculated times of death.

b. At 9:30 AM the body's measured temperature was 82.9°F. Use this body temperature to estimate the time of death.

c. Compare the estimated times of death from part (a) and (b). Are your answers fairly close? Determine a more accurate estimated time of death.

d. Assuming the body remains in a room with a constant temperature of 70°F, determine the temperature of the body after 24 hours.

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e. **When will the temperature of the body drop to 60°F? Explain your reasoning.**

At 6:00 PM, a coroner was called to the home of a person who had died. The body was found at the bottom of the stairs in a hallway, where the constant temperature is 65°F. At 6:30 PM the body's measured temperature was 95.9°F, and at 7:00 PM the body's measured temperature was 93.7°F.

2. **A witness claims that the person fell down the stairs around 5:30 PM, and she immediately called for an ambulance. Is her statement consistent with the forensic evidence? Explain your reasoning.**

Ask Yourself:

Are your answers to the questions reasonable?

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ACTIVITY

5.4

Writing a Function to Model a Problem Situation

Different liquids freeze at various temperatures because of the difference in their molecular structure.

Andre bought several 2-liter bottles of soda, but he did not have enough room in his refrigerator. He left the bottles outside to cool. Before he put each bottle outside, they were at room temperature, 72°F.

The outside temperature was 25°F. After 30 minutes, he brought one bottle of soda inside and measured the temperature. The temperature of the bottle of soda was 60°F.

A bottle of soda will freeze at 28°F. How long can Andre leave the bottles of soda outside before they will freeze?

Think about:

How can the transformation function form determine the function that models this problem situation?

1. **List everything you know about the function to model the temperature of a bottle of soda over time.**

2. **Write a function to model the temperature of a bottle of soda over time.**

3. **How long can Andre leave the bottles of soda outside before they freeze?**

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ACTIVITY **5.5**

Determining a Regression Model

The breaking strength in tons, $b(x)$, of steel cable with diameter in inches, x, is given in the table.

Diameter (inches)	0.25	0.50	0.75	1.00	1.25	1.50	1.75
Breaking Strength (tons)	5.2	9.8	18.6	29.3	54.2	84.4	122.7

1. Graph the points, analyze the data, and write the regression equation that best models the data.

2. The design of an elevator requires use of steel cable with a diameter of 1.2 inches. Predict the breaking strength of this steel cable.

3. You are a summer intern at an engineering firm. A client plans to construct a two-lane suspension bridge that is approximately 2000 feet in length and will be restricted to passenger vehicles only. Make a decision about the diameter of steel cable suitable for this project. Then, write a letter to the client that includes your recommendation and all supporting information and calculations.

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NOTES

TALK the TALK

Fruit Flies Like a Banana

A population of fruit flies increases exponentially. The initial population of fruit flies is 25. After 4 days, the population of fruit flies is 400.

1. Determine a function that models the growth of the fruit fly population.

2. After how many days will the fruit fly population be 5000?

Some items such as automobiles are worth less over time. The age of an item can be predicted using the formula $t = \frac{\log\left(\frac{V}{C}\right)}{\log(1 - r)}$, where t is the age of the item in years, V is the value of the item after t years, C is the original value of the item, and r is the yearly rate of depreciation expressed as a decimal.

3. A car was originally purchased for \$32,000 and is currently valued at \$20,000. The average rate of depreciation for this car is 12% per year. Estimate the age of the car to the nearest tenth of a year.

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Assignment

Write

What are the attributes of a data set that indicate that it can be modeled by an exponential function?

Remember

Exponential and logarithmic equations can be used to model situations in the real world and solve for unknowns.

Practice

1. Allison found an equation to calculate the depreciated value of a vehicle. The equation is $t = \frac{\log\left(\frac{V}{C}\right)}{\log(1 - r)}$, where V represents the value of the vehicle after t years, C represents the original value of the vehicle, and r represents the average rate of depreciation as a decimal. Stephanie found a different equation used to calculate depreciation. The equation she found is $\frac{V}{C} = (1 - r)^t$.
   a. Use properties of logarithms to show that the equation Stephanie found is equivalent to the equation Allison found.
   b. Allison originally purchased her car for \$45,000. It is currently valued at \$36,520. The average rate of depreciation for the car is 16%. Determine the age of Allison's car using the logarithmic equation she found.
   c. Three years ago, Stephanie purchased her truck for \$52,000. The truck has an average rate of depreciation of 18.5%. Determine the current value of Stephanie's truck using the logarithmic equation Allison found.
   d. Four years ago, Kayla purchased a car for \$36,000. The current value of the car is \$17,400. Determine the average rate of depreciation for the car.
2. The equation used to determine the amount, A, in an account after t years of continuous compounding is $A = Pe^{rt}$, where P represents the principal (or original) amount in the account and r represents the annual interest rate as a decimal.
   a. Six years ago, Dimitri invested \$5000 in an account with continually compounded interest at an annual interest rate of 5.2%. Determine the current amount of money in the account.
   b. Kris invests \$2500 in an account in which the interest is compounded continuously at an annual interest rate of 6%. Determine the amount of money that will be in the account after 10 years.
   c. Five years ago, Vaughn invested money into an account in which the interest is compounded continuously at an annual interest rate of 4%. The account is currently valued at \$1099.26. Determine the amount of money Vaughn invested to the nearest dollar.
   d. Determine the number of years it will take for Cindy's initial investment of \$10,000 to reach a value of \$13,000 if she invests the money in an account in which the interest is compounded continuously at an annual interest rate of 6.5%.
   e. Determine the annual interest rate needed for an initial investment of \$4000 to reach an amount of \$6000 in 8 years if the money is invested in an account in which the interest is compounded continuously.
   f. How many years would it take for the amount in a continuously compounded account to triple in value if the annual interest rate was 6.3%?

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Stretch

Christine has been training for a marathon. Immediately after her morning run, her heart rate is 180 beats per minute. One minute later, her heart rate has fallen to 147 beats per minute, and 2 minutes after her run, it is down to 124 beats per minute. Christine's resting heart rate is 70 beats per minute.

1. Determine the type of function that would best model the given data. Explain your reasoning.
2. Write the function $R(t)$ to model Christine's heart rate t minutes after her morning run.
3. Graph the function $R(t)$.
4. Use the function $R(t)$ to predict Christine's heart rate 5 minutes after her morning run.

Review

1. In music, the cent, c, is a unit of measure that is used to measure the differences in frequencies between musical notes. The formula $c = 1200 \log_2 \left(\frac{f_1}{f_2}\right)$ can be used to determine the number of cents, c, between a musical note of higher frequency f_1 and a musical note of a lower frequency f_2.
   a. Determine the number of cents between the notes A5 and D3 if A5 has a frequency of 880 Hz (hertz) and D3 has a frequency of 146.83 Hz.
   b. Determine the frequency (in hertz) of the note F2 if the notes F2 and B2 are separated by 600 cents and B2 has a frequency of 123.47 Hz. The note B2 has a higher frequency than the note F2.
2. Solve $\log_3 (x + 6) + \log_3 x = 3$. Check your work.
3. A scientist begins with a population of 20 bacteria in a culture. She records the population every hour. The year 0 corresponds to the date of the hour the study began.
   a. Does the population represent an example of exponential decay, exponential growth, or neither? Explain your reasoning.
   b. Write a function to represent the population of bacteria, B, as a function of the hour, h.
   c. Use your function to complete the table.

Hour	Bacteria Population
0	20
1	50
2	125
3	
4	
5	

4. Solve $\frac{a-4}{5a} = \frac{1}{5a+1}$.

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Exponential and Logarithmic Equations Summary

KEY TERMS

- logarithmic equation
- Zero Property of Logarithms
- Logarithm with Same Base and Argument
- Product Rule of Logarithms
- Quotient Rule of Logarithms
- Power Rule of Logarithms
- Change of Base Formula

LESSON 1 All the Pieces of the Puzzle

When you evaluate a logarithmic expression (logarithm), you are determining the value of the exponent in the corresponding exponential expression.

$$\text{base}^{\text{exponent}} = \text{argument} \Leftrightarrow \log_{\text{base}}(\text{argument}) = \text{exponent}$$

The variables of a logarithmic equation have the same restrictions as the corresponding variables of the exponential equation. The base, b, must be greater than 0 but not equal to 1; the argument must be greater than 0; and the value of the exponent has no restrictions.

A **logarithmic equation** is an equation that contains a logarithm. To write a logarithmic equation, sometimes it is helpful to consider the exponential form first and then convert it to logarithmic form.

You can use the relationship between the base, argument, and exponent to solve for any unknown in a logarithmic equation. To solve for any unknown in a simple logarithmic equation, begin by converting it to an exponential equation.

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Argument Is Unknown	Exponent Is Unknown	Base Is Unknown
$\log_4 y = 3$	$\log_4 64 = x$	$\log_b 64 = 3$
$4^3 = y$	$4^x = 64$	$b^3 = 64$
$64 = y$	$4^x = 4^3$	$b^3 = 4^3$
	$x = 3$	$b = 4$

You can convert a logarithmic equation to an exponential equation regardless of which term is unknown.

A logarithm can be any real number, even an irrational number. You can estimate the value of a logarithm that is not an integer by using a number line as a guide.

To estimate $\log_3 33$ to the tenths place, identify the closest logarithm whose argument is less than 33 and the closest logarithm whose argument is greater than 33 on the number line that represents base 3.

The closest logarithm whose argument is less than 33: $\log_3 27$

The logarithm you are estimating: $\log_3 33$

The closest logarithm whose argument is greater than 33: $\log_3 81$

You know that $\log_3 27 = 3$ because $3^3 = 27$ and $\log_3 81 = 4$ because $3^4 = 81$. This means the estimate of $\log_3 33$ is between 3 and 4.

$$\begin{array}{ccccc} \log_3 27 & < & \log_3 33 & < & \log_3 81 \\ 3 & < & x & < & 4 \end{array}$$

Next, estimate the decimal digit. Because 33 is closer to 27 than to 81, the value of $\log_3 33$ is closer to 3 than to 4.

In this case, 3.2 is a good estimate for $\log_3 33$.

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LESSON 2

Mad Props

Logarithms by definition are exponents, so they have properties that are similar to those of exponents and powers.

The Zero Property of Powers states that any base with an exponent of zero is equal to one. The corresponding logarithmic property called the **Zero Property of Logarithms**.

$$10^0 = 1 \Rightarrow \log 1 = 0$$

There is an exponent rule that says that any number raised to the first power is equal to the base. The rule **Logarithm with Same Base and Argument** states that the logarithm of any value that is the same as its base is equal to 1.

$$b^1 = b \ \log_b(b) = 1$$

The Product Rule of Powers states that if $x = b^m$ and $y = b^n$ then $xy = b^{m+n}$. The corresponding logarithmic property is called **Product Rule of Logarithms**.

$$\log_b(xy) = \log_b x + \log_b y$$

The Quotient Rule of Powers states that $\frac{b^m}{b^n} = b^{m-n}$, if $b \neq 0$. The corresponding logarithmic property is called **Quotient Rule of Logarithms**.

$$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

The Power to a Power Rule states that $(b^m)^n = b^{mn}$. The corresponding logarithmic property is called **Power Rule of Logarithms**.

$$\log_b x^n = n \cdot \log x$$

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The properties of logarithms can be used to rewrite logarithmic expressions.

For example, consider the logarithmic expression.

$$\log_7 \left(\frac{3y^4}{x^3}\right)$$

$$\log_7 (3y^4) - \log_7 (x^3) \quad \text{• Quotient Rule of Logarithms}$$

$$4\log_7 (3y) - 3\log_7 x \quad \text{• Power Rule of Logarithms}$$

LESSON 3

More Than One Way to Crack an Egg

The **Change of Base Formula** allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base.

The Change of Base Formula states:

$$\log_b c = \frac{\log_a c}{\log_a b}, \text{ where } a, b, c > 0 \text{ and } a, b \neq 1$$

Many calculators can evaluate only common logs and natural logs. So, the Change of Base Formula can be helpful to evaluate logs of other bases.

For example, consider the expression $\log_5 30$.

$$\log_5 30 = \frac{\log 30}{\log 5} \approx 2.11$$

One other characteristic of logarithms is that when $\log_b a = \log_b c$ we can say that $a = c$. The converse is also true. Together they can be used to rewrite expressions that have expressions as an exponent.

$$\begin{aligned} 9^{(x+4)} &= 27 \\ (3^2)^{(x+4)} &= 3^3 \\ 3^{2x+8} &= 3^3 \\ 2x + 8 &= 3 \\ x &= -2.5 \end{aligned}$$

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Another way to solve equations that have a variable in the exponent is to apply a logarithm to both sides of the equation.

$$\begin{aligned} 24^x &= 5 \\ \log 24^x &= \log 5 \\ x \cdot \log 24 &= \log 5 \\ x &= \frac{\log 5}{\log 24} \\ x &\approx 0.506 \end{aligned}$$

LESSON 4

Logging On

When solving equations with a certain method, it is important to consider the number of steps involved. The more steps there are, the more chances you have to introduce an error in your calculations. Methods with fewer steps can be more accurate and efficient.

There are multiple methods for solving logarithmic equations.

	Example	First, rewrite as an exponential equation. Then, solve for *x*	First, apply the Change of Base Formula. Then, solve for *x*
Argument Is Unknown	$\log_5 x = 3.1$	$5^{3.1} = x$ $x \approx 146.83$	$\frac{\log x}{\log 5} = 3.1$ $\log x = 3.1 \cdot \log 5$ $\log x \approx 2.167$ $10^{2.167} \approx x$ $x \approx 146.83$
Exponent Is Unknown	$\log_8 145 = x$	$8^x = 145$ $\log 8^x = \log 145$ $x\log 8 = \log 145$ $x = \frac{\log 145}{\log 8}$ $x \approx 2.39$	$\frac{\log 145}{\log 8} = x$ $x \approx 2.39$

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	Example	First, rewrite as an exponential equation. Then, solve for *x*	First, apply the Change of Base Formula. Then, solve for *x*
Base Is Unknown	$\log_x 24 = 6.7$	$x^{6.7} = 24$ $\log x^{6.7} = \log 24$ $6.7 \cdot \log x = \log 24$ $\log x = \frac{\log 24}{6.7}$ $\log x \approx 0.206$ $10^{0.206} \approx x$ $x \approx 1.607$	$\frac{\log 24}{\log x} = 6.7$ $\log x = \frac{\log 24}{6.7}$ $\log x \approx 0.206$ $10^{0.206} \approx x$ $x \approx 1.607$

If a logarithmic equation involves multiple logarithms, you can use the properties of logarithms to rewrite the equation in a form you already know how to solve. Recall the restrictions on the variables for the logarithmic equation, $y = \log_b x$. The variable y can be any real number, the base b must be greater than 0 and not equal to 1, and the argument x must be greater than 0.

$$\begin{aligned} \log 5 + \log x &= 2 \\ \log (5x) &= 2 \\ 10^2 &= 5x \\ 100 &= 5x \\ x &= 20 \end{aligned}$$

LESSON 5

What's the Use?

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Some problem situations can be modeled with either exponential or logarithmic equations. To solve these problems, identify what the question is asking and then use the given model to determine the answer.

For example consider the problem situation.

The pH scale is a scale for measuring the acidity or alkalinity of a substance, which is determined by the concentration of hydrogen ions. The formula for pH is $\text{pH} = -\log \text{H}^+$ where H^+ is the concentration of hydrogen ions in moles per cubic liter.

Vinegar has a pH of 2.2. Determine the concentration of hydrogen ions in vinegar.

$$
\begin{aligned}
2.2 &= -\log H^+ \\
-2.2 &= \log H^+ \\
10^{-2.2} &= H^+ \\
H^+ &\approx 0.006
\end{aligned}
$$

The concentration of hydrogen ions in vinegar is 0.006 moles per cubic liter.

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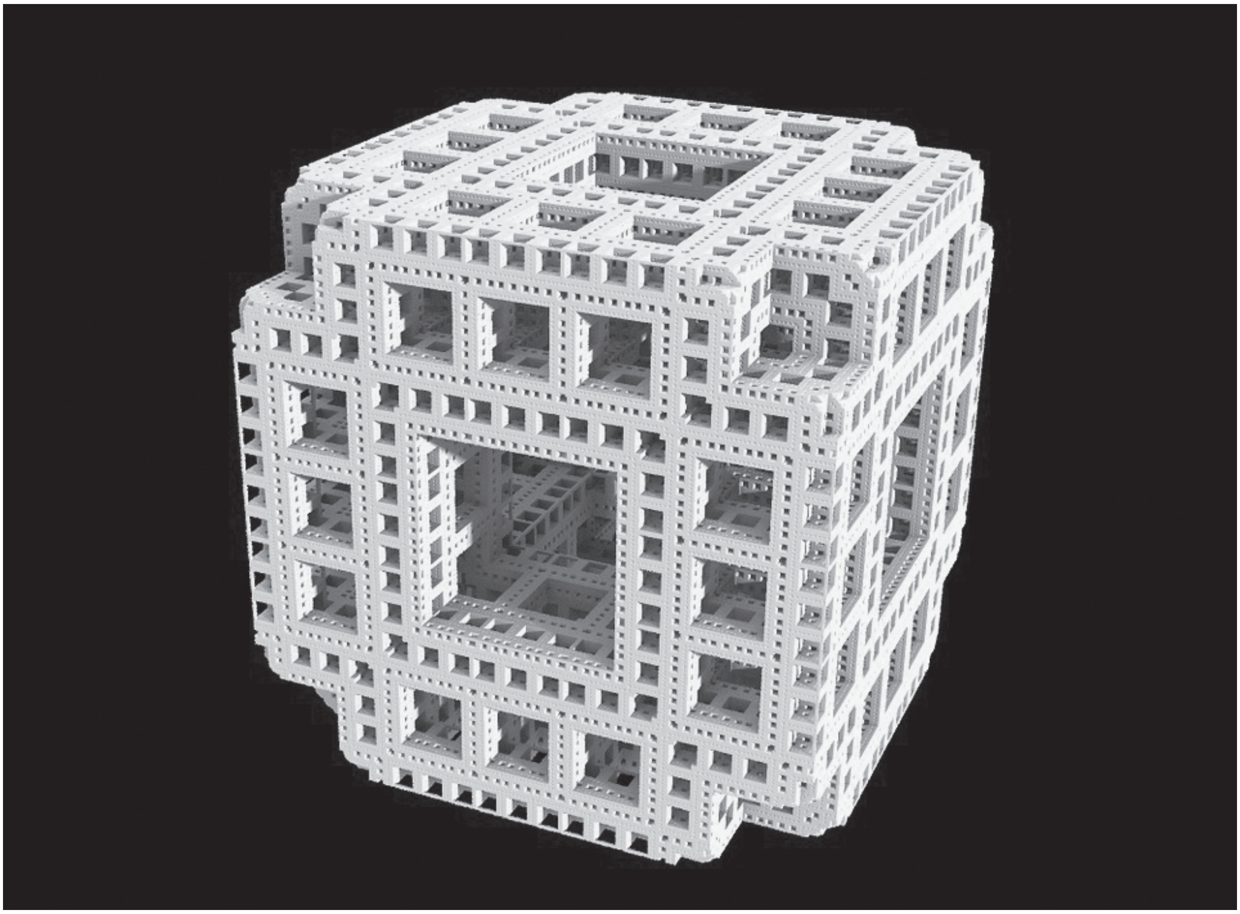
TOPIC 4

Applications of Growth Modeling



A fractal is a complex geometry shape that is constructed by a mathematical pattern.

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Module 3: Inverting Functions

TOPIC 4: APPLICATIONS OF GROWTH MODELING

In this topic, the term *geometric series* is defined. Students explore different methods to compute any geometric series. They use the pattern generated from repeated polynomial long division to write a formula for the sum of any geometric series. A second formula to compute any geometric series is derived, $S_n = \frac{g_1(r^n - 1)}{r - 1}$. Next, students use their prior knowledge of transformation function form to create graphics on the coordinate plane. Finally, students use iteration and repeated reasoning to explore fractals.

Where have we been?

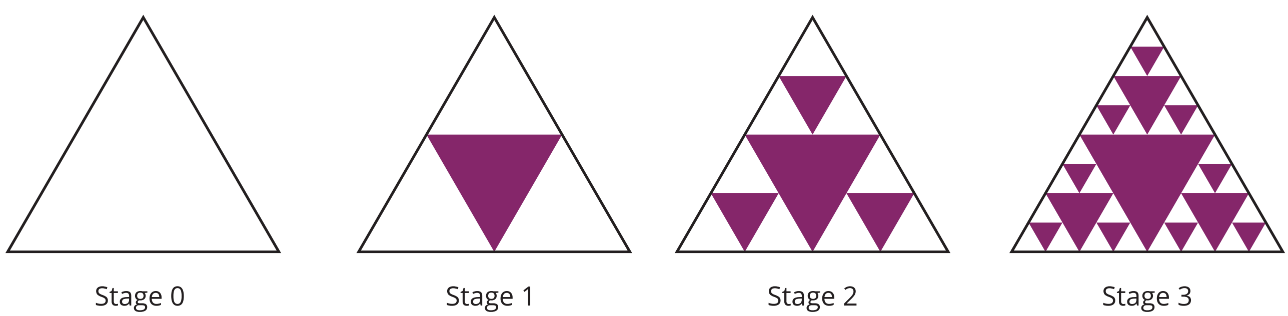
Students have used repeated reasoning and pattern recognition throughout their mathematical careers in both geometry and algebra. They have experience analyzing geometric sequences.

Where are we going?

This topic allows students to flex their creativity in exploring mathematical functions and their transformations. They investigate how applications of iterative operations and repeated reasoning can result in important mathematical products such as fractals, which have been applied to geographic measurement.

The Sierpinski Triangle

The Sierpinski Triangle is a fractal first described by Polish mathematician Wacław Sierpinski in 1915.



To construct the Sierpinski Triangle:

- Begin with an equilateral triangle.
- Connect the midpoints of the sides and remove the center triangle by shading it.
- Repeat Stage 1 on the remaining triangles.

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The Infinite Cat Project

If you like looking at pictures of cats, then the internet is for you. And if you like looking at pictures of cats looking at pictures of cats on the internet, then look into looking at the Infinite Cat Project.

The project showcased about 1800 pictures of cats looking at pictures of cats looking at . . . you get the (very simple) idea.

Recursion like that featured on the Infinite Cat Project is an interesting mathematical topic, and a source of inspiration for self-referential lesson titles.

Talking Points

Geometric series can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

What is the sum of the finite geometric sequence $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27}$?

One formula to compute a geometric series is given by $S_n = \frac{g_1(r^n - 1)}{r - 1}$, where g_1 is the first term in the series, r is the common ratio, and n is the number of terms. Use the formula to determine the sum:

$$S_n = \frac{2\left(\left(\frac{1}{3}\right)^4 - 1\right)}{\frac{1}{3} - 1} = \frac{80}{27}$$

Key Terms

geometric series

A geometric series is the sum of the terms of a geometric sequence.

fractal

A fractal is a complex geometric shape that is constructed by a repeating mathematical pattern. Fractals are infinite and self-similar across different scales.

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1

Series Are Sums

Geometric Series

Warm Up

1. Write the first 6 terms of the geometric sequence determined by $r = -8$ and $g_1 = 1$.

2. Write the explicit formula for the geometric sequence in Question 1.

Learning Goals

- Generalize patterns to derive the formula for the sum of a finite geometric series.
- Compute a finite geometric series.
- Apply an understanding of series to a problem situation.
- Write the formula for a geometric series representing a problem situation.

Key Term

- geometric series

You have investigated and analyzed geometric sequences. How can you reason with the sums of geometric sequences?

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GETTING STARTED

Terese's Trick

Remember:

A geometric sequence is a list of numbers, called terms, in which the ratio of any two consecutive terms is constant.

Terese claims that she has a trick for quickly calculating the sum of the terms in a geometric sequence. She asks members of the class to write any geometric sequence on the board. She boasts that she can quickly tell them how to determine the sum without adding all of the terms. Several examples are shown.

Paul: "OK, so prove it! What is the sum of 1 + 3 + 9 + 27 + 81 + 243 + 729?"	**Terese:** "Multiply 729(3) and subtract 1. Then divide by 2."
Stella: "What is 5 + 20 + 80 + 320 + 1280 + 5120?"	**Terese:** "I will have the answer if I multiply 5120(4), subtract 5, and then divide by 3."
Julian: "Let me see . . . How about 10 + 50 + 250 + 1250?"	**Terese:** "No problem. Multiply 1250(5), subtract 10, and then divide by 4."
Henry: "Hmmm . . . I bet I can stump you with 10 + (−20) + 40 + (−80) + 160."	**Terese:** "Pretty sneaky with the negatives, Henry, but the method still works. Multiply 160(−2) and subtract 10. This time divide by −3."

1. Verify that Terese is correct for each series.

Remember:

The explicit formula for a geometric sequence is $g_n = g_1 r^{n-1}$.

2. What is Terese's trick? Describe in words how to calculate the sum of the terms of any geometric sequence.

3. Use Terese's trick to calculate 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128. Show all work and explain your reasoning.

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ACTIVITY 1.1 Euclid's Method

A **geometric series** is the sum of the terms of a geometric sequence. For example, adding the terms of the sequence 1, 3, 9, 27, 81 creates the geometric series 1 + 3 + 9 + 27 + 81.

Terese's "trick" really isn't a trick. It is known as Euclid's Method. An example of this method, along with a justification for each step, is shown.

Remember:

The E-like symbol is actually the Greek letter sigma, and in mathematical terms it means the "summation" or "sum of".

Worked Example

Compute $\sum_{i=1}^{5} 3^{i-1}$.

$S_5 = 1 + 3 + 9 + 27 + 81$

- Notice that the common ratio is 3.

$$\begin{aligned} 3S_5 &= \phantom{1 + {}} 3 + 9 + 27 + 81 + 243 \\ S_5 &= 1 + 3 + 9 + 27 + 81 \end{aligned}$$

$\times 3$ $\times 3$ $\times 3$ $\times 3$ $\times 3$

- Multiply each term of the original series by the common ratio to write an equation for $3S_n$ Line up each product above the original series.

$$\begin{aligned} 3S_5 &= \phantom{-(1 + {}} 3 + 9 + 27 + 81 + 243 \\ -S_5 &= -(1 + 3 + 9 + 27 + 81) \\ \hline 2S_5 &= -1 + 243 \\ \frac{2S_5}{2} &= \frac{242}{2} \\ S_5 &= 121 \end{aligned}$$

- Subtract to determine $3S_5 - S_5 = 2S_5$.
- Divide by 2 to determine the value of S_5.

In all of the examples in the Getting Started, Terese knew that she could calculate each sum by first multiplying the last term by the common ratio and subtracting the first term. Then she could divide that quantity by 1 less than the common ratio.

In other words, $S_n = \dfrac{\text{(Last Term)(Common Ratio)} - \text{(First Term)}}{\text{(Common Ratio} - 1)}$.

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1. Analyze the worked example.

a. In the worked example, why is it necessary to multiply both sides of the equation by 3? Does the algorithm still work if you multiply by a different number? Explain your reasoning.

b. Why do you always divide by 1 less than the common ratio?

The formula to compute any geometric series becomes $S_n = \frac{g_n(r) - g_1}{r - 1}$, where g_n is the last term, r is the common ratio, and g_1 is the first term.

2. Apply Euclid's method to compute each.

a. $1 + 10 + 100 + \cdots + 1{,}000{,}000$

b. $10 + 20 + 40 + 80 + 160 + 320$

c. $\sum_{k=1}^{8} 5^{k-1}$

d. The sum of a sequence with 9 terms, a common ratio of 2, and a first term of 3.

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ACTIVITY **1.2**

Using Polynomial Long Division to Compute Series

Recall that previously you used long division to determine the quotient of polynomial expressions.

Polynomial Long Division $\longrightarrow$	**Rewritten Using the Reflexive and Commutative Properties of Equality**
Example A $\dfrac{r^3 - 1}{r - 1} = r^2 + r + 1$	$\longrightarrow$ $1 + r + r^2 = \dfrac{r^3 - 1}{r - 1}$
Example B $\dfrac{r^4 - 1}{r - 1} = r^3 + r^2 + r + 1$	$\longrightarrow$ $1 + r + r^2 + r^3 = \dfrac{r^4 - 1}{r - 1}$
Example C $\dfrac{r^5 - 1}{r - 1} = r^4 + r^3 + r^2 + r + 1$	$\longrightarrow$ $1 + r + r^2 + r^3 + r^4 = \dfrac{r^5 - 1}{r - 1}$

Each Example represents a geometric series, where r is the common ratio and $g_1 = 1$. Each geometric series can be written in summation notation.

Example D $\quad n = 3 \quad \displaystyle\sum_{i=1}^{3} r^{i-1}$ or $\displaystyle\sum_{i=0}^{2} r^{i}$

Example E $\quad n = 4 \quad \displaystyle\sum_{i=1}^{4} r^{i-1}$ or $\displaystyle\sum_{i=0}^{3} r^{i}$

1. **For Examples D and E, explain why the power of the common ratio in the summation notation is different, yet still represents the series.**

2. **Identify the number of terms in the series in Example C, and then write the series in summation notation.**

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3. **Use the pattern generated from repeated polynomial long division to write a formula to compute any geometric series $1 + r + r^2 + r^3 + \cdots + r^{n-1}$ where n is the number of terms in the series, r is the common ratio, and $g_1 = 1$.**

$\sum_{i=0}^{n} r^i =$ ________________________

Worked Example

You can show a proof of $S_n = \frac{r^n - 1}{r - 1}$ where S_n is a series in the form $r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$ with n-terms and a common ratio r.

$$S_n = r^0 + r^1 + r^2 + r^3 + \cdots + r^{n-1}$$

$$\begin{aligned} rS_n &= \phantom{r^0 +{}} r^1 + r^2 + r^3 + \cdots + r^{n-1} + r^n \\ S_n &= r^0 + r^1 + r^2 + \cdots + r^{n-2} + r^{n-1} \end{aligned}$$

$\times r$ $\times r$ $\times r$ $\times r$ $\times r$

- Multiply each term by r to write an equation for $r \times S_n$. Line up each product above the original series.

$$\begin{aligned} rS_n &= \phantom{-(1 +{}} \cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}} + r^n \\ -\quad S_n &= -(1 + \cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \cdots + \cancel{r^{n-1}}) \\ \hline rS_n - S_n &= -1 + r^n \\ S_n(r - 1) &= r^n - 1 \end{aligned}$$

- Subtract $rS_n - S_n$. Eliminate terms that subtract to 0.

$$\frac{S_n(r - 1)}{(r - 1)} = \frac{(r^n - 1)}{(r - 1)}$$

$$S_n = \frac{r^n - 1}{r - 1}$$

- Divide by $(r - 1)$, to determine the value of S_n.

4. **Identify the number of terms, the common ratio, and g_1 for each series. Then compute each sum.**

   a. $1 + 2^1 + 2^2 + 2^3 + 2^4$

   b. $1 + 5 + 25 + 125 + 625$

   c. $1 + (-2) + 4 + (-8) + 16 + (-32)$

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5. **Angus and Perry each wrote the geometric series $7 + 14 + 28 + 56 + 112 + 224 + 448 + 896$ in summation notation and then computed the sum. Verify that both methods produce the same sum.**

Angus

I know that $g_n = g_1 r^{n-1}$. The number of terms is 8, the common ratio is 2, and the first term is 7, so I can write the series as $\sum_{i=1}^{8} 7 \cdot 2^{i-1}$.

I know the last term is 896, so I can use Euclid's Method to compute the sum.

$$\frac{896 \cdot 2 - 7}{2 - 1}$$

Perry

I can rewrite the series as $7(1 + 2 + 4 + 8 + 16 + 32 + 64 + 128)$.

I know the common ratio is 2, so I can rewrite the series using powers as $7(2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7)$.

The number of terms is 8, so I can write the series in summation notation as $7 \sum_{i=1}^{8} 2^{i-1}$.

Then, I can compute the series as $7\left(\frac{2^8 - 1}{2 - 1}\right)$.

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The formula to compute a geometric series that Perry used is $S_n = \frac{g_1(r^n - 1)}{r - 1}$.

Recall Euclid's method to compute a geometric series is $S_n = \frac{g_n(r) - g_1}{r - 1}$.

Worked Example

You can use the fact that $g_n = g_1 r^{n-1}$ to verify that these two formulas are equivalent.

$S_n = \frac{g_n(r) - g_1}{r - 1}$	• Given Euclid's method.
$= \frac{g_1 r^{n-1}(r) - g_1}{r - 1}$	• Substitute $g_n = g_1 r^{n-1}$.
$= \frac{g_1 r^n - g_1}{r - 1}$	• Perform multiplication.
$= \frac{g_1(r^n - 1)}{r - 1}$	• Factor out g_1.

6. When is it appropriate to use each formula?

7. Rewrite each series using summation notation.

a. $4 + 12 + 36 + 108 + 324$

b. $64 + 32 + 16 + 8 + 4 + 2 + 1$

8. Compute each geometric series.

a. $\sum_{i=1}^{4} 6^{i-1}$

b. $\sum_{i=0}^{4} \left(\frac{1}{3}\right)^i$

c. $10\sum_{i=0}^{4} 3^i$

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9. **Analyze the table of values.**

x	$f(x)$	$\frac{f(x+1)}{f(x)}$
0	3	
1	4.5	
2	6.75	
3	10.125	
4	15.1875	
5		

a. **Complete the table and describe any patterns you notice.**

b. **Assume the geometric sequence continues. Determine $f(0) + f(1) + \cdots + f(9)$. Show all work and explain your reasoning.**

c. **Explain why the ratio of any two consecutive terms in a geometric sequence is always a constant.**

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ACTIVITY 1.3

Calculating Credit Card Payments

Vince wants to purchase a laptop with high screen resolution for his gaming hobby. He charges the $1000 purchase to a credit card with 19% interest.

The credit card company requires a minimum monthly payment of the greater of these two.

- 2% of the balance on the card
- $15.00

To determine how long it will take to pay off the credit card when paying the minimum balance, Vince calls the company. He learns that when making a monthly payment, 75% of the minimum payment goes toward interest and the remaining portion of the minimum monthly payment goes toward the principal.

1. Determine the percent of the payment that is paid toward interest and principal for each monthly payment.

a. Monthly payment is 2% of balance.

b. Monthly payment is 10% of balance.

c. Monthly payment is 25% of balance.

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Let's consider Vince's monthly payment.

Worked Example

To calculate the minimum monthly payment, you need to calculate 2% of the balance.

$$\begin{aligned}\text{minimum payment} &= (0.02)(\text{balance})\\ &= 0.02(1000)\\ &= 20\end{aligned}$$

If the credit card balance is \$1000.00, the minimum monthly payment is \$20.00.

The amount paid toward interest is 1.5% of the balance.

$$\begin{aligned}\text{amount paid toward interest} &= 0.015(\text{balance})\\ &= (0.015)(1000)\\ &= 15\end{aligned}$$

If the minimum monthly payment is \$20.00, the amount paid toward interest is \$15.00.

The amount paid toward principal is 0.5% of the balance.

$$\begin{aligned}\text{amount paid toward principal} &= (0.005)(\text{balance})\\ &= (0.005)(1000)\\ &= 5\end{aligned}$$

If the minimum monthly payment is \$20.00, the amount paid toward principal is \$5.00.

The remaining balance on the credit card is the current balance minus the amount paid toward principal.

$$\begin{aligned}\text{remaining balance} &= (\text{current balance}) - (\text{amount paid toward principal})\\ &= 1000 - 5\\ &= 995\end{aligned}$$

So, if Vince makes a monthly payment of \$20.00 on the \$1000.00 balance, the balance after the monthly payment is \$995.00.

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2. **Calculate the monthly payment details for the first 12 months of minimum payments. The first row of the table reflects the calculations from the worked example.**

Number of Months (n)	Balance Before Monthly Payment ($)	Minimum Monthly Payment ($)	Amount Paid Toward Principal ($)	Amount Paid Toward Interest ($)	Balance After Monthly Payment ($)
0	1000.00	20.00	5.00	15.00	995.00
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
n					

To write the explicit formulas of each column in the n-row, consider the initial value and the rate of change.

3. **Consider the worked example and the answers to each part to complete the n-row of the table.**

   a. **Write the explicit formula for the geometric sequence represented in the "Balance Before Monthly Payment" column.**

   b. **Write the formula to calculate the minimum monthly payment.**

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c. **Write the formula to calculate the amount paid toward principal.**

d. **Write the formula to calculate the amount paid toward interest.**

e. **Write the formula to calculate the balance after minimum payment.**

Vince knows how much he is paying each month, but it would be helpful if he knew how much he had paid in both interest and principal over a certain amount of time instead of on any given month.

4. **Calculate each.**

a. **The total monthly payment in the first 12 months**

b. **The amount paid toward principal in the first 12 months**

c. **The amount paid toward interest in the first 12 months**

Think about:

You developed two different formulas to compute a geometric series. Which one are you going to use in this situation?

5. **Write a formula for the geometric series that represents each payment.**

a. **The total monthly payment**

b. **The total payment toward principal over time**

c. **The total payment toward interest over time**

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6. **Use the formulas you created in Question 5 to complete the table.**

	Total Amount Paid Toward Principal	Total Amount Paid Toward Interest
2 years		
5 years		

7. **Assume the credit card company does not require a monthly payment of $15.00. Determine how long it will take to pay off the balance of the credit card completely.**

8. **Does your answer to Question 7 seem reasonable? Explain your reasoning.**

9. **Determine the amount of time it will take to pay off the credit card balance, taking into account the minimum payment of $15.00.**

   a. **After how many months will the minimum monthly payment become $15.00?**

   b. **What will be the balance on Vince's credit card when the minimum monthly payment becomes $15.00?**

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c. **How much will Vince pay toward principal with every $15.00 payment?**

d. **How much will Vince pay toward interest with every $15.00 payment?**

e. **Suppose Vince continues to make $15.00 payments until the end of the loan. How many months will Vince have to make the $15.00 minimum payment to pay off the remaining balance?**

f. **How many years will it take Vince to pay off the entire balance?**

10. **How much money will Vince end up spending on his $1000.00 laptop?**

Keep in mind, Vince paid 2% of the balance for awhile, and then he made $15 monthly payments until the balance was paid in full.

11. **How much money will Vince spend in interest to pay for his $1000.00 laptop?**

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NOTES

TALK the TALK

Extra Credit

1. **When considering applying for a credit card, what details should you look for? How would you plan to pay off your bill? Write a paragraph to explain.**

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Assignment

Write

Describe a geometric series.

Remember

The formula to compute any geometric series is $S_n = \frac{g_n(r) - g_1}{r - 1}$, where g_n is the last term, r is the common ratio, and g_1 is the first term.

Practice

1. Two popular Florida tourist attractions have been competing for visitors since they each opened 15 years ago. Fantasy World had 100,000 visitors in their 1st year and their number of visitors increased by 2% each year over the 15-year period. Vacation Land had 90,000 visitors in their 1st year and their number of visitors increased by 4% each year over the same period.
   a. Determine which tourist attraction had the most visitors since the two attractions opened 15 years ago.
   b. Fantasy World has had an admission price of \$20 per person since they opened. If the owners keep their price the same, they expect to maintain a 2% yearly increase in attendance. If they lower the price of admission to \$18 per person, the owners expect their yearly attendance to increase by 3% each year beginning next year. Should the owners of Fantasy World lower the price of admission to \$18 over the next 10 years? Explain your reasoning.
2. Ryan and Morgan have competed in the Boston Marathon for 9 consecutive years. In Ryan's 1st year, he ran the marathon in 3.5 hours and he has steadily decreased his time by 3% each year. In Morgan's 1st year, he ran the marathon in 3.3 hours and he has steadily decreased his time by 2% each year.
   a. Which of the 2 runners had the fastest time in their 9th marathon? Round decimals to the nearest hundredth.
   b. Which of the 2 runners had the fastest total time if they combine each of their 9 marathon times? Round decimals to the nearest hundredth.
3. Tamika and her best friend Diane live in different hemispheres. On the 1st day of September, Tamika's home in Illinois used 30 kilowatt-hours (kWh) of electricity and Diane's home in Sydney, Australia, used 50 kilowatt-hours of electricity. For the remaining 29 days of September, Tamika's daily electricity usage increased by 2% while Diane's daily electricity usage decreased by 2.5%.
   a. Determine the total amount of electricity used in each home during the 30 days of September. Round decimals to the nearest hundredth.
   b. In September, Tamika's electricity rates went up from \$0.10 per kilowatt-hour for the 1st 15 days of the month to \$0.11 per kilowatt-hour for the last 15 days of the month. Determine how much Tamika paid for electricity in September.

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Stretch

If the sum of an infinite geometric sequence approaches a number, the series is called convergent. If not, the series is called divergent. Can you determine the sum of an infinite geometric sequence?

What is the sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$? Explain why.

Review

1. Solve for the unknown in each logarithmic equation.
   a. $\log \frac{1}{100} = n$
   b. $\log_n \frac{16}{25} = -2$
   c. $\log_{\frac{1}{6}} 216 = n$
   d. $\log_{16} 8 = n$
2. Estimate $\log_8 490$ to the nearest tenths place. Explain your reasoning.
3. Determine whether the function $f(x) = 2x^3 - 5x^2 + 3x - 7$ is odd, even, or neither. Explain your reasoning.

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2

Paint by Numbers

Art and Transformations

Warm Up

Sketch a graph of each function.

1. $f(x) = |x|$

2. $f(x) = -|x|$

3. $f(x) = |x - 4|$

4. $f(x) = -|x - 4|$

Learning Goals

- Use transformations of functions and other relations to create artwork.
- Write equations for transformed functions and other relations given an image.

You have studied a wide variety of functions and their graphs. Can you manipulate the graphs of functions using transformations to create images out of graphs?

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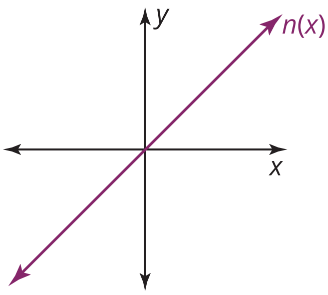
It's Van Gogh Time!

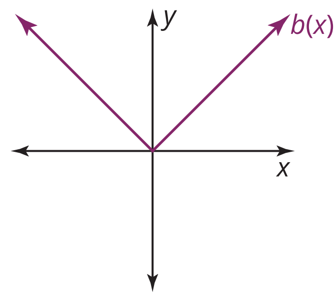
You have studied many types of relations throughout your high school mathematics career. Now it's time to use all of them to create some art.

1. Write the seven basic functions and the basic equation for a circle shown.

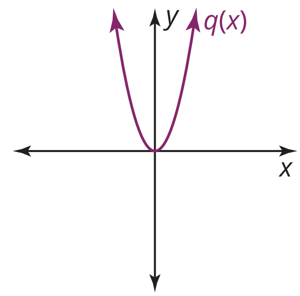
a.



b.



c.



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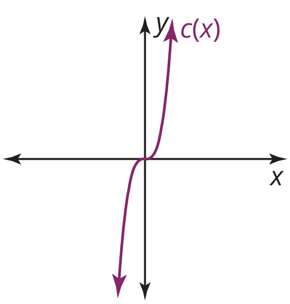
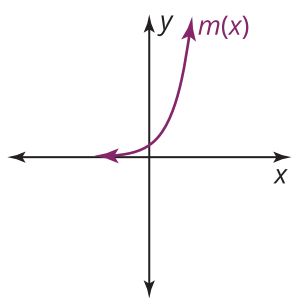
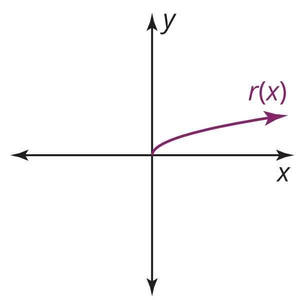


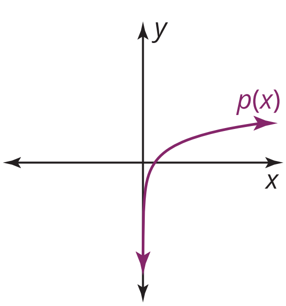
e.



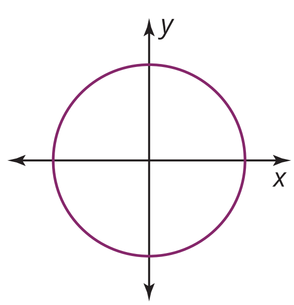
f.



g.



h.



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ACTIVITY 2.1

Using Relations and Functions to Create Art

Recall the effect of each transformation on the graph of $y = f(x)$.

Function Form	Equation Information	Description of Transformation of Graph
$y = f(x) + D$	$D > 0$	Vertical shift up D units
	$D < 0$	Vertical shift down D units
$y = f(x - C)$	$C > 0$	Horizontal shift right C units
	$C < 0$	Horizontal shift left C units
$y = Af(x)$	$\|A\| > 1$	Vertical stretch by a factor of A units
	$0 < \|A\| < 1$	Vertical compression by a factor of A units
	$A < 0$	Reflection across the x-axis
$y = f(Bx)$	$\|B\| > 1$	Horizontal compression by a factor of $\frac{1}{\|B\|}$
	$0 < \|B\| < 1$	Horizontal stretch by a factor of $\frac{1}{\|B\|}$
	$B < 0$	Reflection across the y-axis

1. Use the functions from the Getting Started and your knowledge of the transformation function form to graph the given equations. What image do you see?

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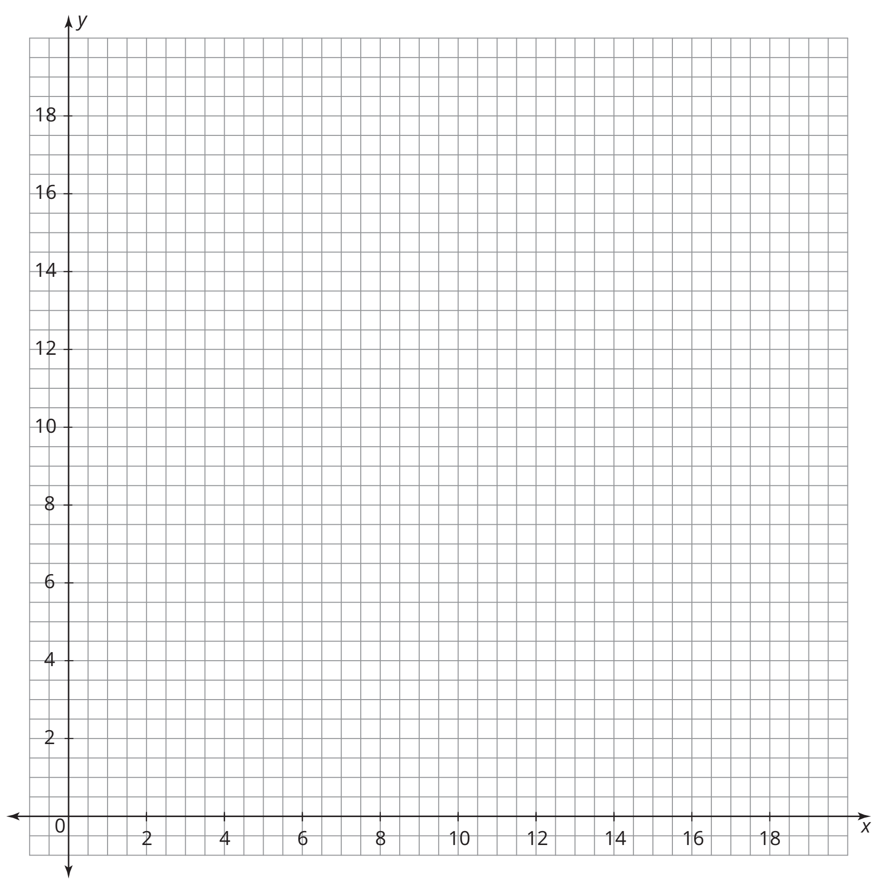
Refer to the Getting Started for the names of basic functions.

Equations	Restrictions
$y = -\frac{1}{10}q(x - 8) + 14$	$2 \le x \le 14$
$y = -3b(x - 4) + 16$	$2 \le x \le 5$
$y = -3b(x - 12) + 16$	$11 \le x \le 14$
$y = m(-x + 5) + 2$	$2 \le x \le 8$
$y = m(x - 11) + 2$	$8 \le x \le 14$
$(x - 6)^2 + (y - 10)^2 = 1$	
$(x - 10)^2 + (y - 10)^2 = 1$	
$y = q(x - 8) + 6$	$7 \le x \le 9$
$y = 7$	$7 \le x \le 9$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$-1 \le x \le 7$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$9 \le x \le 17$
$y = 6.5$	$-1 \le x \le 7$
$y = 6.5$	$9 \le x \le 17$
$y = \frac{1}{4}n(x - 8) + 6.5$	$-1 \le x \le 7$
$y = \frac{1}{4}n(x - 8) + 6.5$	$9 \le x \le 17$



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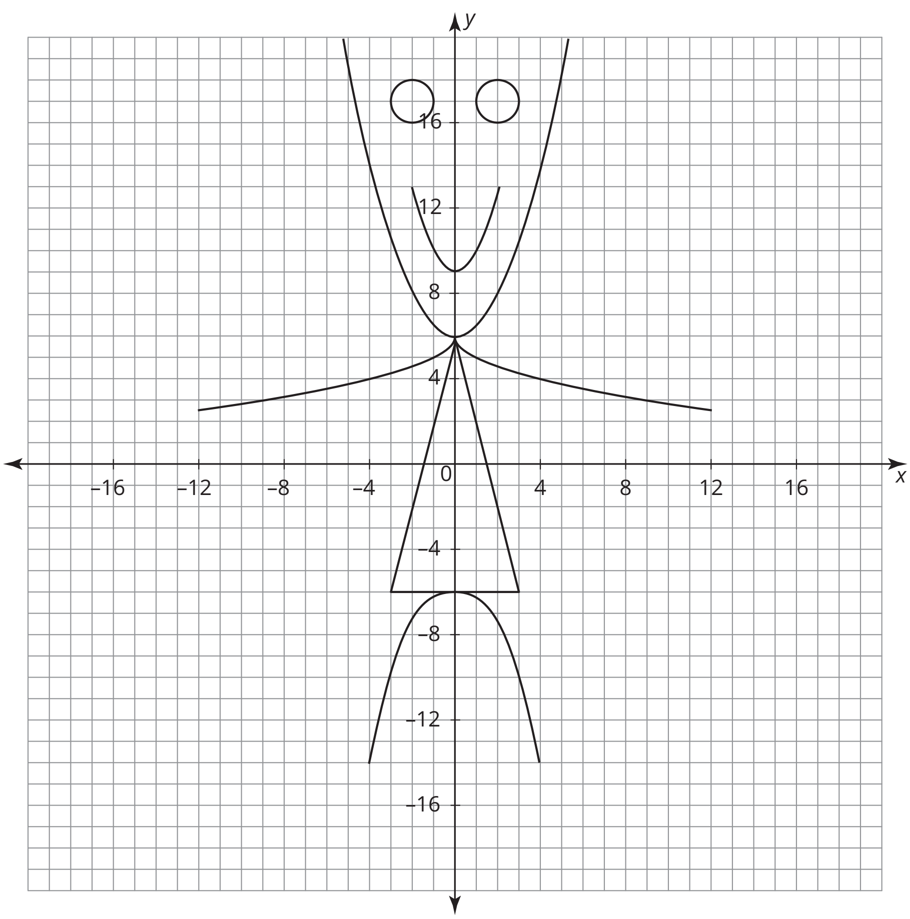
ACTIVITY
2.2 Writing Equations Given Images

1. Use the seven basic functions, the equation of a circle, and your knowledge of the transformational function form to determine the equation of each graph in the picture. Include corresponding domain restrictions where necessary. When possible, write each equation in terms of one of the basic functions or equations given in the Getting Started.



Equation(s) of a circle:

Exponential Equation(s):

Quadratic Equation(s):

Absolute Value Equation(s):

Radical Equation(s):

Linear Equation(s):

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NOTES

TALK the TALK

Design Challenge

1. Create a picture or design using at least five transformed functions or equations. The transformations should include at least one stretch or compression, one translation, and one reflection. You must also include one of each of the following:

   - linear equation
   - absolute value equation
   - quadratic equation
   - exponential equation
   - radical equation

   When possible, write each equation in terms of one of the basic functions or equations given in the Getting Started, and then identify each type. Include corresponding domain restrictions where necessary.

   Equation: Type:

2. Ask a classmate to graph your equations. Does their graph match yours?

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Assignment

Write

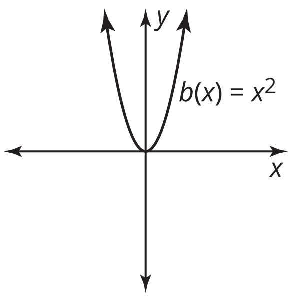
Given $g = f(x)$, describe the effects of the A-, B-, C-, and D-values on the function $g(x) = A \cdot f(B(x - c)) + D$

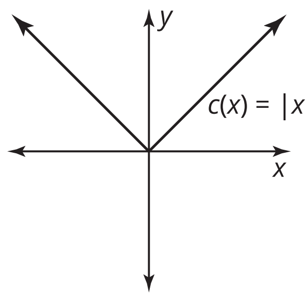
Remember

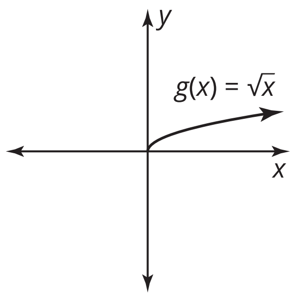
The graph of a function with a restricted domain is one in which only the y-values mapped to included domain x-values are shown.

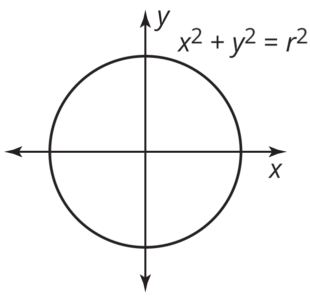
Practice

1. Consider the relations shown.



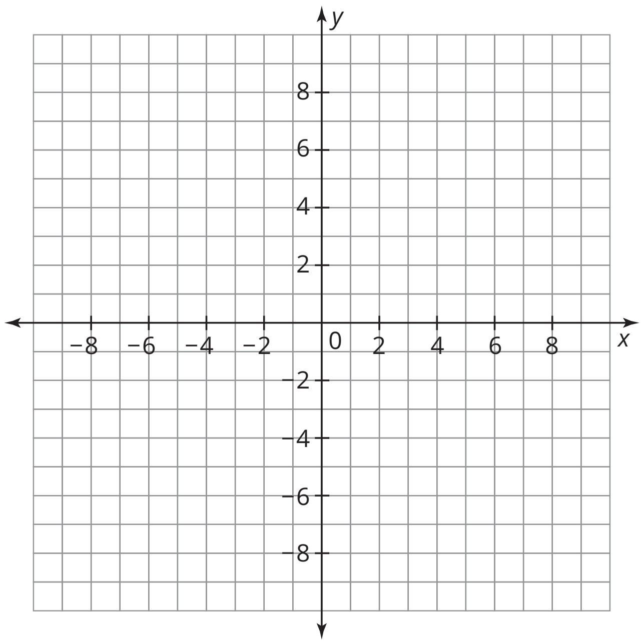






a. Graph each relation to create a picture.

$y = g(x + 1) + 7, \quad -1 \le x \le 1$	$y = g(x) + 7, \quad 0 \le x \le 2$	$y = g(x - 1) + 7, \quad 1 \le x \le 3$
$y = \frac{1}{4}b(x) - 2, \quad -4 \le x \le 4$	$y = \frac{1}{8}b(x), \quad -4 \le x \le 4$	$y = c(x) + 1, \quad -1 \le x \le 1$
$(x + 2)^2 + (y - 4)^2 = 1$	$(x - 2)^2 + (y - 4)^2 = 1$	$x^2 + (y - 2)^2 = 36$



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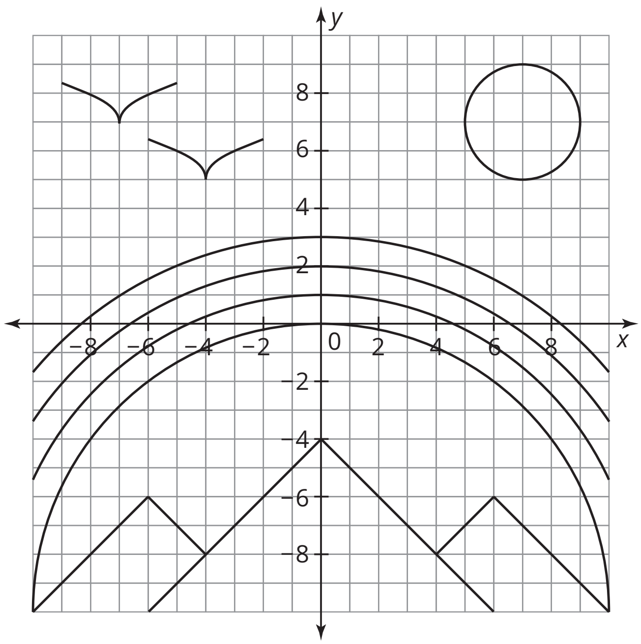
b. Write the equations of the 12 relations used to create this picture. Include any restrictions on the domains.



Stretch

Create a design using a transformation of each equation from the Getting Started. Write each equation and include its corresponding domain.

Review

1. Alyssa and Jade both swim on their high school swim team. At the first swim meet, Alyssa swam the 50 yard freestyle in 31.74 seconds, and she has steadily decreased her time by 1.2% each race. In Jade's 1st meet, she swam the 50 yard freestyle in 30.22 seconds and she has steadily decreased her time in the event by 1.1% each race.
   a. Which of the 2 swimmers had the fastest time in their 10th race? Round decimals to the nearest hundredth.
   b. Which of the 2 swimmers had the fastest total time if they combine each of their 10 freestyle times? Round decimals to the nearest hundredth.

2. Use the properties of logarithms to rewrite each logarithmic expression in expanded form.
   a. $\ln (j^2kl^{10})$
   b. $\log\left(\frac{4x^3}{3y^4}\right)$

3. Use the properties of logarithms to rewrite each logarithmic expression as a single logarithm.
   a. $6 \ln x + \ln 6 - 3 \ln y$
   b. $5(2 \log x + \log 2) - 12 \log x$

4. State the domain for the rational function $f(x) = \frac{x+3}{x^2-9}$. Explain your reasoning.

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3

This Is the Title of This Lesson

Fractals

Warm Up

Write an explicit formula for each geometric sequence.

1. $\{2, 8, 32, 128 \ldots\}$

2. $\left\{\frac{5}{2}, \frac{5}{3}, \frac{10}{9}, \frac{20}{27}, \ldots\right\}$

3. $\{3, 3.6, 4.32, 5.184, \ldots\}$

4. $\left\{x, \frac{1}{2}x, \frac{1}{4}x, \frac{1}{8}x, \ldots\right\}$

Learning Goals

- Build expressions and equations to model the characteristics of self-similar objects.
- Write sequences to model situations and use them to identify patterns.
- Analyze the counterintuitive aspects of fractals.

Key Terms

- fractal
- self-similar
- iterative process

You have analyzed sequences throughout your studies of mathematics and modeled them algebraically. How can you use algebra to model the characteristics of geometric shapes constructed by mathematical patterns?

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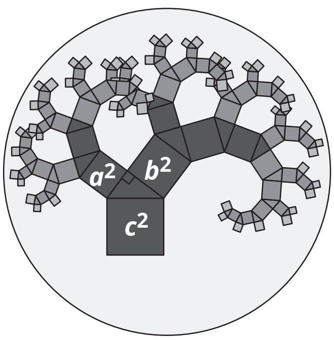
Let It Logo

A **self-similar** object is exactly or approximately similar to a part of itself.

A **fractal** is a complex geometric shape that is constructed by a mathematical pattern. Fractals are infinite and *self-similar* across different scales. Many objects in the real world, such as coastlines, are self-similar, in that parts of them look roughly the same on any scale. Fractals often appear in nature. Examples of phenomena known to have fractal features include river networks, lightning bolts, ferns, snowflakes, and crystals.



1. The high school math club decided to use the image shown as a logo.

a. Describe how you might go about constructing this image.

b. Is the image an example of a fractal? Explain your reasoning.

Fractals are formed by an **iterative process**. The output from one iteration is used as the input for the next iteration. In many situations, this is also known as recursion.

2. Consider the isosceles right triangle shown on the following page.

a. Describe an iterative process that can be performed on the image to create a fractal.

b. Use the image to draw the first four iterations of your process.

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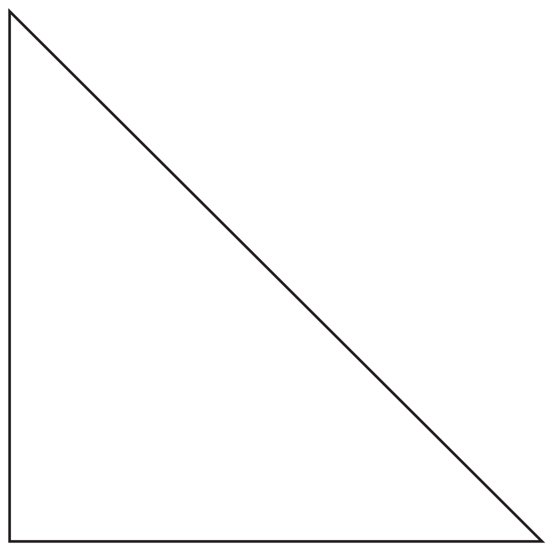

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ACTIVITY
3.1

The Sierpinski Triangle

The Sierpinski Triangle is a fractal first described by Polish mathematician Wacław Sierpiński in 1915.

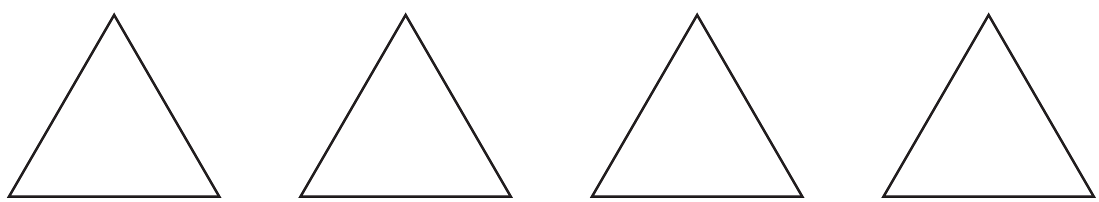
To construct the Sierpinski Triangle:

Stage 0: Begin with an equilateral triangle.

Stage 1: Connect the midpoints of the sides and remove the center triangle by shading it.

Stage 2: Repeat Stage 1 on the remaining triangles.

1. Complete each equilateral triangle to represent each stage of the Sierpinski Triangle.



Stage 0 Stage 1 Stage 2 Stage 3

2. Describe the iterative process to create the Sierpinski Triangle.

3. Determine the number of unshaded triangles at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Number of Unshaded Triangles	1						

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4. Identify the type of sequence represented by the number of unshaded triangles.

5. As the iterative process continues, what happens to the number of unshaded triangles in Sierpinski's Triangle?

6. Let the unshaded area at Stage 0 equal *x* square units. Determine the total area of unshaded triangles at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Area of Unshaded Triangles (square units)	x						

7. Identify the type of sequence represented by the area of unshaded triangles.

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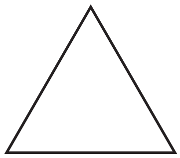
8. As the iterative process continues, what happens to the area of unshaded triangles in Sierpinski's Triangle?

9. Complete the table to determine the perimeter of the Sierpinski Triangle through each iteration. Assume the perimeter is equal to the total perimeters of the unshaded triangles. Let *s* equal the side length of the equilateral triangle in Stage 0.

Stage	Image	Number of Sides Added	Length of Added Side	Total Length Added	Total Perimeter
0		0	N/A	N/A	$3s$
1		3	$\frac{s}{2}$	$\frac{3s}{2}$	$3s + \frac{3s}{2}$
2					
n	N/A				

10. Write a formula using sigma notation for the perimeter of the Sierpinski Triangle at stage *n*.

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ACTIVITY
3.2

The Menger Sponge

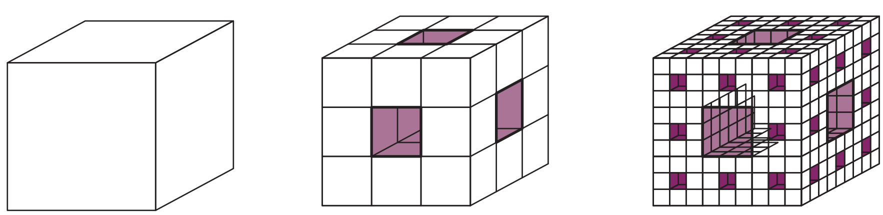
The Menger Sponge is a fractal first described by Austrian mathematician Karl Menger in 1926.

To construct the Menger Sponge:

Stage 0: Begin with a cube.

Stage 1: Divide every face of the cube into 9 squares. This subdivides the cube into 27 smaller cubes. Remove the smaller cube in the middle of each face, and remove the smaller cube in the very center of the larger cube, leaving 20 smaller cubes.

Stage 2: Repeat Stage 1 for the remaining squares on each face.



Stage 0 Stage 1 Stage 2

1. Describe the iterative process to create the Menger Sponge.

2. Determine the number of filled cubes at each stage and complete the table.

Stage (*n*)	0	1	2	3	4	5	*n*
Number of Filled Cubes							

Filled cubes represent solid cubes and not an empty space where a cube was removed.

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3. Identify the type of sequence represented by the number of filled cubes.

4. As the iterative process continues, what happens to the number of filled cubes in the Menger Sponge?

5. Let the volume at Stage 0 equal 1 cubic unit. Determine the total volume of filled cubes at each stage and complete the table.

Stage (n)	0	1	2	3	4	5	n
Volume (cubic units)							

6. Identify the type of sequence represented by the volume of filled cubes.

7. As the iterative process continues, what happens to the volume of filled cubes in the Menger Sponge? Does this situation seem possible? Explain your reasoning.

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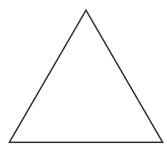
ACTIVITY
3.3

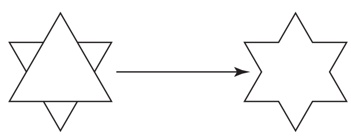
The Koch Snowflake

The Koch Snowflake is a fractal that is created by using equilateral triangles. It is based on the Koch curve, first mentioned in a 1904 paper by the Swedish mathematician Helge von Koch.

In Stage 0, you begin with an equilateral triangle, such as the one shown. This is the first step in the creation of the Koch Snowflake.



In Stage 1, each side of the triangle is divided into thirds. Then, each middle segment becomes the base of a new equilateral triangle, as shown. Finally, the middle segment is removed.



1. How does a side length of a new triangle compare to a side length of the Stage 0 triangle?

2. If a side length of the Stage 0 figure is 1 unit, what is a side length of the Stage 1 figure?

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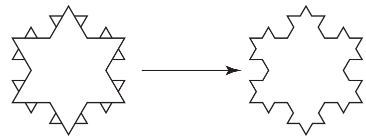
In Stage 2, each side of the figure from Stage 1 is divided into thirds, and the middle segments become the bases of new equilateral triangles. Then, the middle segments are removed.



This process is repeated on the remaining sides.

3. Describe the iterative process to create the Koch Snowflake.

4. Let the side length at Stage 0 equal *s* units. Complete the table.

Stage (n)	Length of a Side	Number of Sides	Total Perimeter
0	s	3	$3s$
1			
2			
3			
4			
5			
n			

5. Identify the type of sequence represented by each characteristic.

a. The length of a side

b. The number of sides

c. The total perimeter

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6. Describe what happens to each characteristic as the iterative process continues.

   a. The length of a side

   b. The number of sides

   c. The total perimeter

7. Does this situation seem possible? Explain your reasoning.

8. Consider the equilateral triangle in Stage 0. Each side length is 1 unit.

   a. Calculate the altitude. Leave your answer in radical form.

   b. Calculate the area of the equilateral triangle. Show all of your work and leave your answer in radical form.

   c. What is the total area of the Stage 0 figure rounded to the nearest hundredth?

Think about:

What two special right triangles will the altitude divide the equilateral triangle into?

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9. Consider one of the smaller triangles that is added to the Stage 0 figure.

a. Calculate the altitude.

To prevent rounding errors, leave your answers in radical form until the last step.

b. Calculate the area of this triangle.

c. Calculate the total area of the Stage 1 figure. Round your answer to the nearest hundredth.

10. Consider one of the smaller triangles that was added to the Stage 1 figure.

a. Calculate the altitude.

b. Calculate the area of this triangle.

c. Calculate the total area of the Stage 2 figure. Round your answer to the nearest hundredth.

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11. Complete the table shown for Stage 0 through Stage 2.

Stage	Number of New Triangles	Area of One New Triangle	Total Area in Radical Form	Total Area (nearest hundredth)
0				
1				
2				

12. Use your table to answer each question.

a. Predict the number of new triangles in the Stage 3 figure, the area of one new triangle, and the total area of the figure. Explain your reasoning. Record your results in the table.

b. What happens to the number of new triangles as the stage number increases?

c. What happens to the area of one new triangle as the stage number increases?

d. What happens to the total area as the stage number increases? Explain your reasoning.

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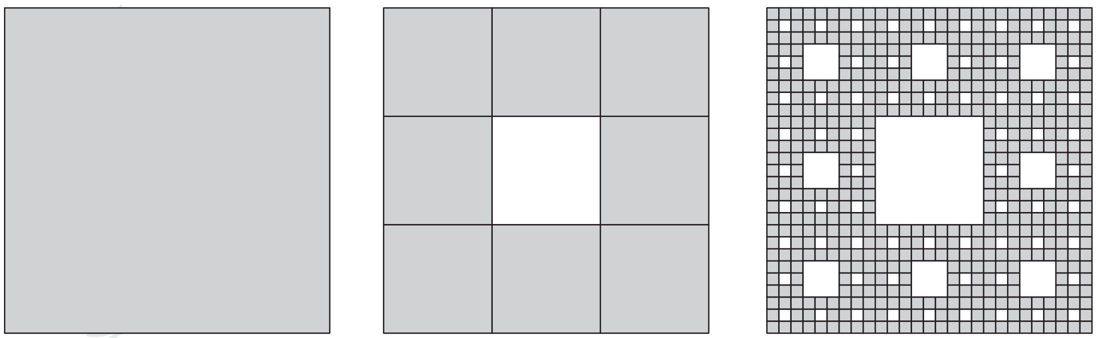
TALK the TALK

A Dog in a Station Wagon

A fractal called the Sierpinski Carpet is shown at Stage 0, Stage 1, and after 3 iterations.

The Sierpinski Carpet:



Stage 0 | Stage 1 | After 3 iterations

1. **Describe the iterative process used to create the Sierpinski Carpet.**

2. **Let the shaded area at Stage 0 equal x square units. Determine the total area of shaded squares for Stage n.**

3. **Describe any similarities or differences between the Sierpinski Carpet and the Sierpinski Triangle.**

4. **Describe how the Sierpinski Carpet is related to the Menger Sponge.**

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Assignment

Write

Write a definition for each term in your own words.

1. fractal
2. self-similar
3. iterative process

Remember

You can analyze the characteristics of a fractal in its stages to complete tables of values, identify infinite geometric sequences and patterns, describe end behaviors, write formulas, and make predictions.

Practice

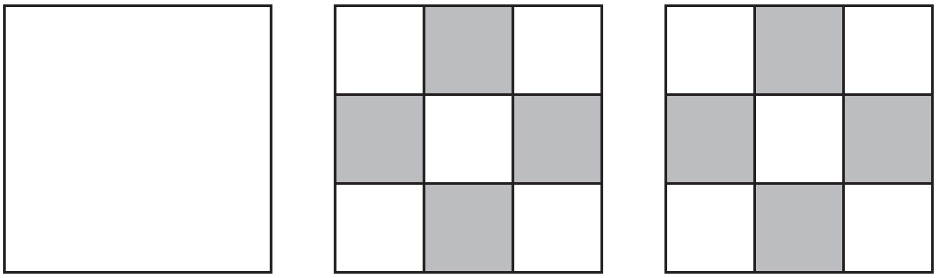
1. The following rules are used to create a certain fractal.

   Stage 0: Begin with a square.

   Stage 1: Divide the square into 9 congruent squares. Shade the square directly above the center square, the square directly below the center square, the square directly to the right of the center square, and the square directly to the left of the center square.

   Stages 2 and up: Repeat Stage 1 for the unshaded squares in the figure.

   a. Complete Stage 2 of the fractal. Stage 0 and Stage 1 are given.



Stage 0 Stage 1 Stage 2

   b. Determine the number of unshaded squares at each stage and complete the table.

Stage	0	1	2	3	4	5	n
Number of Unshaded Squares							

   c. Identify the type of sequence represented by the number of unshaded squares at Stage n.

   d. Write a function to represent the number of unshaded squares as a function of the stage, n. Describe the type of function you used.

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e. Determine the total unshaded area at each stage and complete the table. The area of the initial square in Stage 0 is 1 cm^2.

Stage	0	1	2	3	4	5	n
Total Unshaded Area (cm^2)							

f. Identify the type of sequence represented by the total unshaded area at Stage n.

g. Write a function to represent the total unshaded area as a function of the stage, n. Describe the type of function you used.

h. Describe the amount of unshaded area as n increases. Explain your reasoning.

i. Determine the additional amount of shaded area at each stage and complete the table. Explain your reasoning.

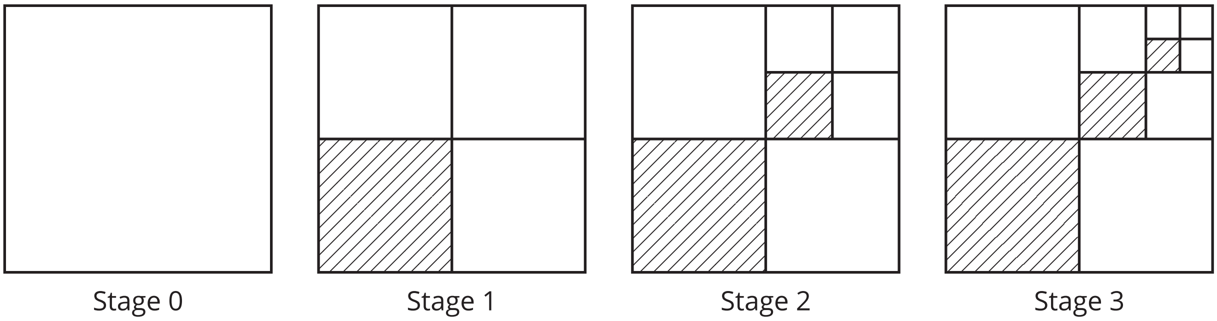
Stage	1	2	3	4	5	n
Additional Shaded Area (cm^2)						

j. Write a formula in sigma notation to represent the amount of shaded area at Stage n.

k. Describe the amount of shaded area as n increases.

Stretch

Consider the pattern given. The initial square in Stage 0 is 1cm^2



a. Describe how the next stage of the pattern is created.

b. Identify the type of sequence represented by the area of each new shaded square at Stage n.

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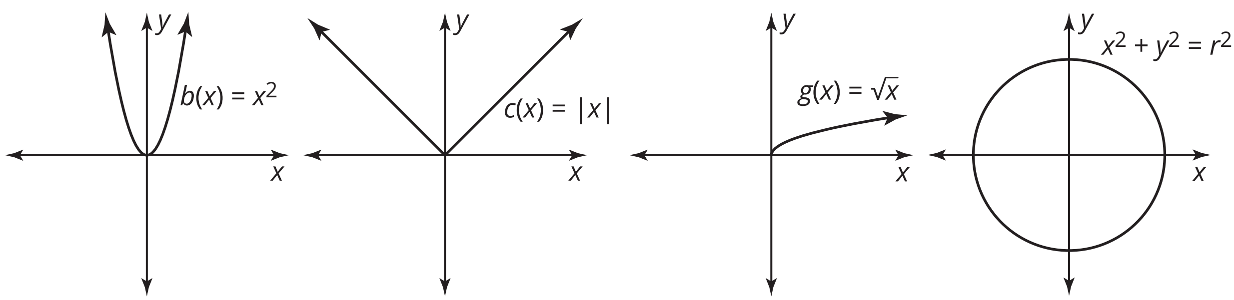
c. Write a function to represent the the area of each new shaded square as a function of the stage, n. Describe the type of function you used.

d. Use the formula for the sum of the first n terms of a geometric series, $S = a\left(\frac{1 - r^n}{1 - r}\right)$, where a is the first term of the sequence and r is the common ratio, to calculate the sum of the first 5 terms of the series.

Review

1. Consider the relations shown.



Graph each relation to create a picture.

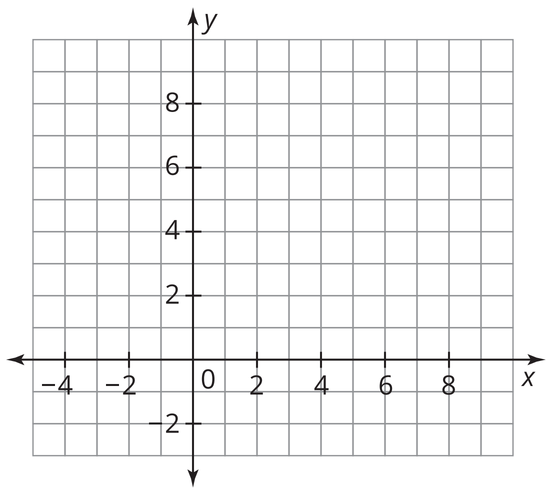
$y = -c(x - 2) + 2, \quad 0 \le x \le 4$

$y = -c(x - 6) + 2, \quad 4 \le x \le 8$

$(x - 4)^2 + (y - 3)^2 = 1$

$y = -0.25b(x - 4) + 6, \quad -1 \le x \le 9$

$y = -0.25b(x - 4) + 8, \quad -1 \le x \le 9$



2. A startup company is selling a new phone application for fitness. The sales, S, can be modeled by the function $S(m) = 50 \cdot 1.35^m$, where m represents the number of months since the app was put on the market.
   a. How many months will it take for the sales of the app to grow to \$2,500?
   b. How many years will it take for the sales of the app to grow to \$100,000?
3. Solve each exponential equation. Round your answer to the nearest hundredth.
   a. $2 + 6^{x+3} = 18$
   b. $\frac{3^{3x-1}}{4} = 9$
4. Subtract $\frac{2x}{x + 7} - \frac{x + 3}{x}$.

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Applications of Growth Modeling Summary

KEY TERMS

- geometric series
- fractal
- self-similar
- iterative process

LESSON 1

Series Are Sums

A **geometric series** is the sum of the terms of a geometric sequence.

Euclid's Method can be used to compute a finite geometric series provided the first term, g_1, the last term, g_n, and the common ratio, r, are known. In which case, the formula $S_n = \frac{g_n(r) - g_1}{r - 1}$ can be used.

For example, consider the geometric series $-5 + (-10) + (-20) + (-40)$.

$$g_1 = -5, n = 4, g_4 = -40, r = 2$$

$$S_n = \frac{g_n(r) - g_1}{r - 1}$$

$$S_4 = \frac{-40(2) - 5}{2 - 1}$$

$$= -75$$

The formula, $S_n = \frac{g_1(r^n - 1)}{r - 1}$, can be used to compute a finite geometric series provided the first term, g_1, the common ratio, r, and the number of terms, n, are known

For example, consider the geometric series $2 + 6 + 18 + 54 + 162$.

$$g_1 = 2, r = 3, n = 5$$

$$S_n = \frac{g_1(r^n - 1)}{r - 1}$$

$$S_5 = \frac{2(3^5 - 1)}{3 - 1}$$

$$= 242$$

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When solving real-world problems involving geometric series, identify what the problem is asking. Then determine what information is provided so that the correct approach to the solution can be used. Choose the appropriate formula(s) needed, make the correct substitution(s), and solve the problem. Check to see that the question asked has been answered.

For example, consider the following problem situation.

Vince wants to purchase a laptop with high screen resolution for his gaming hobby. He charges the $1000 purchase to a credit card with 19% interest. The credit card company requires a minimum monthly payment of the greater of:

- 2% of the balance on the card, or
- $15.00

To determine the amount of time it will take to pay off the credit card when paying the minimum balance, Vince calls the company. He learns that when making a monthly payment, 75% of the minimum payment goes toward interest and the remaining portion of the minimum monthly payment goes toward the principal.

To write the explicit formulas, consider the initial value and the rate of change. The explicit formulas that represent each amount paid are shown.

The minimum monthly payment: $20(0.995)^{n-1}$

The amount paid toward interest: $15(0.995)^{n-1}$

The amount paid toward principal: $5(0.995)^{n-1}$

Vince wants to know how much he has paid in both interest and principal over a certain amount of time instead of on any given month.

The formulas for the geometric series that represents each amount paid are shown.

The total monthly payment: $S_n = \frac{20(0.995^{12-1})}{0.995 - 1}$

The total payment toward principal over time: $S_n = \frac{5(0.995^{12-1})}{0.995 - 1}$

The total payment toward interest over time. $S_n = \frac{15(0.995^{12-1})}{0.995 - 1}$

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LESSON 2

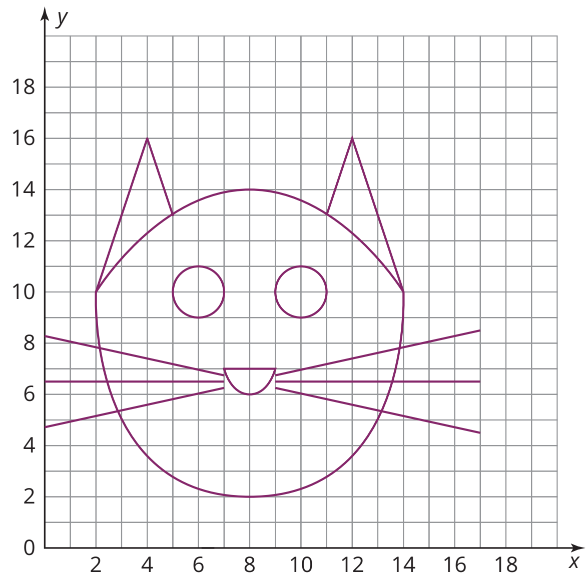
Paint by Numbers

The function $y = f(x)$ is transformed when the function or its argument is multiplied, divided, increased, or decreased by a constant.

You can use the seven basic functions, $n(x) = x$, $b(x) = |x|$, $q(x) = x^2$, $c(x) = x^3$, $m(x) = b^x$, and $r(x) = \log(x)$, as well as the basic equation for a circle and your knowledge of transformation function form to graph given equations and create an image.

For example, the graphed equations shown create the image.

Equations	Restrictions
$y = -\frac{1}{10}q(x - 8) + 14$	$2 \leq x \leq 14$
$y = -3b(x - 4) + 16$	$2 \leq x \leq 5$
$y = -3b(x - 12) + 16$	$11 \leq x \leq 14$
$y = m(-x + 5) + 2$	$2 \leq x \leq 8$
$y = m(x - 11) + 2$ $(x - 6)^2 + (y - 10)^2 = 1$ $(x - 10)^2 + (y - 10)^2 = 1$	$8 \leq x \leq 14$
$y = q(x - 8) + 6$	$7 \leq x \leq 9$
$y = 7$	$7 \leq x \leq 9$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$-1 \leq x \leq 7$
$y = -\frac{1}{4}n(x - 8) + 6.5$	$9 \leq x \leq 17$
$y = 6.5$	$-1 \leq x \leq 7$
$y = 6.5$	$9 \leq x \leq 17$
$y = \frac{1}{4}n(x - 8) + 6.5$	$-1 \leq x \leq 7$
$y = \frac{1}{4}n(x - 8) + 6.5$	$9 \leq x \leq 17$



You can also take an image and determine the equations used to create the image using your knowledge of transformations on the basic functions.

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LESSON 3

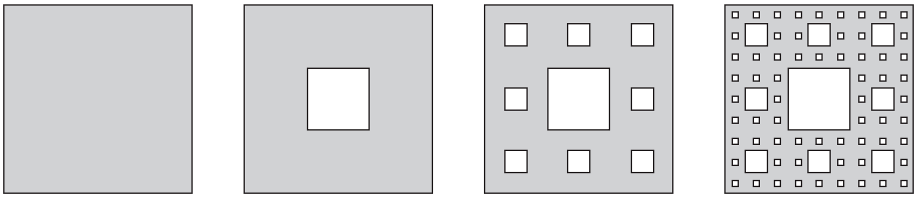
This Is the Title of This Lesson

A **fractal** is a complex geometric shape that is constructed by a mathematical pattern. Fractals are infinite and self-similar across different scales. A **self-similar** object is exactly or approximately similar to a part of itself. Fractals are formed by an **iterative process**. The output from one iteration is used as the input for the next iteration. In many situations, this is also known as recursion.

Relationships between the iterations and the area and perimeter of the fractal can often be represented as an infinite geometric sequence.

For example, consider the Sierpinski Carpet, which is formed in a manner that is similar to the Sierpinski Triangle.

Begin with a square, and then divide it into 9 congruent squares. Remove the center square. Repeat this process. The initial stage and three additional stages of the Sierpinski Carpet are shown. Write a geometric sequence that represents the relationship between the stage and the number of shaded squares in the figure. Then, use the sequence to determine the number of shaded squares in the Stage 6 figure.



Stage (n)	0	1	2	3	n
Shaded squares	2^0 or 1	2^3 or 8	2^6 or 64	2^9 or 512	2^{3n}

The number of shaded squares in each figure represents an infinite geometric sequence. To determine the number of shaded squares in the Stage 6 figure, use the expression 2^{3n}.

$$\begin{aligned} 2^{3n} &= 2^{(3)(6)} \\ &= 2^{18} \\ &= 262,144 \end{aligned}$$

There are 262,144 shaded squares in the Stage 6 figure of the Sierpinski Carpet.

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MODULE 4

INVESTIGATING PERIODIC FUNCTIONS

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The lessons in this module expand your experiences with trigonometric ratios. You have already derived the sine, cosine, and tangent ratios to relate the side lengths of right triangles. In this module, you will define each of these in terms of a function on the coordinate plane. You will integrate your knowledge of the unit circle, radian measures, and the graphical behaviors of trigonometric functions to solve sine, cosine, and tangent equations. Finally, you will use trigonometric functions to model real-world phenomena that exhibit periodic behavior.

TOPIC 1

Trigonometric Relationships

Heartbeats can be periodic functions, though technically they don't continue forever.

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Module 4: Investigating Periodic Functions

TOPIC 1: TRIGONOMETRIC RELATIONSHIPS

Students begin this topic by examining a reference angle and understanding how an angle opens up on the unit circle, defining *standard position, initial ray*, and *terminal ray*. The unit circle is then unrolled along the *x*-axis to demonstrate the key characteristics of periodic functions. Using new understanding of the unit circle, radian measure, and periodic functions, students investigate the sine and cosine functions as well as their characteristics and graphs. Finally, students analyze the characteristics of the tangent graph.

Where have we been?

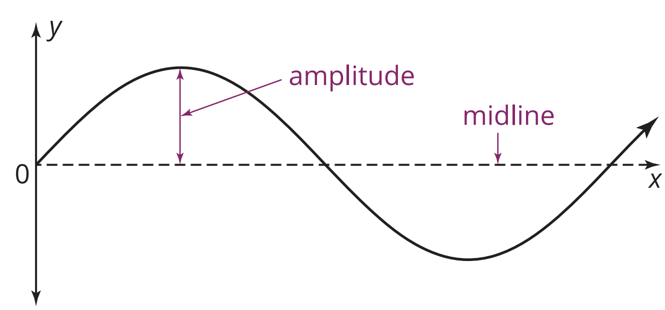
In the previous course, students learned the geometric definitions of the sine, cosine, and tangent ratios and used them to solve problems. Students also defined radian measures and explored the side lengths of special right triangles. They have extensive experience with graphing functions, identifying key characteristics, and performing function transformations.

Where are we going?

Students will use their understanding of the key characteristics of trigonometric functions to model periodic functions from the real world and solve for unknown values. Trigonometric functions are widely used in advanced calculus courses to model real-world scenarios involving circular motion. In these courses, the use of radian measures leads to simple formulas for derivatives and integrals.

Properties of Periodic Functions

The graphs of periodic functions have characteristics that are given special names, such as amplitude and midline.



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Wheel. Of. Ferris.

The Ferris wheel is named after George Washington Gale Ferris, Jr., a Pittsburgh bridge-builder. The first Ferris wheel was designed and constructed by Ferris for the World's Columbian Exposition in Chicago in 1893. It stood 264 feet tall, contained 36 cars, took 20 minutes to complete 2 rotations, and cost 50 cents per ride.

The record for the world's tallest Ferris wheel has been broken many times since 1893. As of 2014, the High Roller in Las Vegas, Nevada, holds the record. It stands a whopping 550 feet tall—over twice as tall as Ferris' original wheel.

Talking Points

Trigonometric functions can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Identify the amplitude of the graph of the function $a(x) = 2\pi \cdot \sin(2\pi x)$.

The amplitude of the basic sine function, $\sin(x)$, is 1. You know that changing the A-value of the function stretches or shrinks the graph vertically.

Thus, the amplitude of $a(x) = 2\pi \cdot \sin(2\pi x)$ is equal to the A-value, which is 2π.

Key Terms

periodic function
A periodic function is a function whose values repeat over regular intervals.

unit circle
A unit circle has a radius of 1 unit.

radians
The unit that describes the measure of an angle theta, θ, in terms of the radius and arc length of a unit circle is called a radian. The ratio of the intercepted arc length of a central angle to the radius is the measure of the central angle in radians.

trigonometric function
Trigonometric functions take angle measures (θ values) as inputs and output real number values, which correspond to coordinates of points on the unit circle.

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1

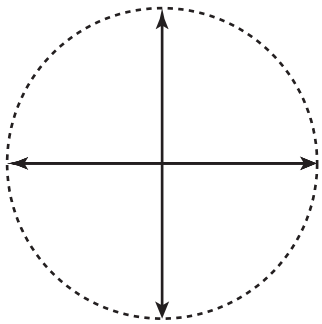
A Sense of Déjà Vu

Periodic Functions

Warm Up

Use a protractor and the axes to draw angles of the given measure in the circle.

1. 45°
2. 30°
3. 180°
4. 270°



Learning Goals

- Model a situation with a periodic function.
- Analyze the period and amplitude of a periodic function.
- Determine the period, amplitude, and midline of a periodic function.

Key Terms

- periodic function
- period
- standard position
- initial ray
- terminal ray
- amplitude
- midline

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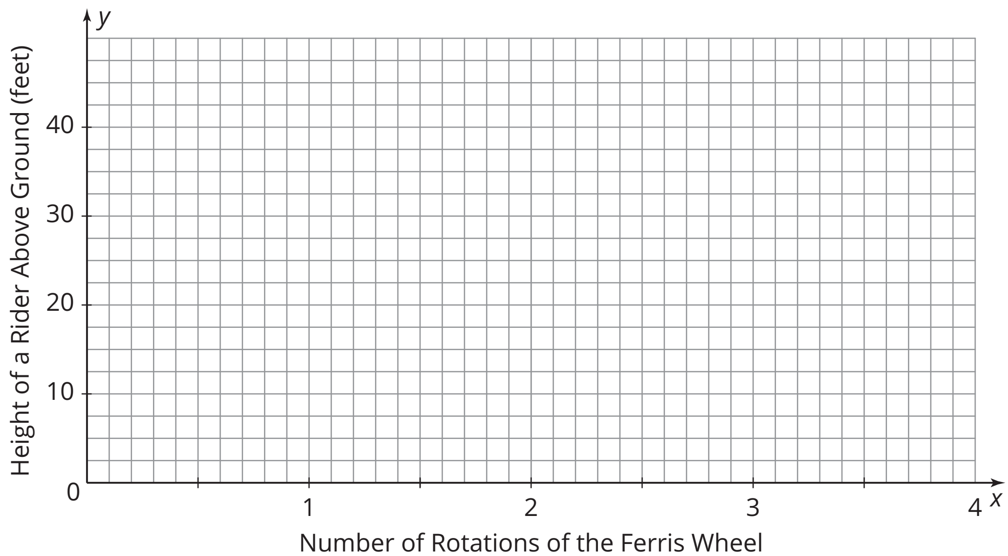
You have learned about many different types of functions. What functions can be defined using points on the circle as the domain?

A Wheel Good Time

One of the most popular amusement park rides is the Ferris wheel. One Ferris wheel has a diameter of 50 feet. Riders board the cars at ground level, and the wheel moves counterclockwise. Each ride consists of four rotations, and you can assume that the Ferris wheel rotates at a constant rate.

1. Create a sketch to model the height of a rider above ground with respect to the number of rotations of the Ferris wheel. Include 4 rotations.



Think about:

Imagine yourself on this Ferris wheel. When will you be on the ground, and when will you be 50 feet above the ground?

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2. **Compete the table to represent the height of a rider above ground as a function of the number of rotations of the Ferris wheel.**

Rotations of the Ferris Wheel	Height of a Rider Above Ground (feet)
0	
$\frac{1}{8}$	
$\frac{1}{4}$	
$\frac{3}{8}$	
$\frac{1}{2}$	
$\frac{5}{8}$	
$\frac{3}{4}$	
$\frac{7}{8}$	
1	

Ask yourself:

What adjustments can you make to your sketch of this situation after completing the table of values?

3. **Describe the characteristics of your graph.**

4. **What do you notice about the shape of the graph for each rotation?**

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ACTIVITY **1.1**

The Underground Ferris Wheel

To model the height of a rider above ground on the Ferris wheel, you used a *periodic function.* A **periodic function** is a function whose values repeat over regular intervals. The **period** of a periodic function is the length of the smallest interval over which the function repeats.

1. Describe the period of the function that models the height of a rider above ground on the Ferris wheel.

At a different amusement park, a Ferris wheel was designed so that half of the wheel is actually below the ground. The diameter of this underground Ferris wheel is still 50 feet. The top of the ride reaches 25 feet above ground and the bottom of the ride reaches 25 feet below ground. Riders board the cars at ground level to the right, and the Ferris wheel moves counterclockwise.

2. Create a sketch to model the height of a rider above ground with respect to the number of rotations of the underground Ferris wheel. Include 4 rotations.

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3. **Complete the table to represent the height of a rider above ground as a function of the number of rotations of the underground Ferris wheel.**

Rotations of the Ferris Wheel	Height of a Rider Above Ground (feet)
0	
$\frac{1}{8}$	
$\frac{1}{4}$	
$\frac{3}{8}$	
$\frac{1}{2}$	
$\frac{5}{8}$	
$\frac{3}{4}$	
$\frac{7}{8}$	
1	

4. **Describe the characteristics of your graph.**

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5. **Describe the period of the function that models the height of a rider above ground on the underground Ferris wheel.**

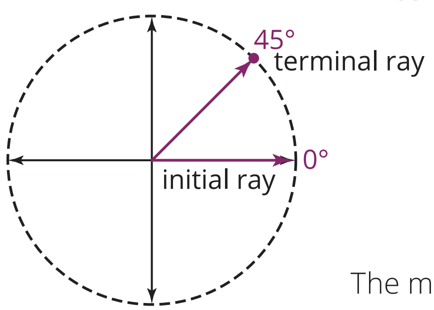
ACTIVITY

1.2 Periodic Functions

In the last two activities, you modeled the height of a rider above ground as a function of the number of rotations of two different Ferris wheels. You can also model the height of a rider as a function using angle measures.

An angle is in **standard position** when the vertex is at the origin and one ray of the angle is on the *x*-axis. The ray on the *x*-axis is the **initial ray**, and the other ray is the **terminal ray**.

1. Use the graph on the next page and complete the steps shown to build a periodic function to model the underground Ferris wheel scenario. The position of the car is the intersection of the terminal ray and the circle.

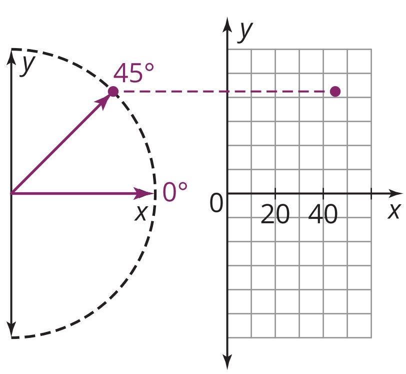


Step 1: Analyze each axis label.

Step 2: Measure a 45° angle in standard position.

Mark and label a point on the Ferris wheel as shown.

The measure of an angle in standard position is the amount of rotation from the initial ray to the terminal ray. When the rotation is counterclockwise, the angle measure is positive. When the rotation is clockwise, the angle measure is negative.



Step 3: Use a straightedge to line up the point on the Ferris wheel with the appropriate location on the coordinate plane. Plot the point.

Step 4: Repeat Steps 2 and 3 for each angle measure: 0°, 30°, 60°, 90°, 180°, 270°, 360°.

Step 5: Draw a smooth curve to connect the points of your graph.

Step 6: Continue the curve to represent angle measures greater than 360°.

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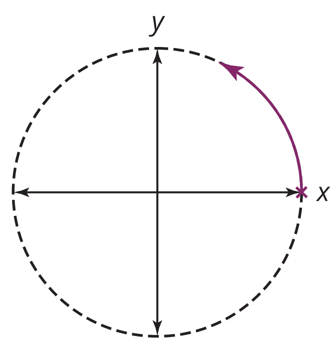

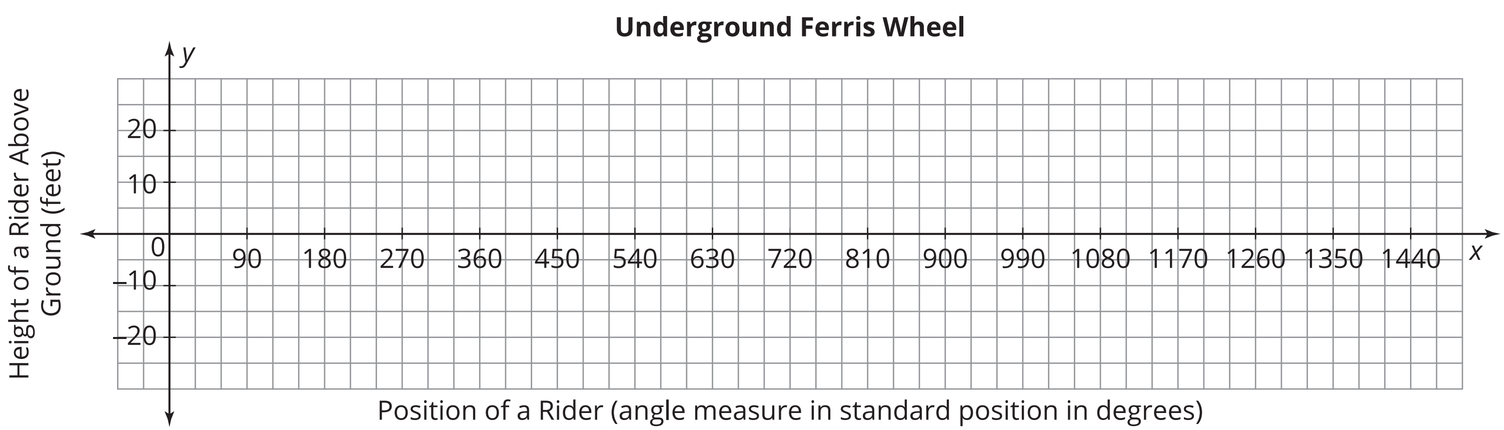

2. **Determine the period of the function you graphed. What does this value represent in terms of this problem situation?**

3. **Determine any maximum or minimum values of your graph. What does each value represent in terms of this problem situation?**

Remember:

You can describe angle measures greater than 360°.

4. **At certain angle measures, a rider is at the highest or lowest point.**

   a. **List 4 angle measures associated with a rider being at the highest point.**

   b. **List 4 angle measures associated with a rider being at the lowest point.**

5. **Describe the symmetries you see in the graph of the function. Explain how these are related to the symmetries associated with the Ferris wheel.**

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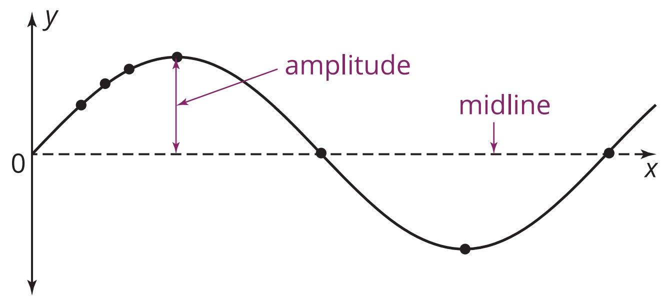
The graphs of periodic functions have characteristics that are given special names, such as *amplitude* and *midline*.



The **amplitude** of a periodic function is one-half the absolute value of the difference between the maximum and minimum values of the function.

The **midline** of a periodic function is a reference line whose equation is the average of the minimum and maximum values of the function.

6. Determine the amplitude of each function you graphed in this lesson. Show your work.

7. Identify the midline of each function you graphed in this lesson.

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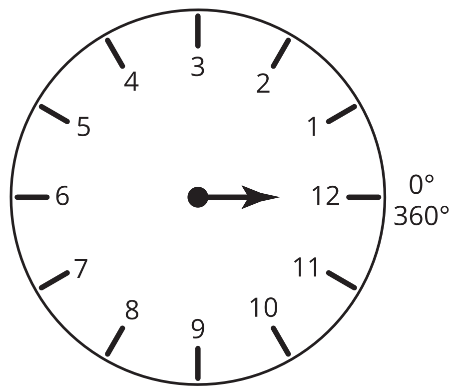
NOTES

TALK the TALK

Rock. Around. The Clock Tonight.

Consider a typical day in the life of you. What do you do every day? At what time?

1. Plot at least 5 of your daily events and their hours on the altered clock shown.



2. Determine the degree measures of each of your events, using 0° to represent 12:00 midnight and 360° to represent 12:00 noon.

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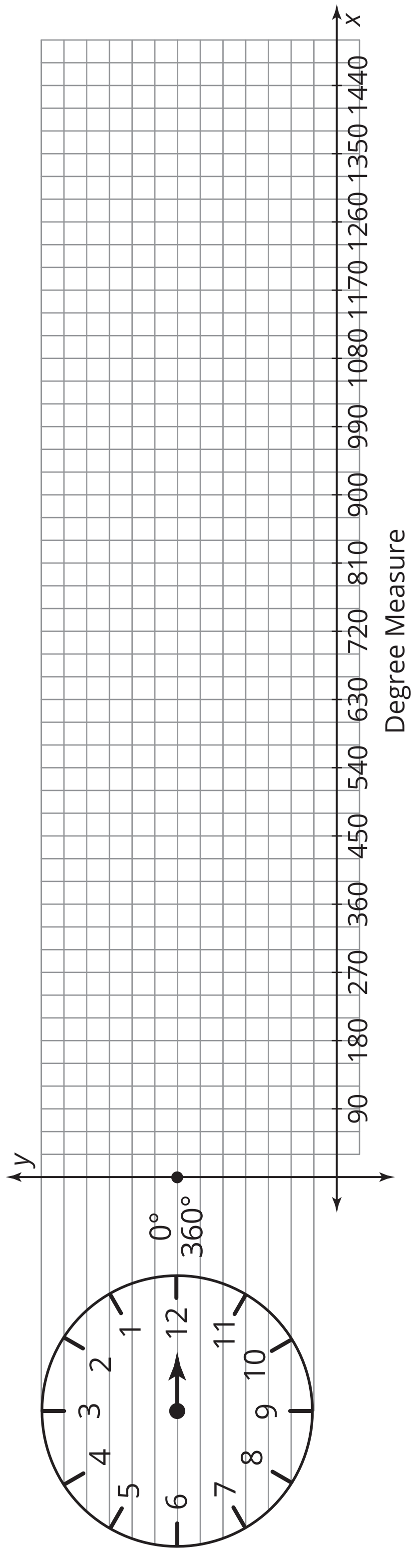
3. **Graph the curve using the same method you used in Activity 1.2. Then, plot your events on the graph. The first point has been plotted for you.**



NOTES

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NOTES

4. Compare your graphs with your classmates' graphs.

   a. What do you notice?

   b. How do you distinguish between AM and PM on your graph?

   c. How can you tell from your graph whether an event happened at 8:00 or 10:00?

   d. How can you tell from your graph when an event happens at the same time every day?

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Assignment

Write

Write the term that best completes each statement.

1. The terminal ray of an angle in standard position is the ray with its endpoint at the origin that is not the ________________.
2. The ________________ of a periodic function is one half the absolute value of the difference between the maximum and minimum values of the function.
3. An angle is in ________________ when the vertex is at the origin and one ray of the angle is on the *x*-axis.
4. A ________________ is a function whose values repeat over regular intervals.
5. The ________________ of a periodic function is a reference line whose equation is the average of the minimum and maximum values of the function.
6. The ________________ of a periodic function is the length of the smallest interval over which the function repeats.
7. The measure of an angle in standard position is the amount of rotation from the initial ray to the ________________.

Remember

A periodic function is a function whose values repeat over regular intervals. The period of a periodic function is the length of the smallest interval over which the function repeats.

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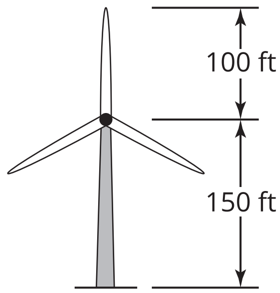
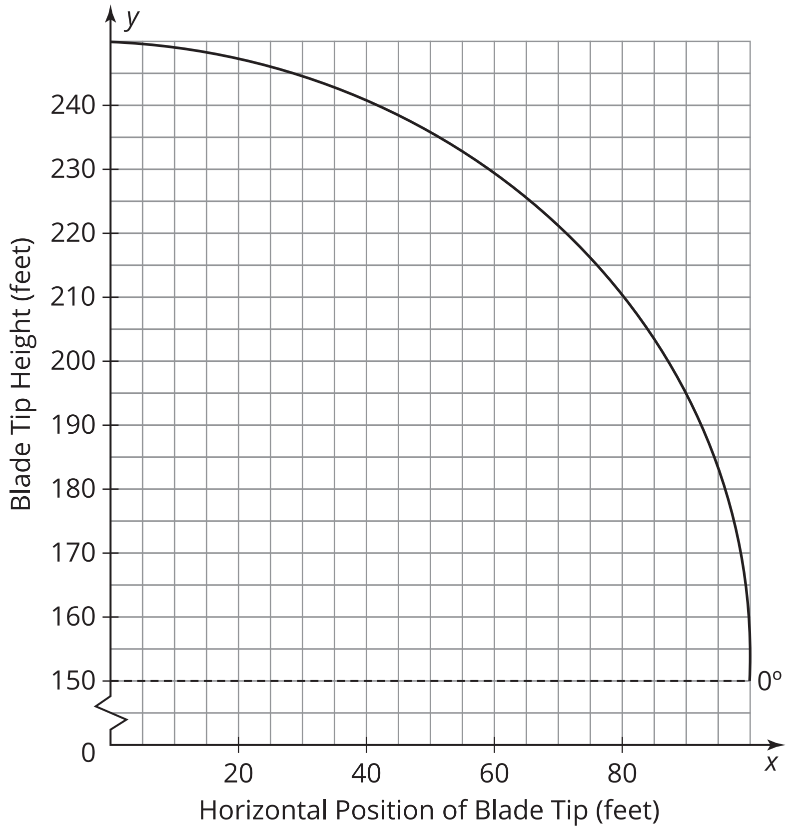
Practice

1. Wind turbines harness the power of the wind to generate electricity.
   One particular wind turbine consists of three 100-foot-long rotor blades that rotate around the top of a 150-foot vertical shaft.

   a. Use a protractor, a straightedge, and the given graph to estimate the height of the blade tip when the blade is at angles of 0°, 30°, 45°, 60° and 90°. The arc on the graph represents a portion of the blade tip's path. Assume the blade rotates counterclockwise and the blade is at an angle of 0° when the blade tip is directly to the right of the top of the vertical shaft. This position has been labeled as 0° on the graph.





   b. Complete the table using your knowledge of the symmetry of circles.

Blade Angle	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°
Tip Height (feet)											

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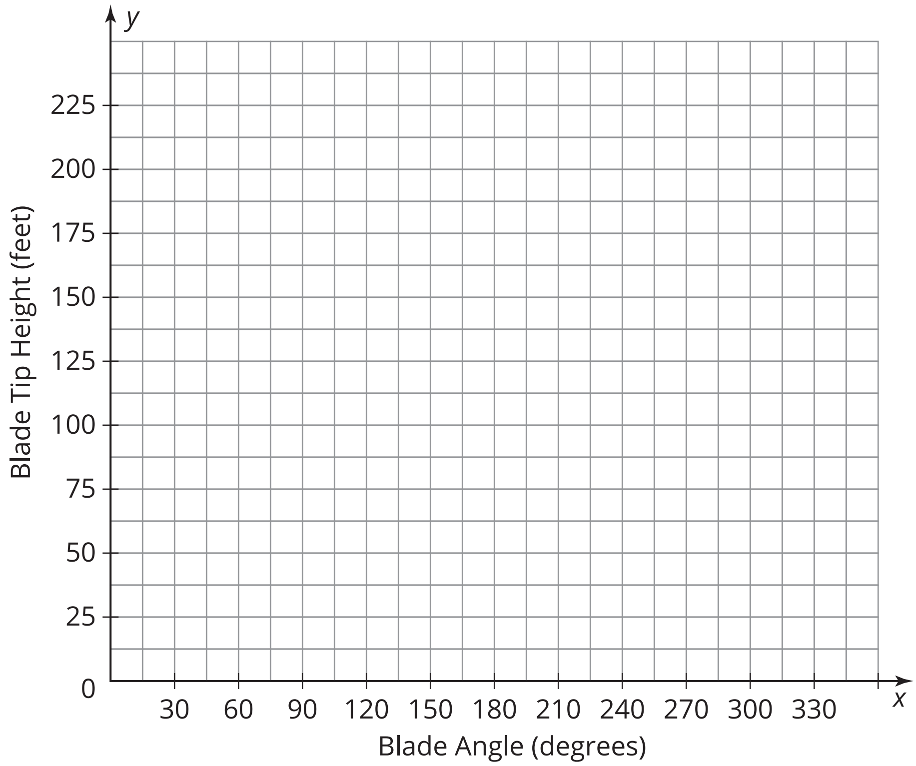
c. Graph the blade tip height as a function of the blade angle.



d. Determine the equation of the midline for the periodic function you graphed in part (c). Sketch and label the midline as a dashed line on the graph in part (c).

e. Determine the amplitude of the function you graphed in part (c). Explain your reasoning.

f. Determine the period of the function you graphed in part (c). Explain your reasoning.

g. Determine the height of the blade tip when the blade angle is 570°. Explain your reasoning.

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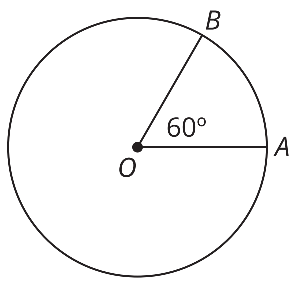
Stretch

1. The circle with center *O* has a radius of 1 unit.



a. Determine the arc length *AB*.

b. Determine the ratio of the arc length to the radius of the circle.

c. The ratio of the arc length to the radius of the circle is the measure of the central angle in radians. Determine the measure of a central angle of 135° in radians.

Review

1. The following rules are used to create a certain fractal, the Cantor set.

   Stage 0: Begin with a line segment.

   Stage 1: Divide the line segment into thirds and then erase the middle third.

   Stages 2 and up: Repeat Stage 1 for the line segments in the figure.

   a. Complete Stage 2 of the fractal. Stage 0 and Stage 1 are given.

   Stage 0

   Stage 1

   Stage 2

Stage	Total Length of Line Segments (in.)
0	
1	
2	
3	
4	
5	
n	

   b. Determine the total length of the line segments at each stage and complete the table. The length of the initial line segment in Stage 0 is 1 in.

   c. Identify the type of sequence represented by the total length of the line segments at Stage n.

   d. Write a function to represent the total length of the line segments as a function of the stage, n. Describe the type of function you used.

2. The following rules are used to create a certain fractal, the von Koch curve.

   Stage 0: Begin with a line segment.

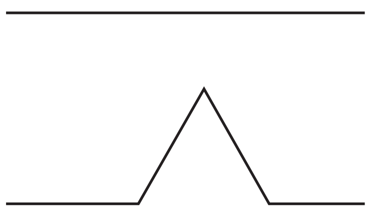
   Stage 1: Replace the middle segment with an equilateral triangle, and remove the side of the triangle corresponding to the initial straight line.

   Stages 2 and up: Repeat Stage 1 for the line segments in the figure.

   a. Complete Stage 2 of the fractal. Stage 0 and Stage 1 are given.

   Stage 0

   Stage 1

Stage	Number of Line Segments
0	
1	
2	
3	
4	
5	
n	

   b. Determine the number of line segments at each stage and complete the table.

   c. Identify the type of sequence represented by the number of line segments at Stage n.

   d. Write a function to represent the number of line segments as a function of the stage, n. Describe the type of function you used.

3. Identify the number of real zeros of each polynomial.

   a. $2x^4 + x^3 + 3x^2 + 3x - 9 = 0$

   b. $x^5 + 2x^4 + 11x^3 + 22x^2 + 24x + 48 = 0$

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2

The Knights of the Round Table

Radian Measure

Warm Up

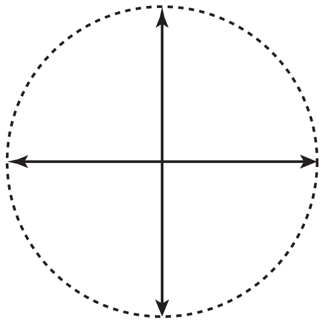
Use a protractor and the axes to draw angles of the given measure in the circle.

1. 50°

2. 25°

3. 135°

4. 225°



Learning Goals

- Determine the radian measure of angles.
- Convert between angle measures in degrees and angle measures in radians.
- Estimate the degree measure of central angle measures given in radians.
- Identify reference angles in radians

Key Terms

- theta (θ)
- unit circle
- radians

You have measured angles in degrees and learned that movement along a circle can be modeled by a periodic function. Are there other units of measure that describe angles?

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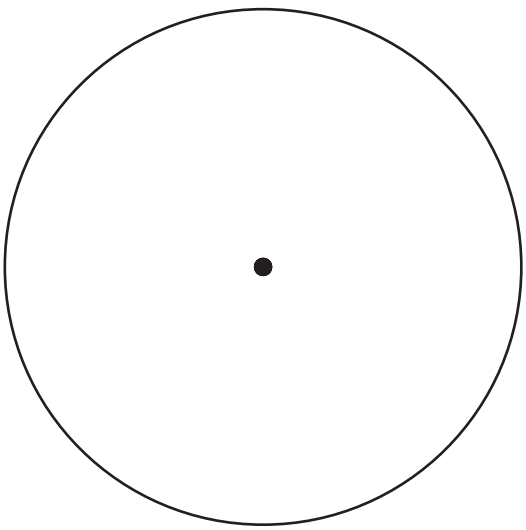
GETTING STARTED

King Arthur's Knights

You may have heard of the legend of King Arthur and his Knights of the Round Table. The round table was used to show that each person sitting at it was equal. In most depictions, the knights also appear to be spaced around the table so that they are an equal distance apart.

But different versions of the legend give different numbers of knights. In many versions, there are 12 knights, but some include 25 or even 150 knights of the round table!



1. **On the circle given, show how 12 knights could be seated at the round table at an equal distance from each other.**

2. **Express the location of each of the twelve knights in terms of the circumference of the circle, $2\pi r$. Write each value in lowest terms.**

3. **Without drawing, describe the locations of the knights if there are 25 Knights of the Round Table spaced an equal distance from each other.**

4. **What if there were 150 knights? What do you think the diameter of the table should be for that many knights to sit comfortably around the table? Justify your answer.**

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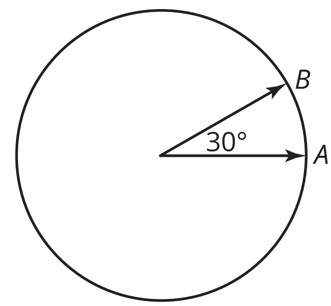
ACTIVITY **2.1**

Measuring Central Angles in Radians

Recall that the measure of an arc of a circle is equal to the degree measure of the central angle that intercepts the arc.

$$m\widehat{AB} = 30°$$



The length of the intercepted arc is given by the expression:

$$\text{arc length} = 2\pi r \cdot \frac{\text{measure of central angle}}{360°}$$

You can identify the central angle measures of a circle in standard position using the symbol **theta**, written as θ. For example, a central angle measure of 30° can be written as $\theta = 30°$.

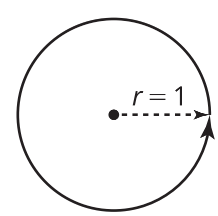
The central angles you will discuss in this lesson are all angles in standard position.

1. Given any circle with a radius of r units:

a. Write an expression in terms of r to describe the arc length for a central angle measure of $\theta = 30°$.

b. Write an expression in terms of r to describe the arc length for a central angle measure of $\theta = 45°$.

A powerful way to measure central angles of a circle is to identify arc lengths of the circle in terms of the radius of a *unit circle.* A **unit circle** has a radius of 1 unit.



2. Consider the unit circle shown.

a. Identify a central angle measure, θ, that represents a complete rotation of the terminal ray around the unit circle.

b. Identify the arc length of this central angle.

c. Identify a central angle measure, θ, and arc length that represent half of a rotation of the terminal ray around the unit circle.

Ask yourself:

What is the arc length when the radius is 1 unit?

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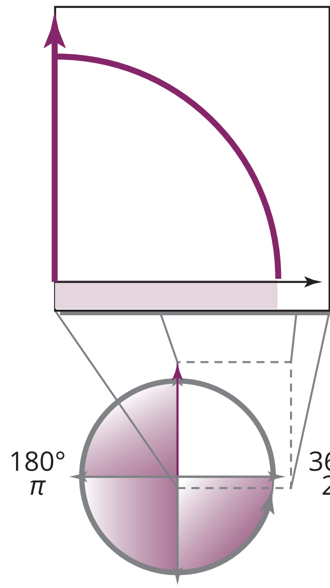
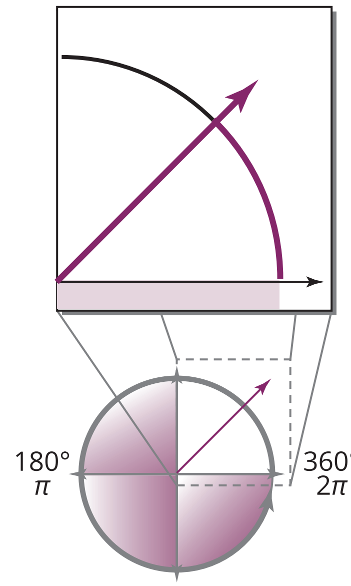
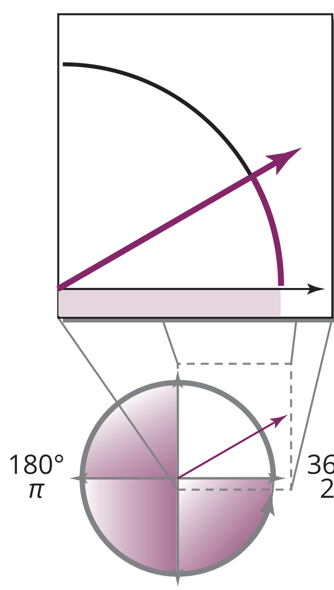
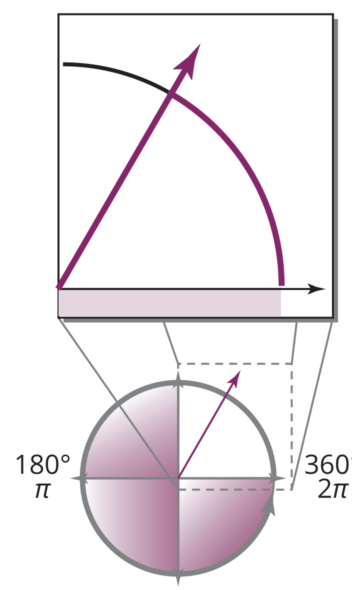
3. Use a protractor to determine each central angle measure, θ, in the unit circle. Then label the angle measures and their corresponding arc lengths in units. Explain how you determined your answers.









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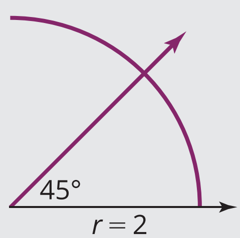
The unit that describes the measure of an angle theta, θ, in terms of the arc length and radius of a unit circle is called a *radian*. The ratio of the intercepted arc length of a central angle to the length of the radius is the measure of the central angle in **radians**.

There are $\frac{2\pi r}{r}$, or 2π, radians in 360° and $\frac{\pi r}{r}$, or π, radians in 180°.

4. Jaylen and Malik each determined the radian measure for a central angle measuring 45° in a circle with a radius of 2 units.

Jaylen

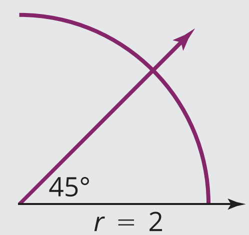


Arc length $= 2\pi(2) \cdot \frac{45°}{360°}$

$= 4\pi \cdot \frac{1}{8} = \frac{\pi}{2}$ units

The radian measure of this angle is $\frac{\pi}{2} \div 2$, or $\frac{\pi}{4}$ radian.

Malik



Arc length $= 2\pi(2) \cdot \frac{45°}{360°}$

$= 4\pi \cdot \frac{1}{8} = \frac{\pi}{2}$ units

The radian measure of this angle is $\frac{\pi}{2}$ radians.

Explain why Malik's reasoning is incorrect.

5. Use what you know about the symmetry of a circle to label each central angle measure in degrees and radians on the unit circle located at the end of the lesson. Explain how you determined the measures, and show your work.

Use your protractor to verify your angle measures.

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ACTIVITY

2.2

Thinking with Radians: Pi Is a Constant

It is important to keep in mind that values such as $\frac{\pi}{4}$ and $\frac{7\pi}{6}$ are constants. Each of these irrational numbers can be rewritten as non-terminating, non-repeating decimals.

$$\frac{\pi}{4} \approx \frac{3.14}{4} \approx 0.785 \qquad \frac{7\pi}{6} \approx \frac{7(3.14)}{6} \approx 3.6633....$$

You can also write whole-number values for radians.

You can use the abbreviation "rad" when describing measures in radians.

1. Estimate the degree measure of each central angle measure given in radians. Explain your reasoning.

a. 3 radians

b. 6 radians

c. 2 radians

d. 4 radians

e. 1 radian

f. 5 radians

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2. **What is the arc length of a central angle that has a measure of 1 radian on the unit circle? Explain your reasoning.**

The formulas you can use to convert the units to measure angles from radians to degrees and degrees to radians are shown.

Notice that the ratios in the formulas are forms of 1.

Radians to Degrees: x radians $\cdot \frac{180°}{\pi \text{ radians}}$

Degrees to Radians: x degrees $\cdot \frac{\pi \text{ radians}}{180°}$

3. **Use the formulas to convert each angle measure in Question 1 to degrees.**

   How close were your estimates?

4. **Corinne made the statement regarding radian measures.**

> Corinne
>
> The complement of an angle measure θ in radians is $\left(\frac{\pi}{2} - \theta\right)$ radians.

Explain why Corinne is correct. Write a similar statement using degrees.

5. **What is the supplement of an angle measure θ in radians? Explain your reasoning.**

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NOTES

TALK the TALK

Degrees Are Rad Too

You can use degrees or radians as units of measure to describe angles.

1. Convert each angle in degree measure to radian measure.

a. 500°

b. 390°

c. 150°

2. Convert each angle in radian measure to degree measure.

a. $\frac{\pi}{10}$

b. $\frac{7\pi}{6}$

c. $\frac{14\pi}{15}$

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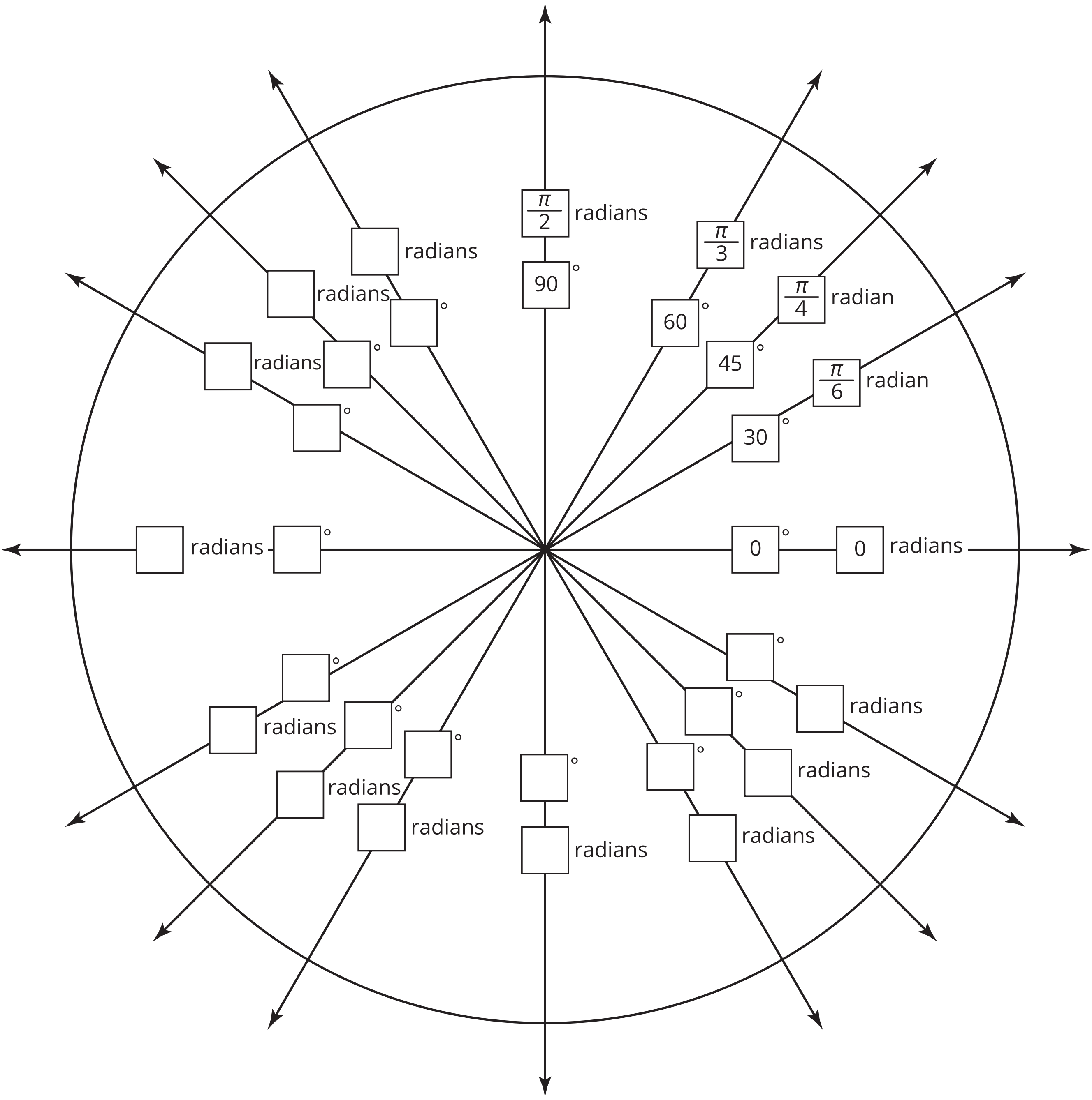
Central Angle Measures in Degrees and Radians



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Assignment

Write

Complete each sentence.

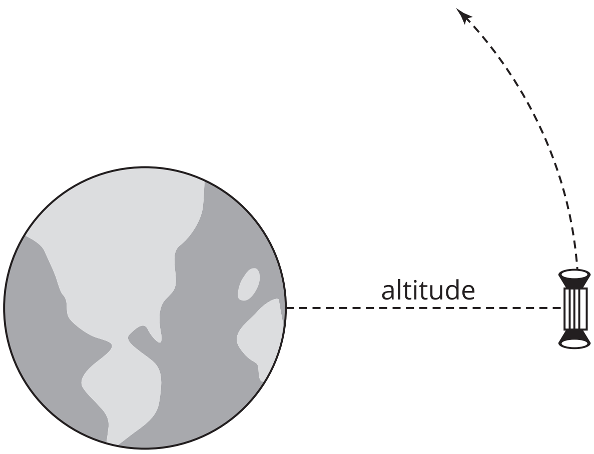
1. A unit circle has a radius of ______________.
2. A symbol used to identify the central angle measure of a circle in standard position is ______________.
3. There are 2π ______________ in 360°.

Remember

The ratio of the intercepted arc length of a central angle to the radius is the measure of the central angle in radians. There are π radians in 180°.

Practice

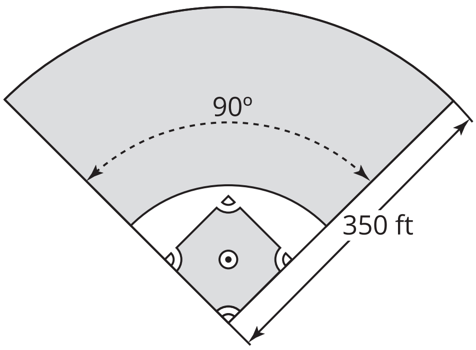
1. A Global Positioning System (GPS) satellite completes 1 orbit of Earth every 12 hours. The satellite follows a circular path with its center at the center of Earth.



   a. Determine the angle of rotation, in radians, that corresponds to 1 complete orbit of the satellite around Earth.
   b. Determine the radius of the circular path the satellite follows during its orbit if Earth's radius is 3,959 miles and the altitude of the satellite is 12,645 miles.
   c. Determine the angle of rotation, in radians, that corresponds to an 8-hour time period.
   d. Determine the distance traveled by the satellite in an 8-hour time period.
   e. The computer onboard the satellite had to be remotely shut down and rebooted in order to repair a software glitch. The satellite traveled a distance of 27,000 miles during that time. How long did it take to shut down and reboot the computer?

2. The outfield fence on a baseball field needs to be replaced. The fence is an arc with its center at home plate and a central angle of 90°. The distance from home plate to any point on the fence is 350 feet.

Outfield Fence



   a. Determine the central angle of the outfield fence in radians.
   b. Determine the length of the outfield fence that needs to be replaced.

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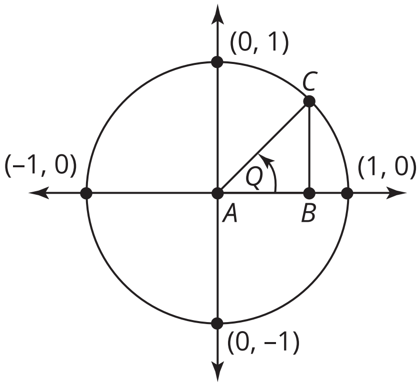
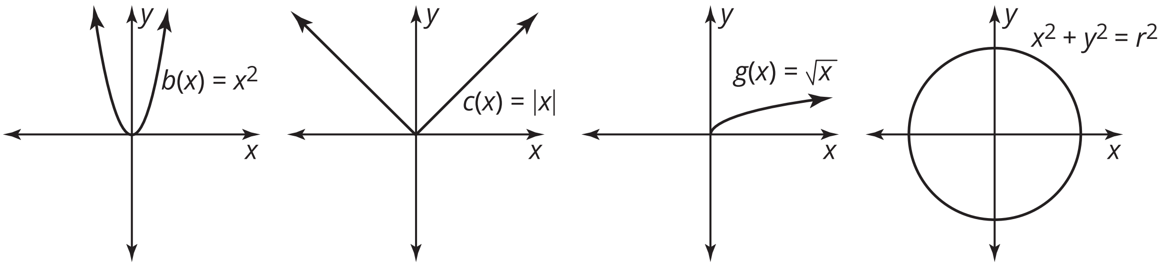
Stretch

1. An automobile tire has a diameter of 28 inches.
   a. What angle does the wheel turn through if the car has moved 2 feet? Give the answer in both radians and degrees.
   b. If the tire makes 10 turns in 1 second, how fast is the car going in miles per hour?
2. A unit circle is shown. Determine the coordinates of points B and C on the triangle for the different measures of θ.
   a. $\theta = \frac{\pi}{6}$ radians
   b. $\theta = \frac{\pi}{4}$ radians
   c. $\theta = \frac{\pi}{3}$ radians



Review

1. Consider the relations shown.



   Graph each relation to create a picture.

   $y = g(-x - 1) + 4, \quad -5 \le x \le -1$
   $y = g(x - 1) + 4, \quad 1 \le x \le 5$
   $y = -2c(x) + 6, \quad -3 \le x \le 3$
   $x^2 + (y - 7)^2 = 1$

2. Identify the number of complex zeros for the polynomial equation.
   a. $12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1 = 0$
   b. $5x^4 + 3x^3 + 3x^2 + 3x - 2 = 0$

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3

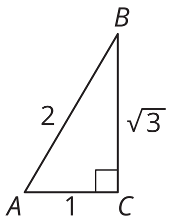
What Goes Around

The Sine and Cosine Functions

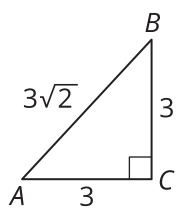
Warm Up

Determine the sine ratio and cosine ratio of $\angle A$ in each triangle.

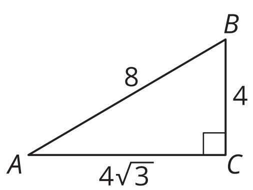
1.



2.



3.



Learning Goals

- Define the sine and cosine functions.
- Calculate values for the sine and cosine of reference angles.
- Define the sine and cosine of an angle as a coordinate of a point on the unit circle.
- Graph and compare the sine and cosine functions.

Key Terms

- sine function
- cosine function
- trigonometric function
- periodicity identity

You have previously explored the relationship of the side lengths in special right triangles, and you know how to determine the sine and cosine ratios of angles in a right triangle. How can these relationships on a unit circle be represented as functions on a coordinate plane?

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GETTING STARTED

The Right Triangle Connection

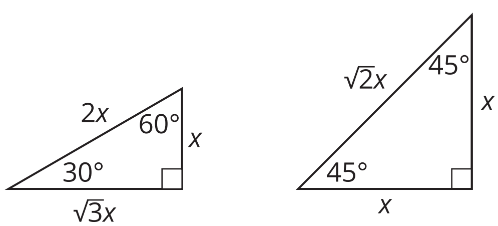
Recall that the sine ratio (sin), given a reference angle, θ, is the ratio of the length of the opposite side to the length of the hypotenuse in a right triangle.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

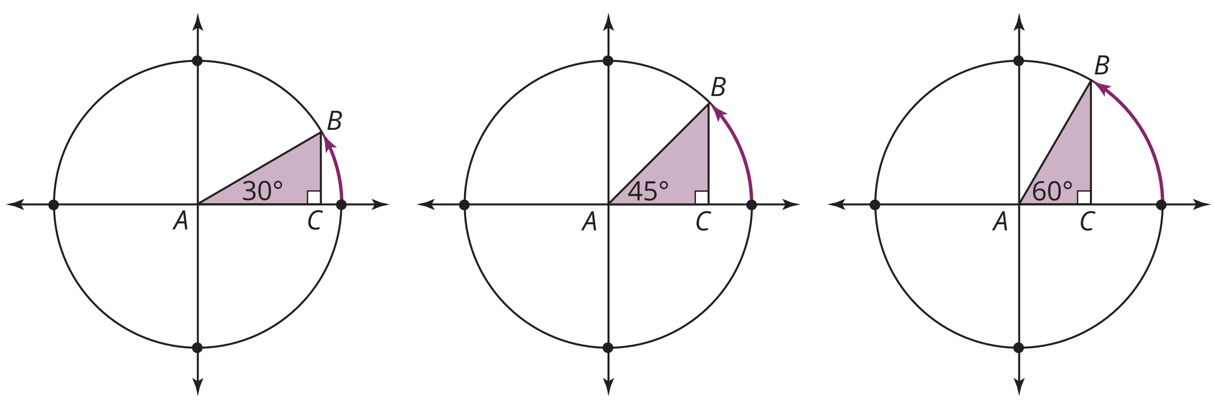
The cosine ratio (cos), given a reference angle, θ, is the ratio of the length of the adjacent side to the length of the hypotenuse.

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

The side-length relationships for a 30° -60° -90° triangle and a 45° -45° -90° triangle are shown.



The diagram shows a right triangle *ABC* placed on a unit circle centered at the origin. The central angle measures $\theta = 30°$, $\theta = 45°$, and $\theta = 60°$ are shown.



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1. What is the length of the hypotenuse *c* in each circle? Label the measures on each triangle.

2. Label the side lengths of the triangles in each diagram in radical form.

3. The hypotenuse of each right triangle represents the terminal ray of a central angle that intersects the unit circle at point *B*.

a. Complete the table to record the sine and cosine of each angle measure, θ, and the coordinates of the point where the terminal ray intersects the unit circle. Explain your reasoning.

θ	$\cos\theta$	$\sin\theta$	Coordinates of Point *B*, (Intersection of Terminal Ray and Unit Circle)
30°			
45°			
60°			

b. Write the coordinates of the intersection of the terminal ray and the unit circle at 0°.

c. Write the coordinates of the intersection of the terminal ray and the unit circle at 90°.

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4. Jorge conjectured that the coordinates of the point where the terminal ray of a central angle θ intersects the unit circle can always be written as ($\cos\theta$, $\sin\theta$). Do you think Jorge's conjecture is correct? Explain your reasoning.

ACTIVITY **3.1**

Unit Circle Coordinates in Quadrant I

In the unit circle, you will label the first quadrant now. Then you will be able to determine the coordinates of the points in the other quadrants in the next activity.

Use the unit circle located at the end of this lesson and your answers to the questions in the Getting Started to complete this activity.

1. Determine the coordinates of the points in the first quadrant on the unit circle. Label the coordinates.

2. Use the unit circle to evaluate each measure.

$\sin(\frac{\pi}{6} \text{ radian}) =$ ________________

$\cos(\frac{\pi}{6} \text{ radian}) =$ ________________

$\sin(\frac{\pi}{4} \text{ radian}) =$ ________________

$\cos(\frac{\pi}{4} \text{ radian}) =$ ________________

3. For each angle measure in Question 2, evaluate the sine and cosine of the complement. Explain your reasoning.

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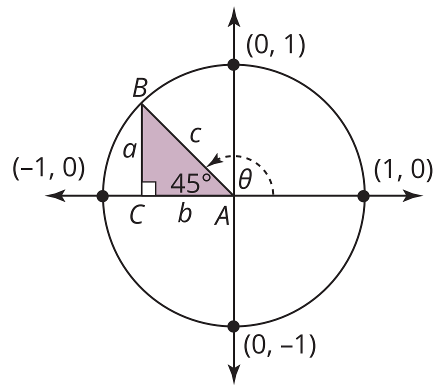
ACTIVITY 3.2 Unit Circle Coordinates Beyond Quadrant I

Now that you have identified values of sine and cosine in the first quadrant, how can you use that knowledge to identify values in other quadrants?



1. The diagram shows a 45° central angle positioned in the second quadrant on the unit circle.

a. State the measure of θ in degrees and in radians. Explain how you determined your answer.

b. Identify the coordinates of the point at which the terminal ray of the angle intercepts the circle. Explain how you determined your answer.

c. What do you notice about the coordinates of this point and the coordinates of the symmetrical point in the first quadrant?

Remember:

When creating a triangle using a terminal ray, the right triangle drawn must always be bound to the *x*-axis.

2. Use what you know about symmetry to label the coordinates of the remaining points on the unit circle located at the end of the lesson.

3. Look back at Jorge's conjecture in the Getting Started. Is his conjecture correct? Explain your reasoning.

Think about:

Does the cosine have a negative or a positive value? Does the sine have a negative or positive value?

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4. **Describe when the values of cosine and sine are positive and negative in the unit circle. Label this information on the unit circle at the end of the lesson.**

5. **Ray makes this conclusion.**

> Ray
> Many different central angle measures have the same sine or cosine values.

Provide examples to support Ray's conclusion.

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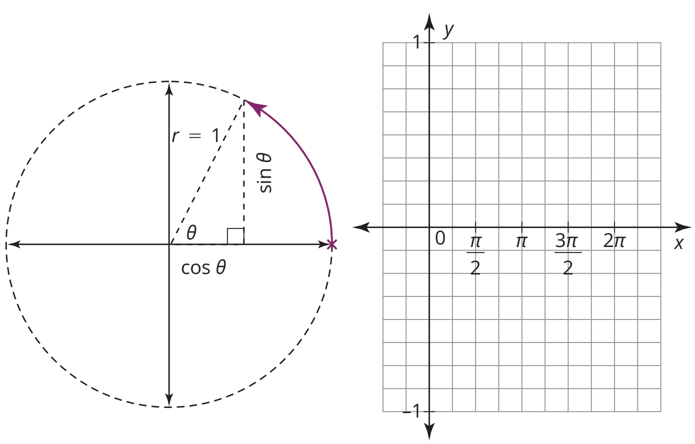
ACTIVITY
3.3 Sine and Cosine Functions

Let's consider how to represent the values from your unit circle as functions on a coordinate plane.

1. Use your completed unit circle to graph the function *y* = sin *x*.

Plot the output values from left to right on the graph as you move counterclockwise around the unit circle.

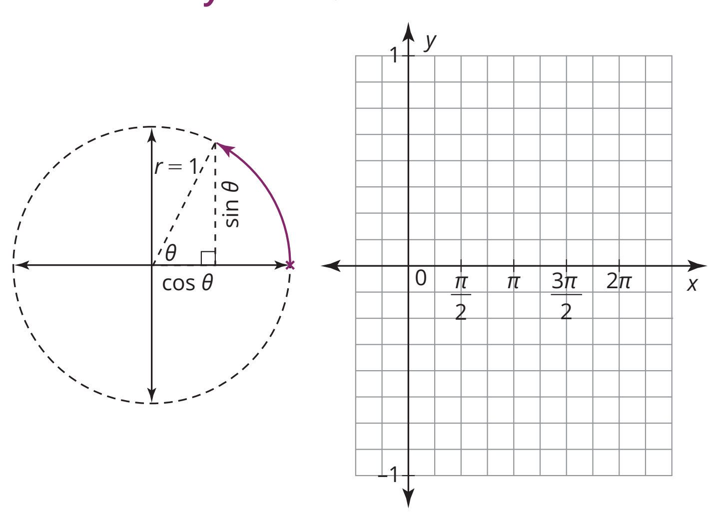
a. As the terminal ray traverses the unit circle counterclockwise in standard position, plot the output value, sin *θ*, that corresponds to the input value, *θ*, which is the radian measure of the central angle, from 0 to 2*π* radians.



b. What coordinate values on the unit circle did you use to create the graph of *y* = sin *x*?

2. Use your completed unit circle to graph the function *y* = cos *x*.

a. As the terminal ray traverses the unit circle counterclockwise in standard position, plot the output value, cos *θ*, that corresponds to the input value, *θ*, which is the radian measure of the central angle, from 0 to 2*π* radians.



b. What coordinate values on the unit circle did you use to create the graph of *y* = cos *x*?

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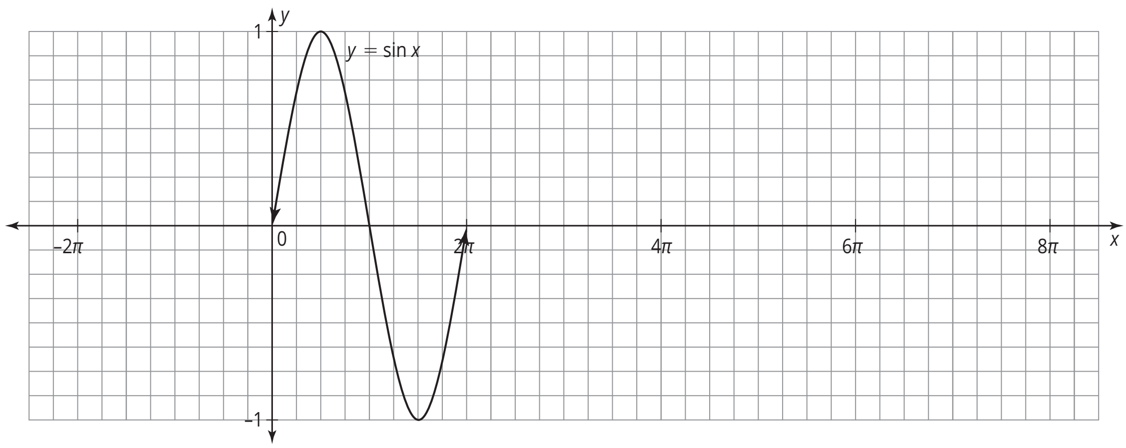
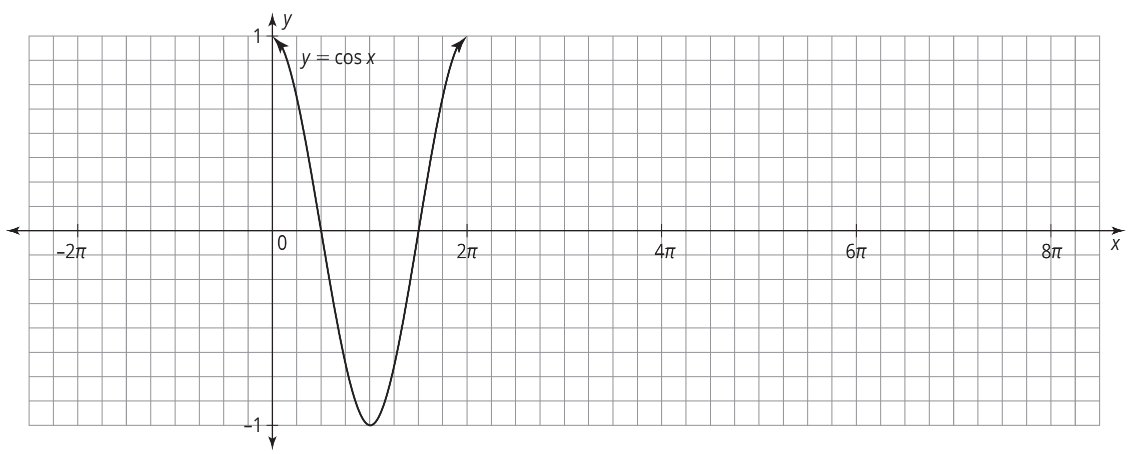
You have graphed the *sine function* and *cosine function*. The **sine function** and **cosine function** are periodic **trigonometric functions**. Each of these trigonometric functions takes angle measures (θ values) as inputs and outputs real number values, which correspond to coordinates of points on the unit circle.

3. Consider the functions $y = \sin x$ and $y = \cos x$.





a. Extend the graphs of the functions $y = \sin x$ and $y = \cos x$ over the domain $0 \leq x \leq 8\pi$.

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b. **Determine the values of sin *x* and cos *x* at 4π, 6π, and 8π radians.**

c. **Describe how you can determine each value from part (b) on the unit circle for each function.**

4. **Now consider a domain of $-2\pi \le x \le 8\pi$ for the functions $y = \sin x$ and $y = \cos x$.**

a. **Extend the graphs of the functions $y = \sin x$ and $y = \cos x$ in Question 3 through $x = -2\pi$.**

b. **Determine each sine value.**

$\sin(-\frac{\pi}{2}) =$ ____________________

$\sin(-\pi) =$ ____________________

$\sin(-\frac{3\pi}{2}) =$ ____________________

$\sin(-2\pi) =$ ____________________

c. **Determine each cosine value.**

$\cos(-\frac{\pi}{2}) =$ ____________________

$\cos(-\pi) =$ ____________________

$\cos(-\frac{3\pi}{2}) =$ ____________________

$\cos(-2\pi) =$ ____________________

Ask yourself:

Can the sine and cosine functions output any real number, given any angle measure input?

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5. Consider the values of $\sin(x + 2\pi)$. How do these values compare to the values of $\sin x$?

6. Consider the values of $\cos(x + 2\pi)$. How do these values compare to the values of $\cos x$?

The period of the sine function is 2π radians, and the period of the cosine function is 2π radians. Thus, you can write two *periodicity identities*:

- $\sin(x + 2\pi) = \sin x$
- $\cos(x + 2\pi) = \cos x$

Each of these is called a **periodicity identity** because they are each based on the period of the function, 2π.

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NOTES

TALK the TALK

Comes Around

1. Complete the table.

Angle Measure (θ)		cos θ	sin θ	Angle Measure (θ)		cos θ	sin θ
radians	degrees			radians	degrees		
0	0°	1	0	$\frac{7\pi}{6}$	210°		
$\frac{\pi}{6}$	30°			$\frac{5\pi}{4}$	225°		
$\frac{\pi}{4}$	45°			$\frac{4\pi}{3}$	240°		
$\frac{\pi}{3}$	60°			$\frac{3\pi}{2}$	270°		
$\frac{\pi}{2}$	90°			$\frac{5\pi}{3}$	300°		
$\frac{2\pi}{3}$	120°			$\frac{7\pi}{4}$	315°		
$\frac{3\pi}{4}$	135°			$\frac{11\pi}{6}$	330°		
$\frac{5\pi}{6}$	150°			2π	360°		
π	180°						

2. Compare and contrast the functions $y = \sin x$ and $y = \cos x$. Describe the similarities and differences between the two functions.

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NOTES

3. **Identify each of the characteristics for $y = \sin x$ and $y = \cos x$.**

	$y = \sin x$	$y = \cos x$
y-intercept(s)		
Domain		
Range		
Period		
Minimum Output Value		
Maximum Output Value		
Amplitude		
Midline		

4. **Describe the intervals of increase and decrease for both the sine and cosine functions. Explain your reasoning.**

5. **Identify the x-intercepts for each function.**

   a. **x-intercepts for $y = \sin x$**

   b. **x-intercepts for $y = \cos x$**

6. **Use the language of transformations to explain how the sine and cosine functions are related.**

Ask yourself:

How can you write this relationship mathematically?

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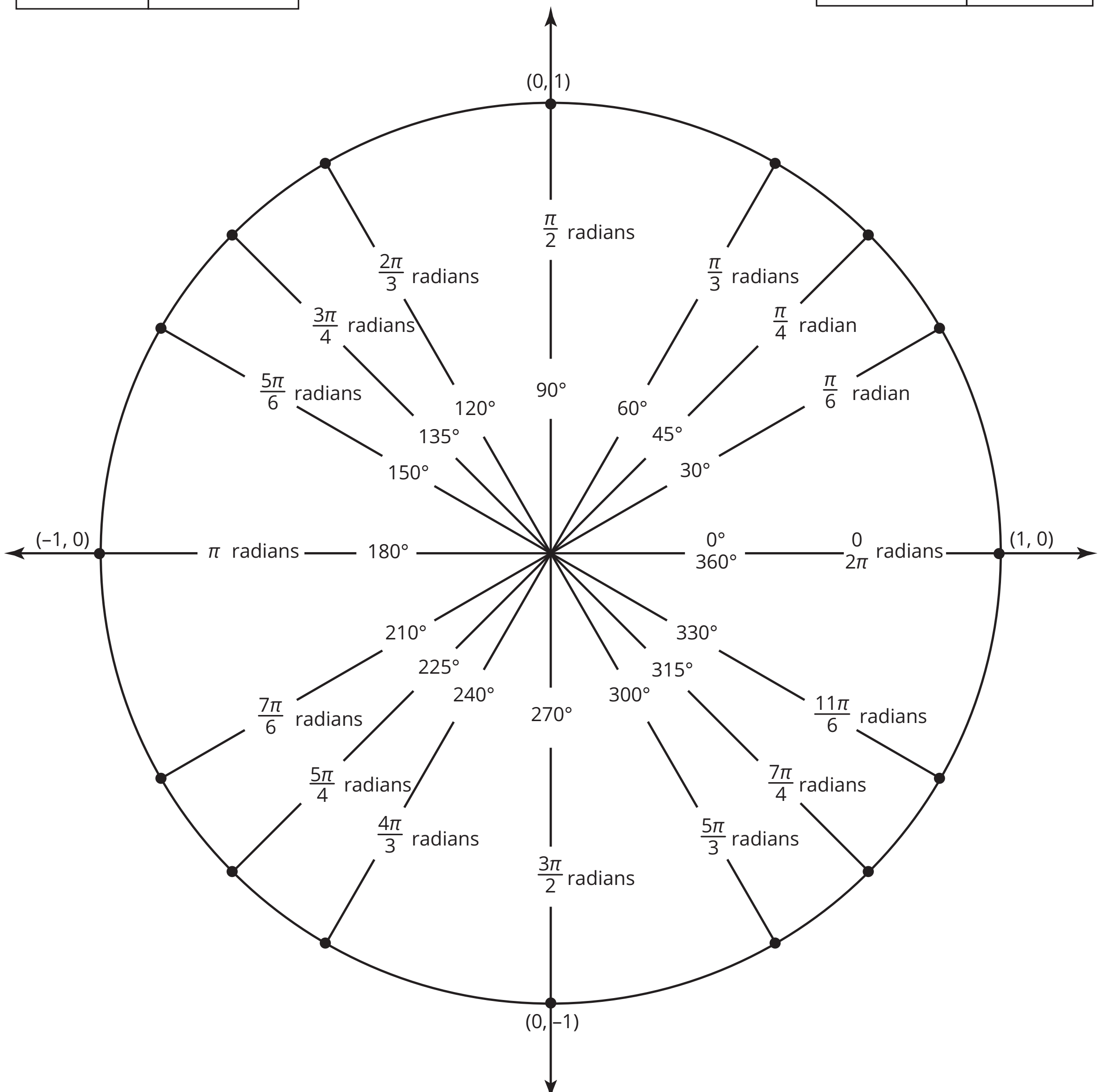
Sine and Cosine on the Unit Circle

Quadrant II

sin θ	
cos θ	

Quadrant I

sin θ	
cos θ	



Quadrant III

sin θ	
cos θ	

Quadrant IV

sin θ	
cos θ	

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Assignment

Write

Write a definition for each term in your own words.

1. sine function
2. cosine function
3. trigonometric function
4. periodicity identity

Remember

The cosine of the central angle measure of a unit circle is the x-coordinate of the point where the terminal ray intersects the unit circle and the sine of the same central angle measure is the y-coordinate of the same point.

The sine function, $y = \sin x$, and cosine function, $y = \cos x$, are periodic trigonometric functions that take angle measures (θ values) as inputs and outputs real number values, which correspond to coordinates of points on the unit circle. The period of each function is 2π radians, therefore $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$.

Practice

1. Determine θ and $\cos\theta$ when $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta$ is negative. Restrict values for θ such that $0 \le \theta \le 2\pi$.
2. Determine θ and $\sin\theta$ when $\cos\theta = -\frac{\sqrt{2}}{2}$ and $\sin\theta$ is negative. Restrict values for θ such that $0 \le \theta \le 2\pi$.
3. Determine 3 values for θ such that $\sin\theta = -\frac{\sqrt{3}}{2}$.
4. Determine 3 values for θ such that $\cos\theta = \frac{\sqrt{2}}{2}$.
5. Determine 3 values for θ such that $\cos\theta = 0$.
6. Determine the value of each ratio.

   a. $\sin\left(\frac{15\pi}{4}\right)$

   b. $\cos\left(\frac{17\pi}{6}\right)$

   c. $\sin\left(\frac{25\pi}{6}\right)$

   d. $\cos\left(\frac{19\pi}{4}\right)$

Stretch

1. Determine θ and $\sin\theta$ when $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{3}}{2}$. Restrict values for θ such that $0 \le \theta \le 2\pi$.
2. Complete the table of values for the functions shown.

		Function			
		$\sin\theta$	$2\sin\theta$	$\sin\theta + 1$	$\sin(\theta + \pi)$
Angle Measure (θ)	0				
	$\frac{\pi}{6}$				
	$\frac{\pi}{4}$				
	$\frac{\pi}{3}$				
	$\frac{\pi}{2}$				

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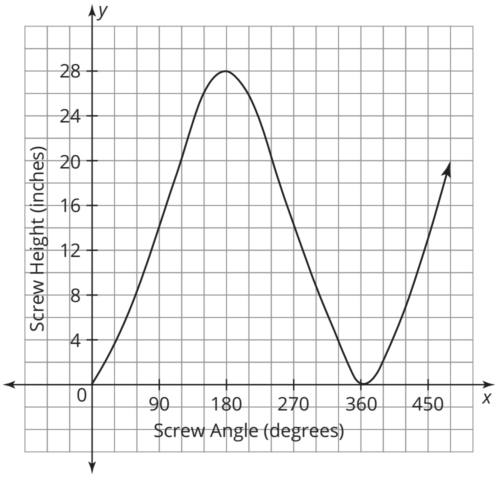
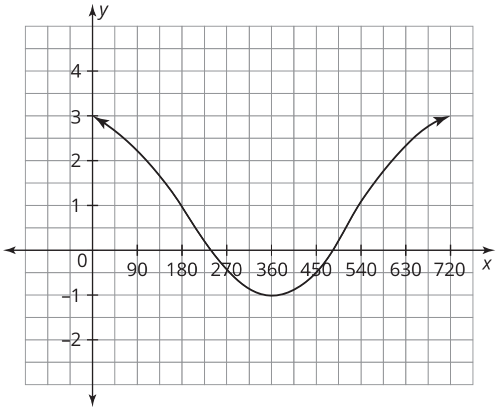
Review

1. Ahmed is riding his bike. The tires on the bike have a diameter of 28 inches. He runs over a screw, but is able to keep riding the bike. Assume the tire rotates clockwise and the screw is at an angle of 0° when it is at ground level. The graph shows the height of the screw above the ground as a function of the angle of the screw.



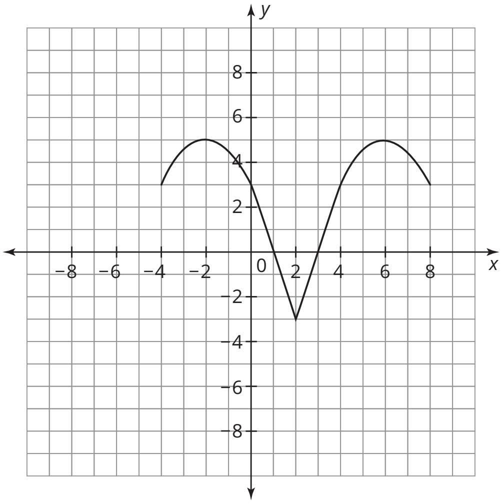
   a. Determine the amplitude of the function.
   b. Determine the period of the function.
   c. Determine the height of the screw when the screw angle is 630°.

2. Consider the periodic function shown, with x in degrees.



   a. Determine the amplitude of the function.
   b. Determine the period of the function.
   c. Determine the value of the function when $x = 900°$.

3. Write the equations of the two relations used to create this bird in terms of the function $f(x) = x^2$. Include any restrictions on the domains.



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4

The Sines They Are A-Changin'

Transformations of Sine and Cosine Functions

Warm Up

Determine the output for each function when $x = \frac{\pi}{2}$.

1. $y = 2 \sin x$
2. $y = \cos x - 3$
3. $y = \sin(2x)$

Learning Goals

- Transform the graphs of the sine and cosine functions.
- Determine the amplitude, frequency, and phase shift of transformed functions.
- Graph transformed sine and cosine functions using a description of the period, phase shift, and amplitude.

Key Terms

- frequency
- phase shift

You have graphed the basic sine and cosine functions and reasoned that adding 2π to the argument of each function translates the function onto itself. How do the *A*-, *B*-, *C*-, and *D*-values in the transformation function form, $g(x) = Af(B(x - C)) + D$ affect the graphs of the basic sine and cosine functions?

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GETTING STARTED

The Sines They Are *A*-Changin'

The table shows the characteristics of the graphs of the sine and cosine functions.

	$y = \sin x$	$y = \cos x$
y-intercept	(0, 0)	(0, 1)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
Minimum Output Value	-1	-1
Maximum Output Value	1	1
Amplitude	1	1
Midline	$y = 0$	$y = 0$

Recall that transformations performed on any function $f(x)$ to form a new function $g(x)$ can be described by the transformation function form.

$$g(x) = Af(B(x - C)) + D$$

Ask

In general, what effect does multiplying a function $y = f(x)$ by a constant, A, have on the graph of the function?

1. Which characteristics of the transformed function $y = A \sin x$ differ from those of the basic function $y = \sin x$ if $|A| > 0$? Which characteristics remain the same? Explain your predictions.

	$y = A \sin x$	
	Will Change	**Won't Change**
y-intercept		
Domain		
Range		
Period		
Minimum Output Value		
Maximum Output Value		
Amplitude		
Midline		

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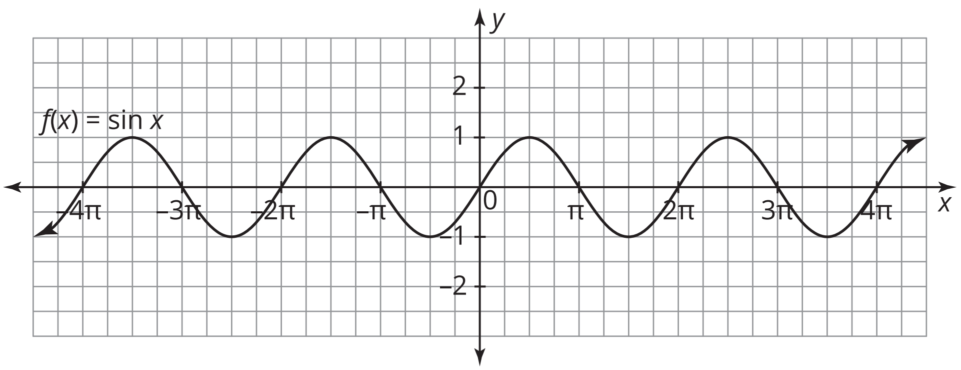
ACTIVITY 4.1

Amplitude

Let's investigate how the A-value affects the graph of $y = \sin x$.

1. A graph of the function $f(x) = \sin x$ is shown. Sketch the graphs of the functions $g(x) = 2 \sin x$ and $h(x) = \frac{1}{2} \sin x$ on the same coordinate plane.



Ask yourself:

How is each value of the basic function affected by the transformation?

2. What similarities and differences do you notice about the three functions with respect to their periods, intercepts, and maximum and minimum values?

3. How do your graphs of the transformed functions compare with your predictions in the Getting Started?

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Remember:

The amplitude of a sine or cosine function is one-half the absolute value of the difference between the maximum and minimum values of the function.

4. Determine the maximum, minimum, and amplitude of each function you graphed.

a. $g(x) = 2 \sin x$

b. $h(x) = \frac{1}{2} \sin x$

5. Determine the maximum, minimum, and amplitude of each cosine function.

a. $f(x) = \cos x$

b. $g(x) = 3 \cos x$

c. $h(x) = \frac{1}{4} \cos x$

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ACTIVITY

4.2 Period and Frequency

Let's consider what effect multiplying the argument of a sine or cosine function by a constant, B, has on the graph of the function. The transformed function can be written as $y = \sin(Bx)$ or $y = \cos(Bx)$

1. Which characteristics of the transformed function $y = \cos(Bx)$ differ from those of the basic function $y = \cos x$ if $|B| > 0$? Which characteristics remain the same? Explain your predictions.

	$y = \cos(Bx)$	
	Will Change	Won't Change
y-intercept		
Domain		
Range		
Period		
Minimum Output Value		
Maximum Output Value		
Amplitude		
Midline		

Ask yourself:

In general, what effect does multiplying the argument of a function $y = f(x)$ by a constant, B, have on the graph of the function?

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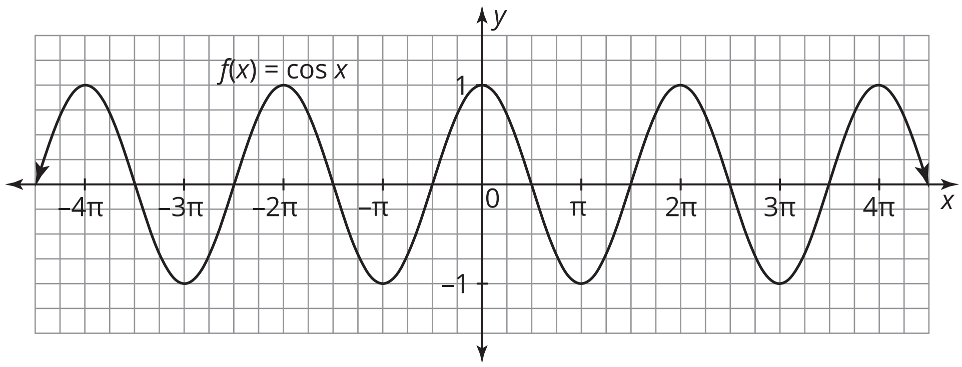
2. A graph of the function $f(x) = \cos x$ is shown. Sketch the graphs of the functions $g(x) = \cos(4x)$ and $h(x) = \cos\left(\frac{1}{2}x\right)$ on the same coordinate plane.



3. What similarities and differences do you notice about the three functions with respect to their periods, intercepts, and maximum and minimum values?

4. How do your graphs of the transformed functions compare with your predictions in Question 1?

5. How do the equations of the functions you graphed relate to the similarities and differences in the graphs?

Recall that the period of a periodic function is the length of the smallest interval over which the function repeats.

Think about:

What pattern do you see in the periods and the *B*-values?

6. Determine the period of each function from the graph.

a. $f(x) = \cos(x)$

b. $g(x) = \cos(4x)$

c. $h(x) = \cos\left(\frac{1}{2}x\right)$

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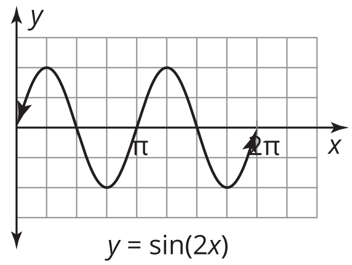
The B-value stretches or compresses a periodic function horizontally, so changes to the B-value have an effect on the period of the function.

Worked Example

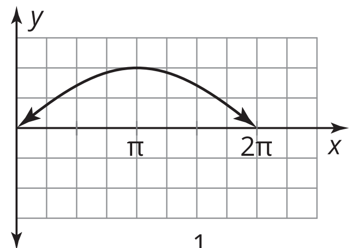
The basic function $y = \sin x$ has a period of 2π radians. You can determine the period of the transformed sine function by interpreting the B-value.

When the B-value is 2, there are 2 repetitions of the function in the original period, so the period is $\frac{1}{|B|} \cdot 2\pi$, or $\frac{1}{2} \cdot 2\pi = \pi$ radians.

When the B-value is $\frac{1}{2}$, there is $\frac{1}{2}$ of a repetition of the function in the original period, so the period is $\frac{1}{|B|} \cdot 2\pi$, or $2 \cdot 2\pi = 4\pi$ radians.



$y = \sin(2x)$



$y = \sin(\frac{1}{2}x)$

7. Write an expression to describe the period of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$.

Frequency is related to the period of the function. The **frequency** of a periodic function is the reciprocal of the period and specifies the number of repetitions of the graph of a periodic function per unit.

8. Write an expression to describe the frequency of the functions $y = \sin(Bx)$ and $y = \cos(Bx)$. Explain your reasoning.

9. Determine the period and frequency of each sine function.

a. $f(x) = \sin(3x)$

b. $g(x) = \sin\left(\frac{2}{3}x\right)$

c. $h(x) = \sin\left(\frac{1}{4}x\right)$

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ACTIVITY
4.3

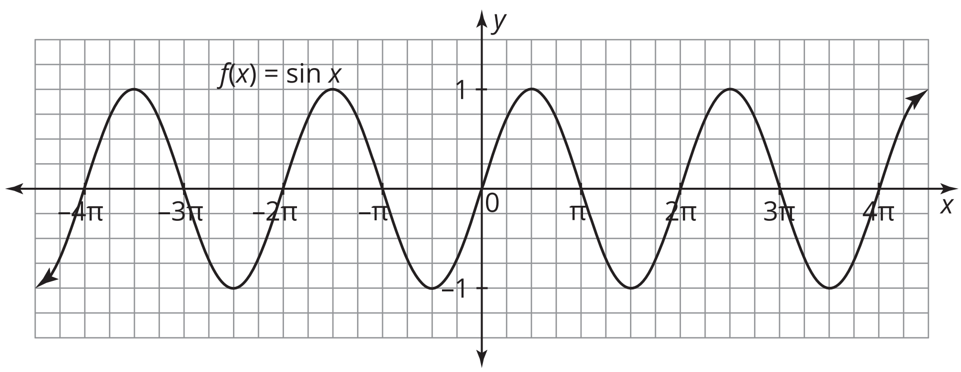
Phase Shifts and More

Now consider what effect subtracting a constant, C, from the argument of a sine or cosine function has on the graph of the function. The transformed function can be written as $y = \sin(x - C)$ or $y = \cos(x - C)$.

1. Sketch graphs of the functions shown over the domain $-4\pi \leq x \leq 4\pi$.

a. $g(x) = \sin\left(x + \frac{\pi}{2}\right)$

b. $h(x) = \sin(x - \pi)$



2. What similarities and differences do you notice about the three functions in terms of their maximums, minimums, periods, and amplitudes?

3. How do the equations of the functions you graphed relate to the similarities and differences in the graphs?

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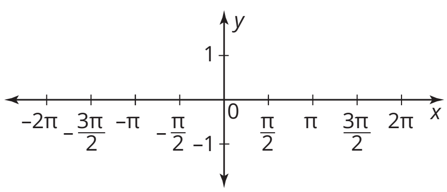
Transforming a periodic function by subtracting a *C*-value from the argument of the function results in horizontal translations of the function. These transformations act just as they have on other functions you have studied. For periodic functions, horizontal translations are called **phase shifts**.

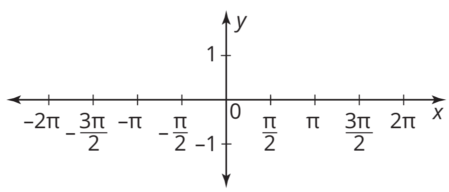
4. Predict the effect of adding a constant, *D*, to a sine or cosine function $y = f(x)$.

5. Use what you know about transformations to sketch the graph of each function.

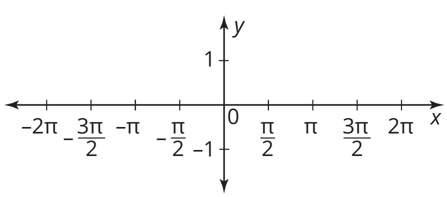
a. $y = -\sin x$



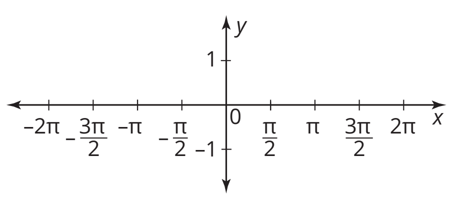
b. $y = \sin(-x)$



c. $y = -\cos x$



d. $y = \cos(-x)$



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e. Compare and contrast the graphs you sketched. What do you notice?

6. Complete the table to describe the graph of each function as a transformation of $y = f(x)$.

Sine or Cosine Function	Equation Information	Description of Transformation of Sine of Cosine Graph	Effect on Period, Amplitude, Midline, Phase Shift
$y = Af(x)$	$\lvert A\rvert > 1$	vertical stretch by a factor of $\lvert A\rvert$ units	
	$0 < \lvert A\rvert < 1$	vertical compression by a factor of $\lvert A\rvert$ units	
	$A < 0$	reflection across the x-axis	
$y = f(Bx)$	$\lvert B\rvert > 1$	horizontal compression by a factor of $\frac{1}{\lvert B\rvert}$	
	$0 < \lvert B\rvert < 1$	horizontal stretch by a factor of $\frac{1}{\lvert B\rvert}$	
	$B < 0$	reflection across the y-axis	
$y = f(x - C)$	$C > 0$	horizontal shift right C units	
	$C < 0$	horizontal shift left C units	
$y = f(x) + D$	$D > 0$	vertical shift up D units	
	$D < 0$	vertical shift down D units	

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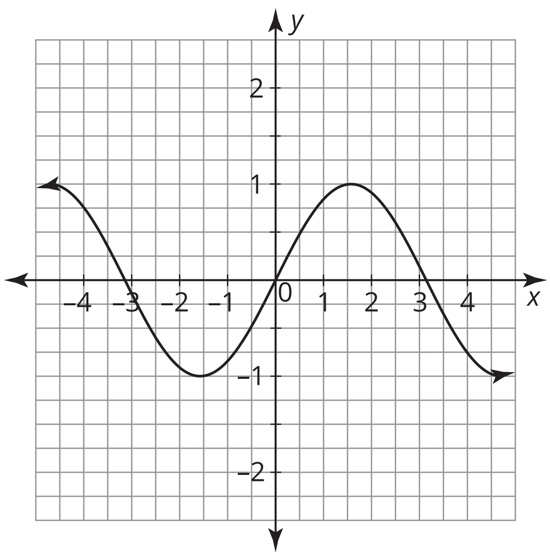
NOTES

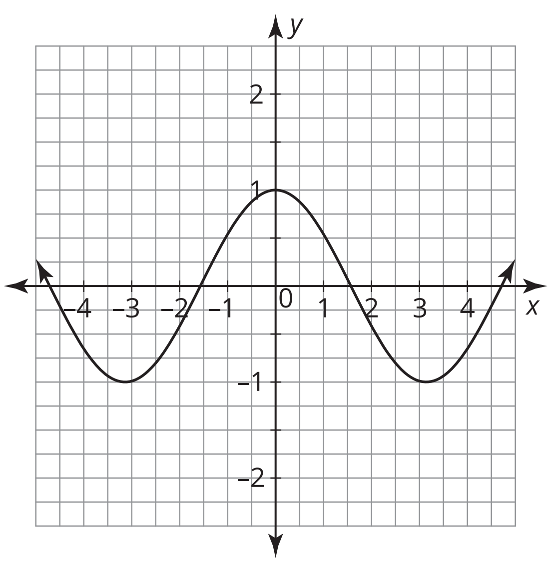
TALK the TALK

Shifting Perspectives

1. Identify the function graphed.



2. Identify the function graphed.



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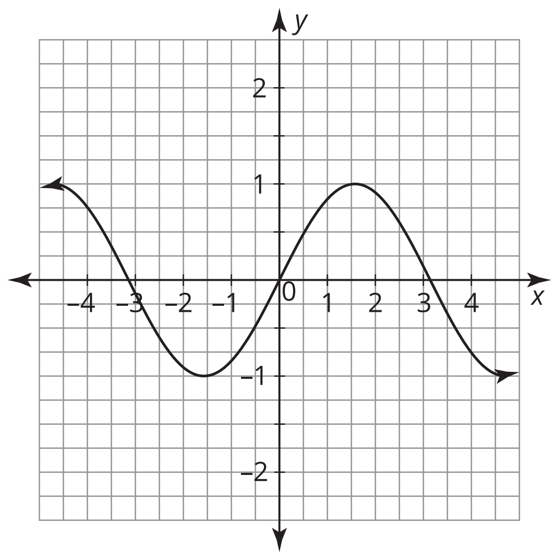
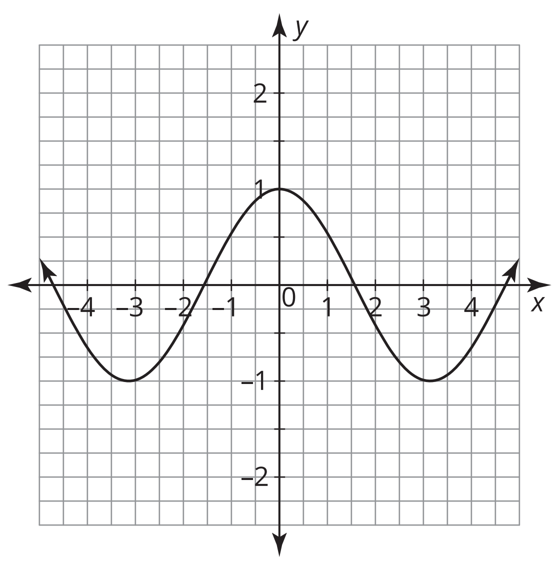
NOTES

3. **The graph shows $y = \sin x$. Rewrite the sine function as two different transformed cosine functions.**



4. **The graph shows $y = \cos x$. Rewrite the cosine function as two different transformed sine functions.**



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Assignment

Write

Write the word(s) that best completes each statement.

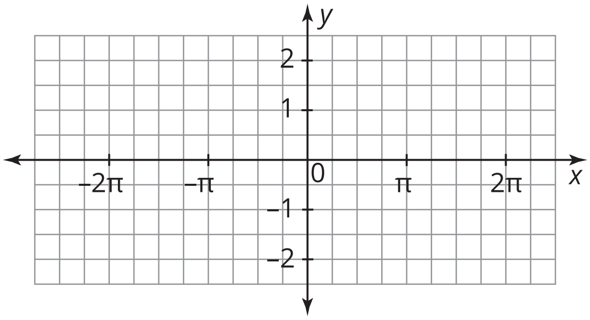
1. The ______________ of a periodic function is the reciprocal of the period and specifies the number of repetitions of the graph of a periodic function per unit.
2. For periodic functions, a horizontal translation is called a ______________.

Remember

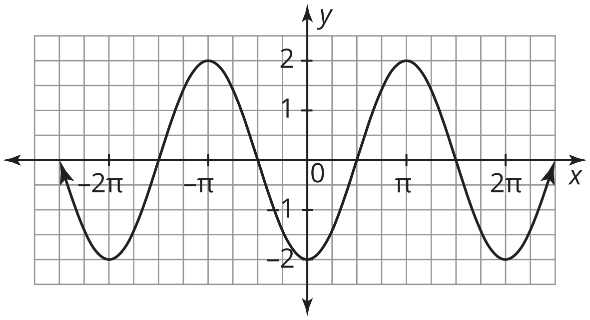
Given the transformed functions $y = A\sin(B(x - C)) + D$ and $y = A\cos(B(x - C)) + D$, the A-value affects the range, minimum and maximum output values, and the amplitude of the basic function, the B-value affects the period and frequency of the basic function, the C-value is interpreted as the phase shift, and the D-value affects the midline.

Practice

1. To create the function $m(x)$, the function $f(x) = \sin x$ is first reflected across the x-axis. Then, the amplitude is increased to 1.5 and the period was changed to π radians.
   a. Graph the function $m(x)$.
   b. Write the function $m(x)$.



2. Consider the given graph of a trigonometric function.
   a. Write the function $g(x)$ that matches the given graph if the function $g(x)$ is a transformation of the function $f(x) = \sin x$.
   b. Determine the amplitude, period, frequency, and phase shift of $g(x)$.
   c. Write the function $h(x)$ that matches the given graph if the function $h(x)$ is a transformation of the function $f(x) = \cos x$.
   d. Determine the amplitude, period, frequency, and phase shift of $h(x)$.



3. The function $f(x) = \sin x$ has been horizontally stretched by a factor of 2 and shifted up 3 units to create the function $t(x)$. Write the function $t(x)$.
4. The function $f(x) = \cos x$ has been vertically compressed by a factor of $\frac{1}{4}$ and shifted $\frac{3\pi}{2}$ radians to the right to create the function $p(x)$. Write the function $p(x)$.

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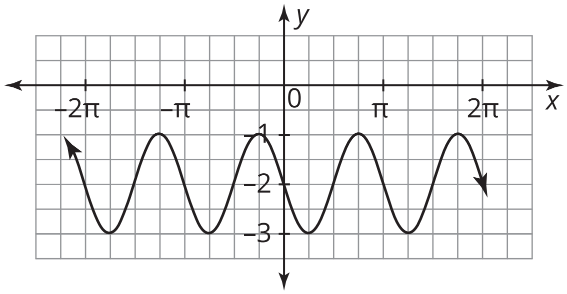
Stretch

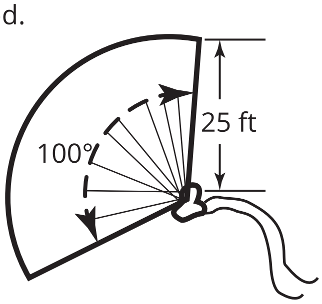
1. Consider the given graph of a trigonometric function.
   a. Write the function $g(x)$ if $g(x)$ is a transformation of $f(x) = \sin x$.
   b. Determine the amplitude, period, frequency, and phase shift of $g(x)$.
   c. Write the function $h(x)$ if $h(x)$ is a transformation of $f(x) = \cos x$.
   d. Determine the amplitude, period, frequency, and phase shift of $h(x)$.



2. The tangent of θ is the ratio of sin θ to cos θ. Use your knowledge of the unit circle and the sine and cosine functions to determine tan θ for each value of θ.
   a. 0 radians
   b. $\frac{\pi}{6}$ radians
   c. $\frac{2\pi}{3}$ radians
   d. $\frac{5\pi}{4}$ radians
   e. $\frac{3\pi}{2}$ radians
   f. $\frac{11\pi}{6}$ radians

Review

1. A satellite in a low Earth orbit completes one orbit every 90 minutes. The satellite follows a circular path with its center at the center of the earth. The satellite is at an altitude of 160 kilometers. The radius of the earth is 6371 kilometers.
   a. Determine the angle of rotation, in radians, that corresponds to a 15-minute time period.
   b. Determine the distance traveled by the satellite in a 15-minute time period.
2. Archie is watering his lawn with a sprinkler attached to a hose. The outer path of the spray is an arc with the center at the sprinkler and a central angle of 100°. The distance from the sprinkler to any point on the outer path is 25 feet. Determine the central angle of the outer path in radians and the length of the outer path of the spray.



3. An owner of two large commercial buildings is trying to make the buildings more environmentally friendly. She has the building's bathroom facilities revamped with more modern energy saving equipment. She also places signs in the buildings encouraging the occupants to conserve water. On the first day after the building reconstruction is complete, Building A used 21,150 gallons of water and Building B used 24,325 gallons of water. For the remaining 29 days of the first month, Building A's water usage decreased by 0.5% each day while Building B's water usage decreased by 0.75% each day.
   a. Determine the total amount of water used by each building during the first month. Round decimals to the nearest hundredth.
   b. The cost of water for the state Building A is located in is \$.00785 per gallon. After day 15 of the first month after reconstruction, the state raised its rates to \$.00795. Determine how much the owner paid for water for the first month after reconstruction was done at the building.

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5

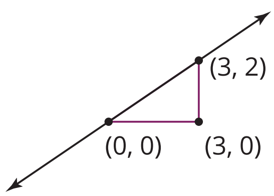
Farmer's Tan

The Tangent Function

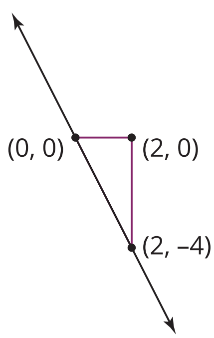
Warm Up

Determine the slope of each line.

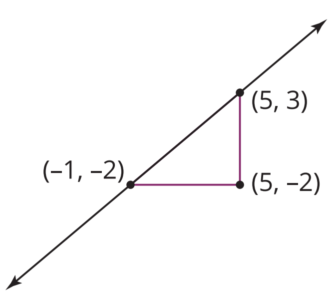
1.



2.



3.



Learning Goals

- Build the graph of the tangent function using the ratio $\frac{\sin\theta}{\cos\theta}$.
- Analyze characteristics of the tangent function, including period and asymptotes.
- Calculate values of the tangent function for common angles.
- Identify transformations of the tangent function.
- Use symmetry to determine the sine, cosine, and tangent of angle measures given by θ, $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ in radians.

Key Term

- tangent function

You have learned about the tangent ratio as defined in a right triangle. How can you build the tangent function using a unit circle?

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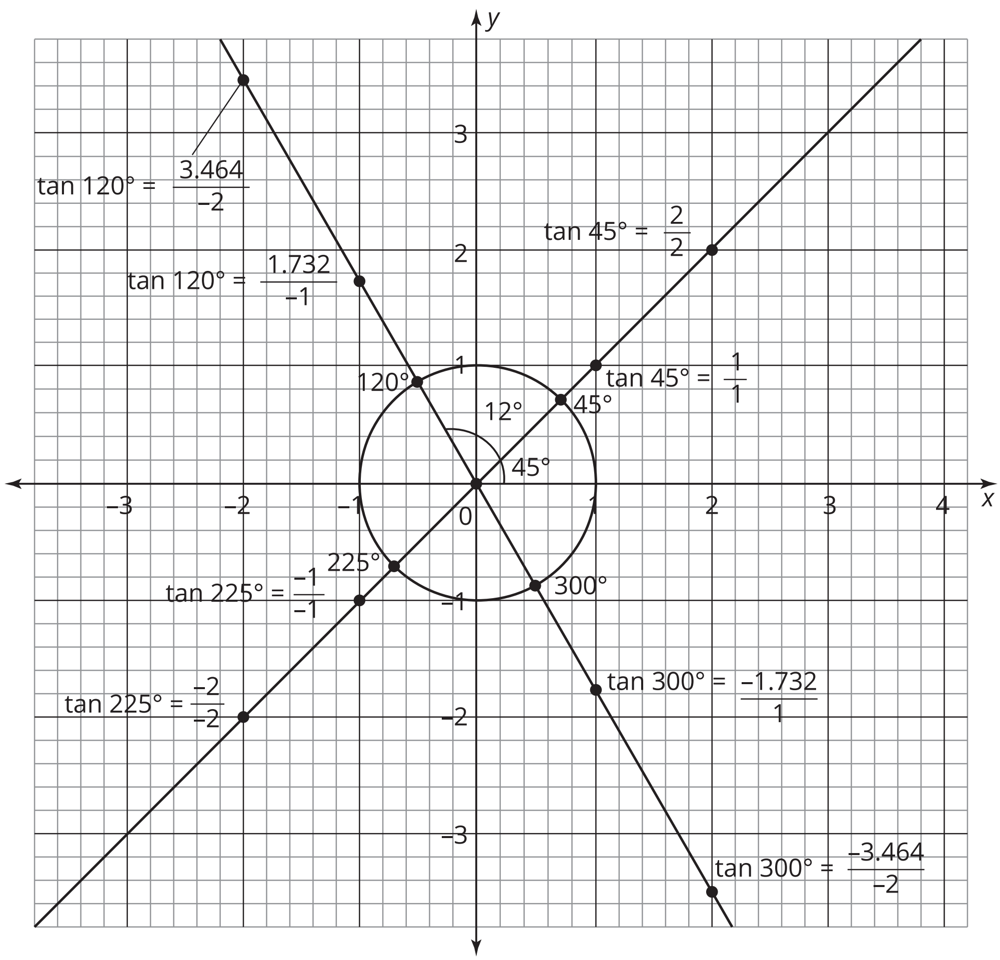
GETTING STARTED

Let's Not Forget About Similarity

You know that the tangent of an acute angle in a right triangle is the ratio of the opposite side to the adjacent side. You can also determine the tangents of angles directly from the coordinate plane, using similar right triangles and the unit circle centered at the origin.



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1. **Consider the unit circle shown.**

   a. **How are similar triangles used to determine the tangent of a 45° angle?**

   b. **How are similar triangles used to determine the tangent of a 120° angle?**

2. **Explain why the tangent of a 45° angle is the same as the tangent of a 225° angle, and why the tangent of a 120° angle is the same as the tangent of a 300° angle.**

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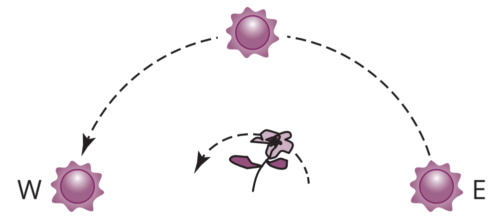
ACTIVITY **5.1**

Constructing the Tangent Function

Many plants exhibit the ability to track sunlight as the Sun moves across the sky during the day. This movement is called phototropism.



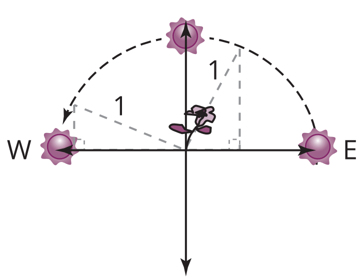
Imagine that flowers face due east in the morning where the Sun rises, and they track the sunlight throughout the day as the Sun moves directly overhead and then to the west.

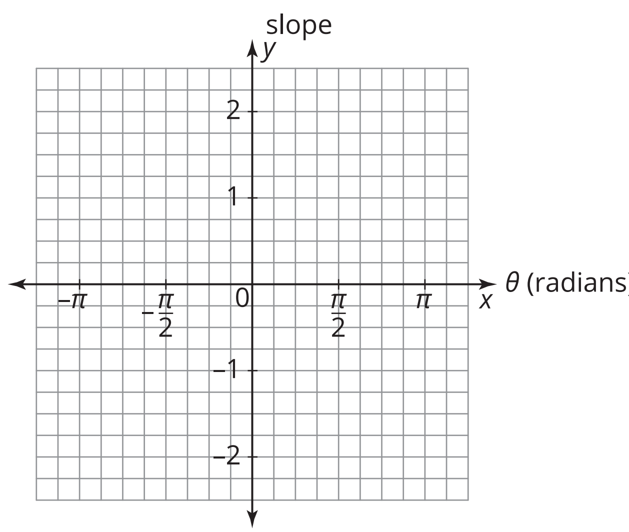
1. Suppose you track the slope of the angle that a flower makes with the ground over the course of a day. Create a visual interpretation of the changing slope on the graph as you answer each question.

a. What is the value of the slope at $\theta = 0$ radians, $\frac{\pi}{4}$ radian, and $\frac{\pi}{2}$ radians on the unit circle? Explain your reasoning.

Ask yourself:

Is the slope defined for all values of theta?





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b. Describe the value of the slope as θ increases from 0 radians and approaches $\frac{\pi}{2}$ radians.

c. What is the value of the slope at $\theta = \frac{3\pi}{4}$ radians and π radians? Explain how you determined each value.

Think about:

Can you use symmetry to determine other slope values?

d. Describe the value of the slope as θ decreases from π radians and approaches $\frac{\pi}{2}$ radians.

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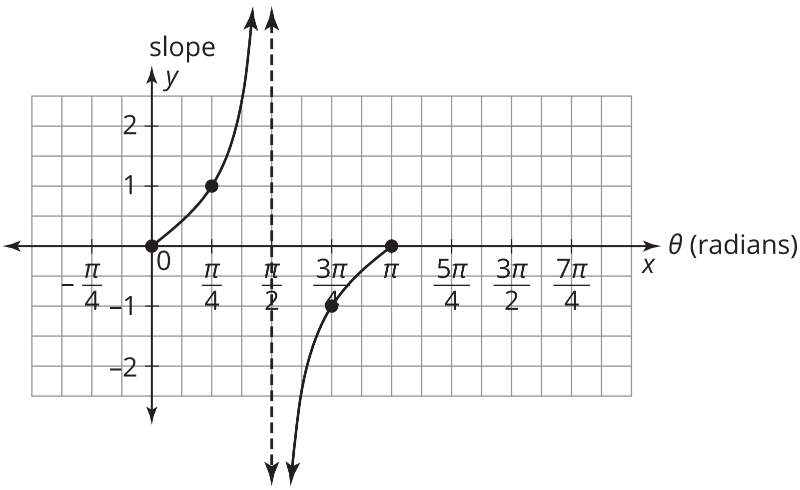
At night, flowers do not continue to follow the Sun after it sets.

But suppose the flower represents the terminal ray of a central angle in standard position. Let's continue to model the change in the slope of the terminal ray as it traverses the unit circle.

2. Use your answers to Question 1 and what you know about symmetry to answer each question.



a. Complete the graph of the slope values from $-\frac{\pi}{2}$ radians to 2π radians.

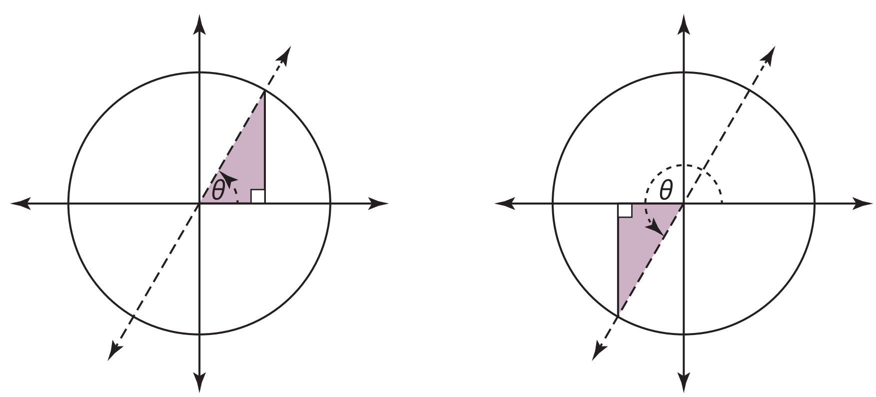
b. For what value(s) of θ is the slope equal to 0?

c. For what value(s) of θ is the slope undefined?

To help you think about slope values, you can remember that the terminal ray is a part of a line.

Worked Example

The triangles shown in the diagram are congruent. The hypotenuse of each triangle represents a terminal ray of a central angle with measure θ.



The slope of the terminal ray shown in Quadrant I is the same as the slope of the terminal ray shown in Quadrant III, because both rays are part of the same line. Both slopes are positive.

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3. **Use the worked example and your completed graph in Question 2 to answer each question.**

What special triangle has a "rise" and "run" that are equal?

   a. **For what value(s) of θ is the slope equal to 1?**

   b. **For what value(s) of θ is the slope equal to −1?**

4. **Use your completed graph to answer each question.**

   a. **Explain why the relation you graphed is a function.**

   b. **Is the function periodic? If so, determine the period of the function. If not, explain why not.**

5. **James said that the period of the function is 2π radians because the graph starting at 2π radians repeats the same values as it does starting at 0 radians. Juli says that the period of the function is $\frac{\pi}{2}$ radians, because there is an asymptote at multiples of $\frac{\pi}{2}$ radians.**

   Who is correct? Explain your reasoning.

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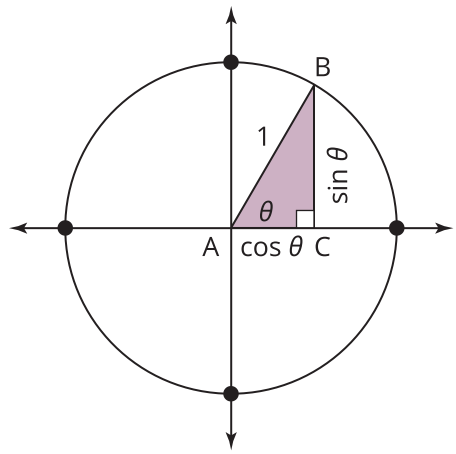
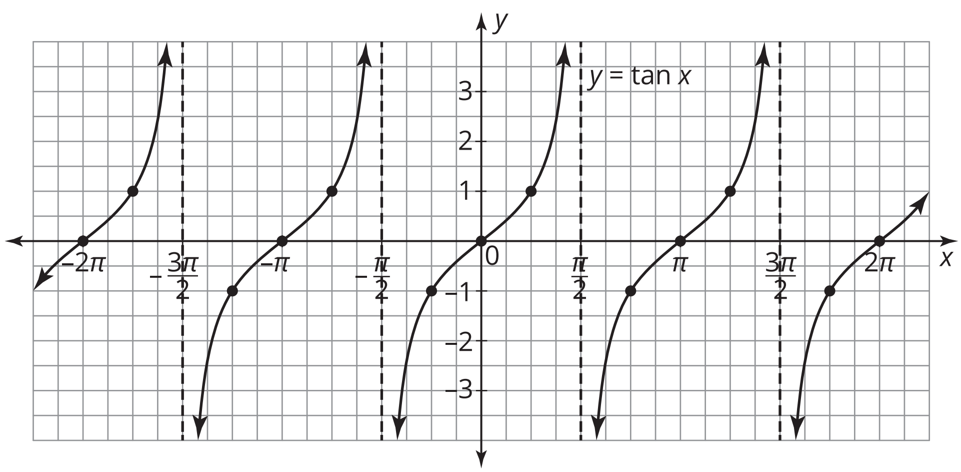
ACTIVITY 5.2

Exploring the Properties of the Tangent Function

The function that you graphed in the previous problem is the **tangent function**. Recall that the tangent ratio (tan) is the ratio of the lengths of the opposite side and the adjacent side in a right triangle. The tangent ratio is equal to the slope of the hypotenuse, which represents the terminal ray of the central angle on the unit circle.





1. How can you write the tangent function in terms of sine and cosine, using the unit circle?

2. In which quadrants is the tangent function positive and negative? Explain your reasoning. Record this information on the Sine, Cosine, and Tangent on the Unit Circle diagram located at the end of the lesson.

3. Use what you know about rational functions to describe the discontinuities in the graph of the tangent function.

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4. **What is the value of $\tan\left(\frac{n\pi}{2}\right)$ for any odd integer value of n?**

5. **Identify the periodicity identity for the tangent function. Explain your reasoning.**

Remember:

You have identified periodicity identities for both the sine function and cosine function.

6. **The table shows some of the characteristics of the sine and cosine functions that you have identified. Complete the table for the tangent function.**

	$y = \sin x$	$y = \cos x$	$y = \tan x$
y-intercept	(0, 0)	(0, 1)	
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	
Range	$[-1, 1]$	$[-1, 1]$	
Period	2π	2π	
Minimum Output Value	-1	-1	
Minimum Output Value	1	1	
Amplitude	1	1	
Midline	$y = 0$	$y = 0$	

7. **Complete the Sine, Cosine, and Tangent on the Unit Circle diagram located at the end of the lesson by labeling the tangent values for each angle measure.**

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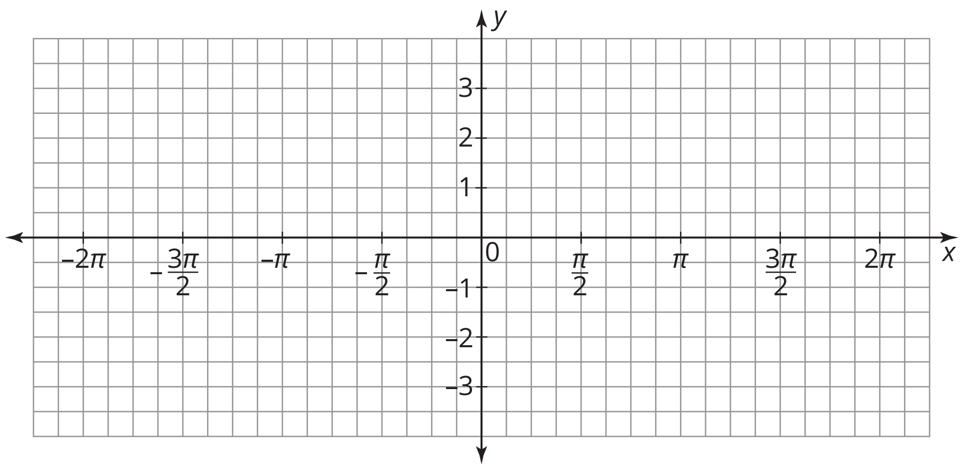
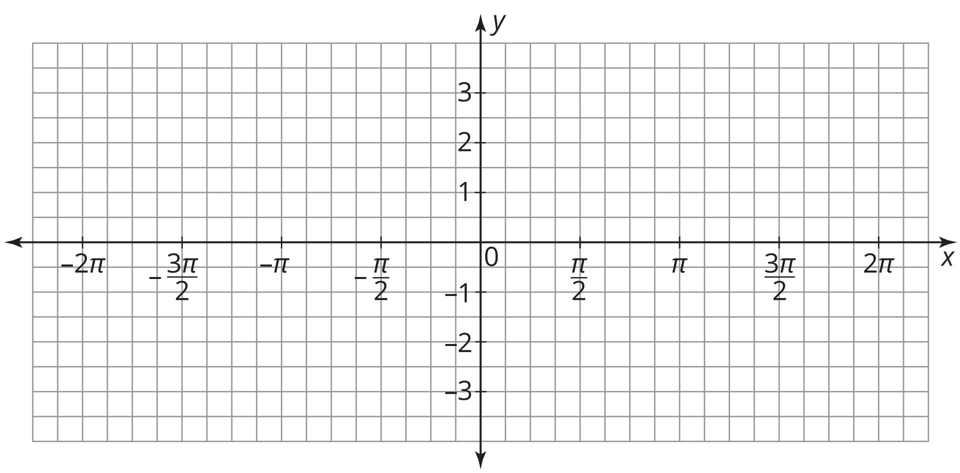
ACTIVITY 5.3 Transformations of the Tangent Function

1. Use what you know about transformations to sketch each graph.

a. $f(x) = -\tan x$



b. $g(x) = \tan(-x)$



c. What do you notice about the graphs in parts (a) and (b)?

Ask yourself:

What happens to the slope of a line when you reflect it across the *x*-axis or the *y*-axis?

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2. Match each equation with its corresponding graph. Explain your reasoning.

a. $y = \tan\left(\frac{1}{2}x\right)$

b. $y = \tan\left(x + \frac{\pi}{2}\right)$

c. $y = \frac{1}{20}\tan x + 1$

d. $y = 2\tan x$

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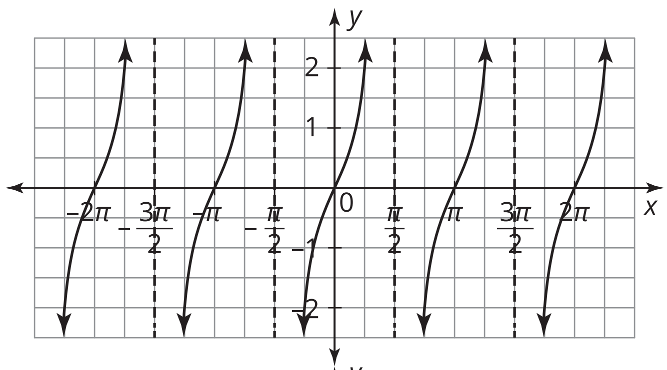

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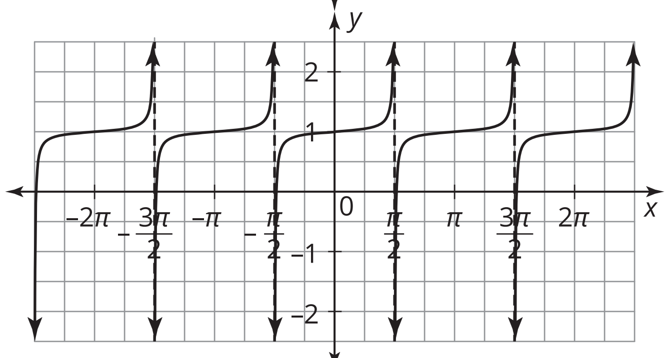

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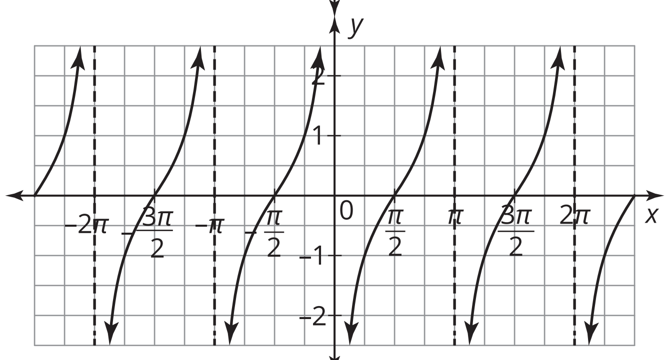

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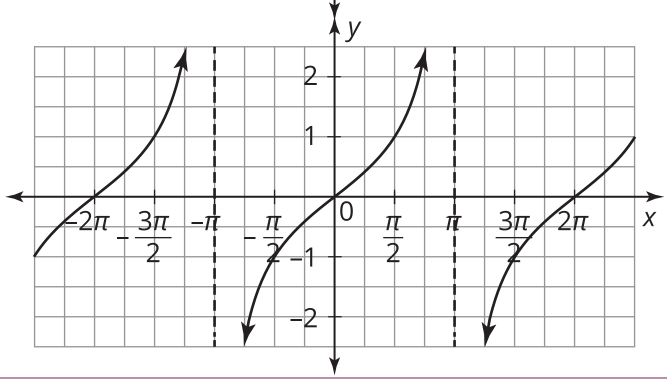

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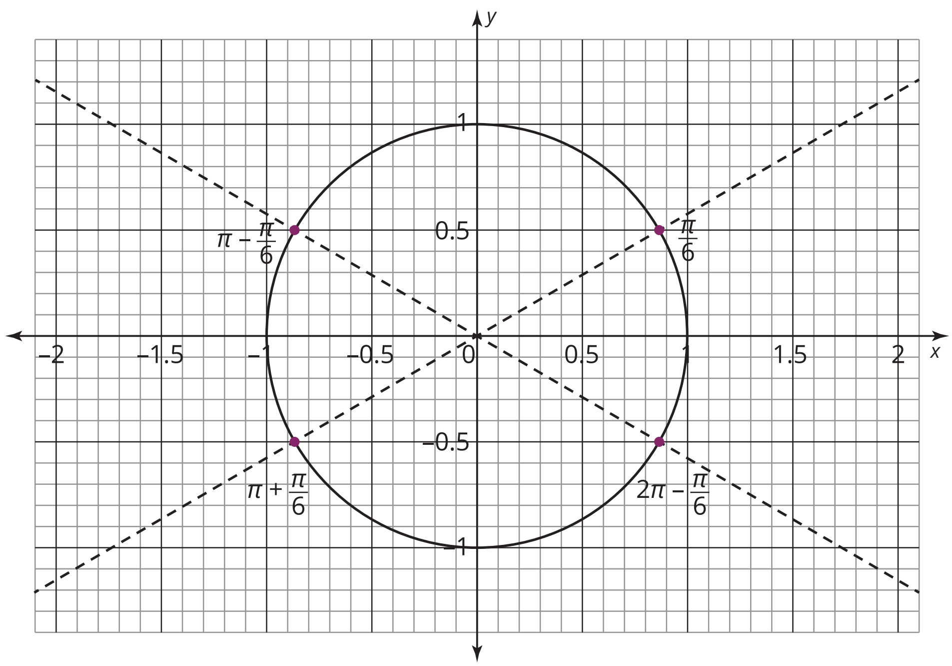
NOTES

TALK the TALK

Oh, Also, Don't Forget About Symmetry

The locations of $\frac{\pi}{6}$, $\pi - \frac{\pi}{6}$, $\pi + \frac{\pi}{6}$, and $2\pi - \frac{\pi}{6}$ are plotted on the unit circle shown.



1. Describe the central angle measure associated with each location, and determine the sine, cosine, and tangent of each angle measure.

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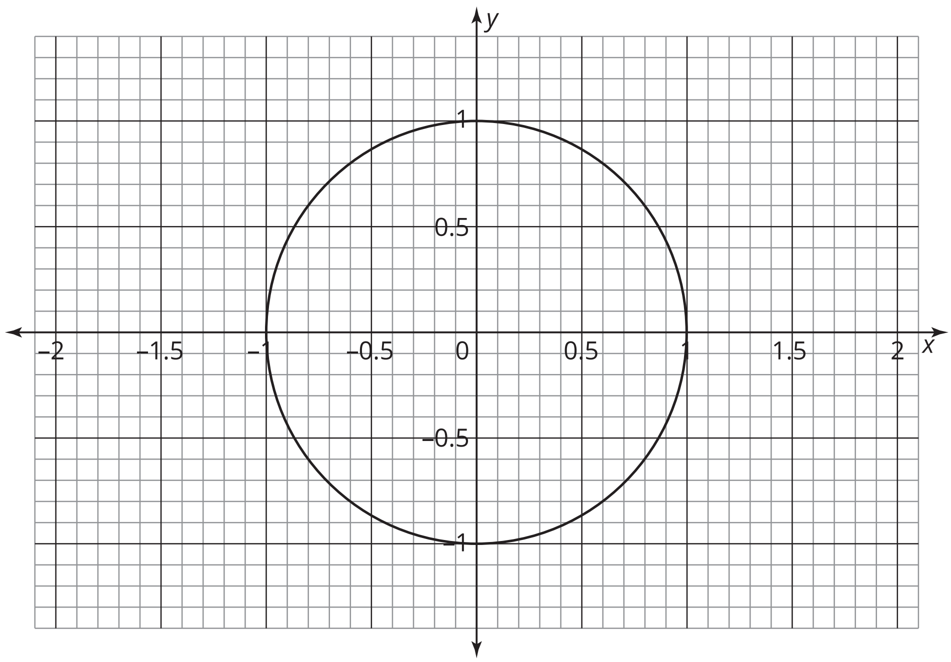
NOTES

2. **Identify the locations of $\frac{\pi}{4}$, $\pi - \frac{\pi}{4}$, $\pi + \frac{\pi}{4}$, and $2\pi - \frac{\pi}{4}$ on the unit circle. Determine the sine, cosine, and tangent of each central angle measure.**



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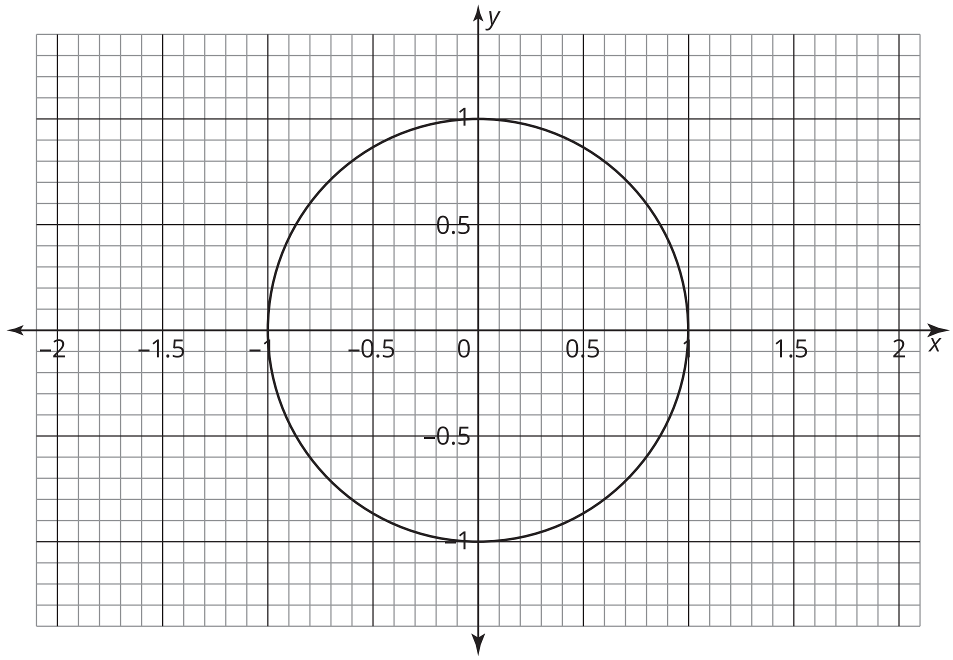
NOTES

3. **Identify the locations of $\frac{\pi}{3}$, $\pi - \frac{\pi}{3}$, $\pi + \frac{\pi}{3}$, and $2\pi - \frac{\pi}{3}$ on the unit circle. Determine the sine, cosine, and tangent of each central angle measure.**



4. **Explain how you can use symmetry to determine the values of trigonometric functions at certain input values.**

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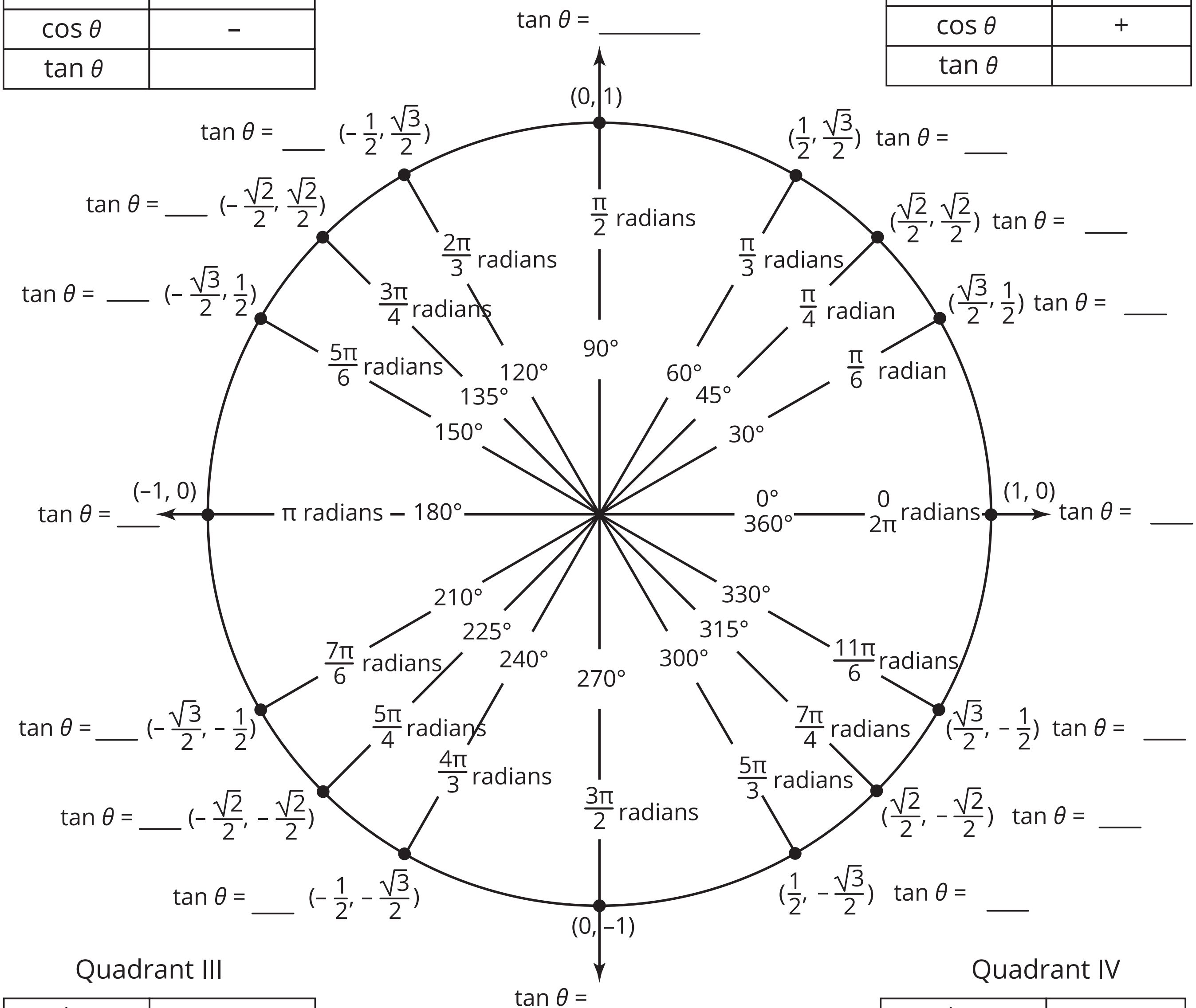
Sine, Cosine, and Tangent on the Unit Circle

Quadrant II

sin θ	+
cos θ	–
tan θ	

Quadrant I

sin θ	+
cos θ	+
tan θ	



Quadrant III

sin θ	–
cos θ	–
tan θ	

Quadrant IV

sin θ	–
cos θ	+
tan θ	

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Assignment

Write

Explain how the tangent function is related to the sine and cosine functions.

Remember

The tangent function is positive when sin θ and cos θ have the same sign, and the tangent function is negative when sin θ and cos θ have different signs.

The period of the function $y = \tan x$ is π radians.

The periodicity identity for the tangent function is written as $\tan(x + \pi) = \tan x$.

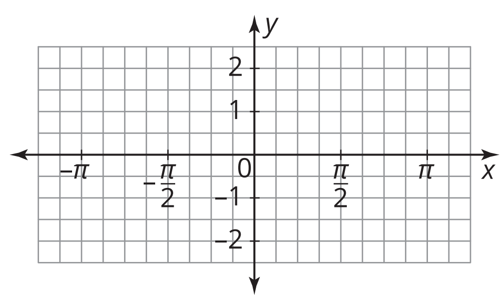
Practice

1. Consider Manuel's incorrect work. Identify the errors and correctly determine $\tan\left(\frac{7\pi}{3}\right)$.

Manuel

$$\begin{aligned}\tan\left(\frac{7\pi}{3}\right) &= \tan\left(\frac{6\pi}{3} + \pi\right)\\ &= \tan\left(\frac{6\pi}{3}\right)\\ &= \tan(2\pi)\\ &= 0\end{aligned}$$

2. Given $\tan\theta = -\sqrt{3}$. Determine 2 values for θ such that $\theta < 0$ and 2 values for θ such that $\theta > 2\pi$.
3. Given $\tan\theta = 1$. Determine 2 values for θ such that $\theta < 0$ and 2 values for θ such that $\theta > 2\pi$.
4. Determine $\tan\left(\frac{13\pi}{6}\right)$.
5. Determine $\tan\left(\frac{11\pi}{4}\right)$.
6. To create the function $g(x)$, the function $f(x) = \tan x$ was reflected across the x-axis and shifted $\frac{\pi}{2}$ radians to the right.
   a. Graph the function $g(x)$.
   b. Write the function $g(x)$.



7. The function $f(x) = \tan x$ has been horizontally stretched by a factor of 4 and shifted down 3 units to create the function $m(x)$. Write the function $m(x)$.

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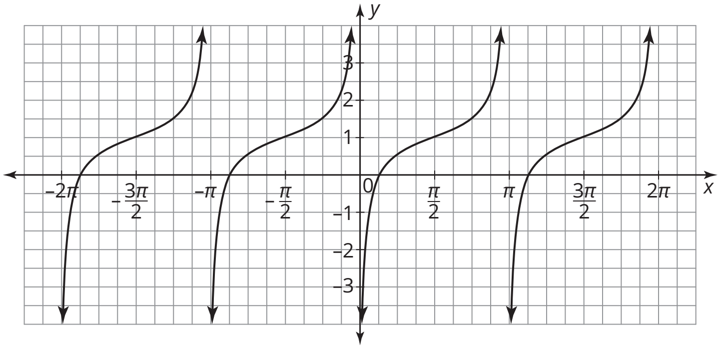
Stretch

1. Consider the graph of a trigonometric function $g(x)$.



   Write the function $g(x)$, a transformation of the function $f(x) = \tan x$.

2. Determine the values of θ in radians that would make each equation true for $0 \le \theta \le 2\pi$.

   a. $\cos \theta = 1$ b. $\cos \theta = 0$ c. $\cos \theta = \frac{\sqrt{3}}{2}$ d. $\cos \theta = -\frac{1}{2}$ e. $\cos(\theta) + 1 = 0$

Review

1. Determine θ and $\cos \theta$ when $\sin \theta = -\frac{\sqrt{2}}{2}$ and $\cos \theta$ is negative. Restrict values for θ such that $0 \le \theta \le 2\pi$.
2. Determine $\sin\left(\frac{15\pi}{4}\right)$.
3. Priscilla and Theo both bought farms the same year, and each dedicated one acre of the land for growing strawberries. The first year of operation, Priscilla's strawberry field yielded 22,000 pounds and Theo's field yielded 19,500 pounds. Since that first year, Priscilla's yield of strawberries has decreased by 1.5% each year while Theo's yield of strawberries has increased by 1.0% each year.
   a. Whose farm yielded more strawberries in the 7th year of production? Round decimals to the nearest hundredth.
   b. Which of the 2 farms had the biggest yield over the first 7 years? Round decimals to the nearest hundredth.
4. Use long division to determine whether $x + 2$ is a factor of $2x^4 + 5x^3 + 5x^2 + 10x + 8$. Show your work.

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Trigonometric Relationships Summary

KEY TERMS

- periodic function
- period
- standard position
- initial ray
- terminal ray
- amplitude
- midline
- theta (θ)
- unit circle
- radian
- sine function
- cosine function
- trigonometric function
- periodicity identity
- frequency
- phase shift
- tangent function

LESSON 1 A Sense of Déjà Vu

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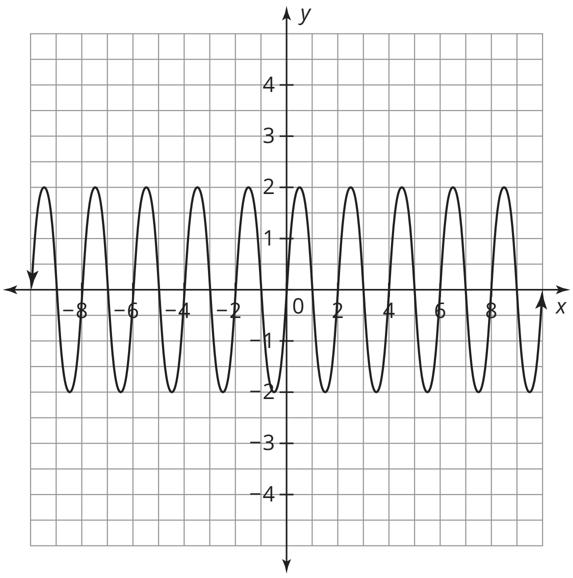
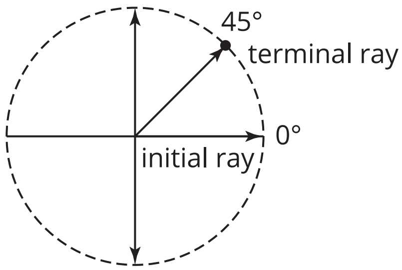
A **periodic function** is a function whose values repeat over regular intervals. The **period** of a periodic function is the length of the smallest interval over which the function repeats.

For example, the smallest interval over which this function repeats is 2. So, the period of the function is 2.

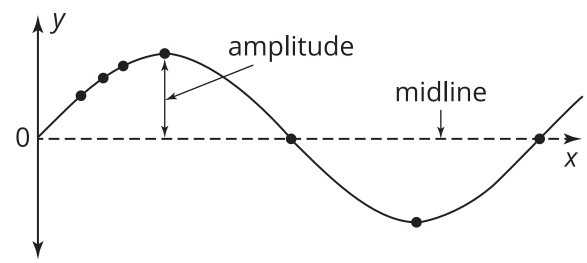


An angle is in **standard position** when the vertex is at the origin and one ray of the angle is on the *x*-axis. The ray on the *x*-axis is the **initial ray**, and the other ray is the **terminal ray**.

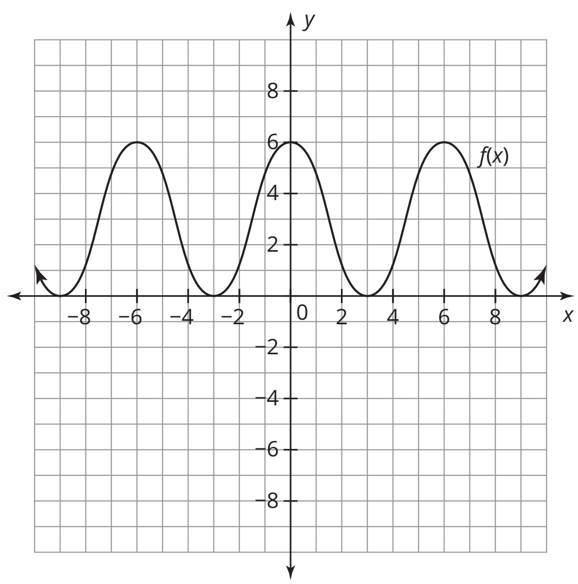


The measure of an angle in standard position is the amount of rotation from the initial ray to the terminal ray. When the rotation is counterclockwise, the angle measure is positive. When the rotation is clockwise, the angle measure is negative.

The **amplitude** of a periodic function is one-half the absolute value of the difference between the maximum and minimum values of the function.
The **midline** of a periodic function is a reference line whose equation is the average of the minimum and maximum values of the function.



For example, the maximum value of the graphed function $f(x)$ is 6, and the minimum value is 0. So, the amplitude is $\frac{1}{2}|6 - 0|$ or 3. The midline is $y = \frac{6 + 0}{2} = 3$.



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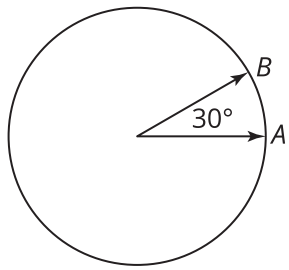
LESSON

2 The Knights of the Round Table

You can identify the central angle measures of a circle in standard position using the symbol **theta**, written as θ. For example, a central angle measure of 30° can be written as $\theta = 30°$.



A powerful way to measure central angles of a circle is to identify arc lengths of the circle in terms of the radius of a unit circle. A **unit circle** has a radius of 1 unit.

The unit that describes the measure of an angle theta, θ, in terms of the radius and arc length of a unit circle is called a **radian**. The ratio of the intercepted arc length of a central angle to the radius is the measure of the central angle in radians. There are $\frac{2\pi}{r}$ or 2π, radians in 360° and $\frac{\pi r}{r}$, or π, radians in 180°.

The formulas you can use to convert measures from radians to degrees and degrees to radians are shown.

$$\text{Radians to Degrees: } x \text{ radians} \cdot \frac{180°}{\pi \text{ radians}}$$

$$\text{Degrees to Radians: } x \text{ degrees} \cdot \frac{180°}{\pi \text{ radians}}$$

For example, to convert 3 radians to degrees, use the formula x radians $\cdot \frac{180°}{\pi \text{ radians}}$.

$$\begin{aligned} x \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} &= 3 \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} \\ &= \frac{540°}{\pi} \\ &\approx 171.89° \end{aligned}$$

The degree measure of an angle with a radian measure of 3 radians is approximately 171.89°.

To convert $\theta = 225°$ to radians, use the formula x degrees $\cdot \frac{180°}{\pi \text{ radians}}$.

$$\begin{aligned} x \text{ degrees} \cdot \frac{180°}{\pi \text{ radians}} &= 225\ 3° \cdot \frac{180°}{\pi \text{ radians}} \\ &= \frac{225\pi}{180 \text{ radians}} \\ &\approx \frac{5\pi}{4 \text{ radians}} \end{aligned}$$

The radian measure of an angle with a degree measure of 225° is $\frac{5\pi}{4}$ radians.

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LESSON 3

What Goes Around

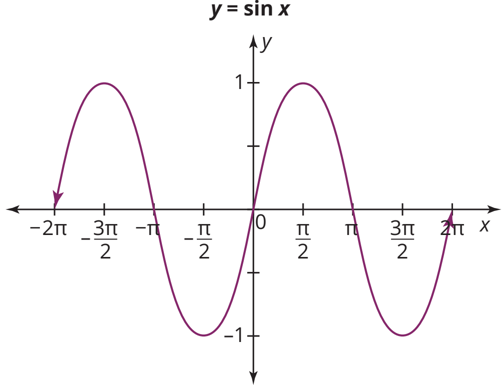
In a unit circle, the ordered pair of the point where the terminal ray of an angle intersects the circle is the value of the cosine and sine of the angle.

The table lists angle measures and their sine and cosine values based on the unit circle.

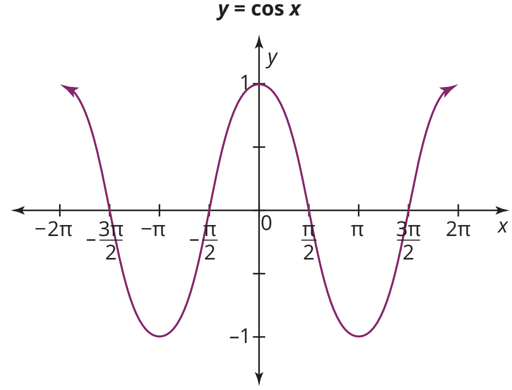
Angle Measure (θ)		cos (θ)	sin (θ)	Angle Measure (θ)		cos (θ)	sin (θ)
radians	degrees			radians	degrees		
0	0°	1	0	$\frac{7\pi}{6}$	210°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{\pi}{6}$	30°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{5\pi}{4}$	225°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{4}$	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{4\pi}{3}$	240°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	60°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3\pi}{2}$	270°	0	-1
$\frac{\pi}{2}$	90°	0	1	$\frac{5\pi}{3}$	300°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{2\pi}{3}$	120°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{7\pi}{4}$	315°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{4}$	135°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{11\pi}{6}$	330°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{5\pi}{6}$	150°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	2π	360°	1	0
π	180°	-1	0				

The **sine function** and **cosine function** are periodic **trigonometric functions**. Each of these trigonometric functions takes angle measures (θ values) as inputs and outputs real number values, which correspond to coordinates of points on the unit circle.

y* = sin *x



y* = cos *x



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The period of the sine function is 2π radians, and the period of the cosine function is 2π radians. Thus, you can write two periodicity identities:

- $\sin(x + 2\pi) = \sin(x)$
- $\cos(x + 2\pi) = \cos(x)$

Each of these is called a **periodicity identity** because they are each based on the period of the function, 2π.

LESSON 4

The Sines They Are A-Changin'

The table shows the characteristics of the graphs of the sine and cosine functions

	$y = \sin(x)$	$y = \cos(x)$
y-intercept	(0, 0)	(0, 1)
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[-1, 1]$	$[-1, 1]$
Period	2π	2π
Minimum Output Value	-1	-1
Maximum Output Value	1	1
Amplitude	1	1
Midline	$y = 0$	$y = 0$

In general, multiplying the sine or cosine function by a constant, A, will change the minimum and maximum values and the range of the function. The transformed function can be written as $y = A\sin(x)$ or $y = A\cos(x)$.

Multiplying the argument of a sine or cosine function by a constant, B, affects the period of the function. The transformed function can be written as $y = \sin(Bx)$ or $y = \cos(Bx)$.

The formula to determine the period of function is $\frac{1}{|B|} \cdot 2\pi$.

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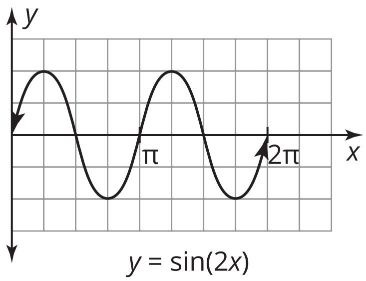
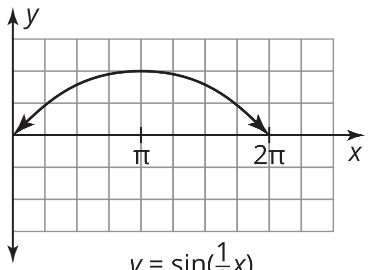
For example, consider the sine functions shown.

When the B-value is 2, the period is $\frac{1}{|B|} \cdot 2\pi$, or $\frac{1}{2} \cdot 2\pi = \pi$ radians.



$y = \sin(2x)$

When the B-value is $\frac{1}{2}$, the period is $\frac{1}{|B|} \cdot 2\pi$, or $2 \cdot 2\pi = 4\pi$ radians.



$y = \sin(\frac{1}{2}x)$

Frequency is related to the period of the function. The **frequency** of a periodic function specifies the number of repetitions of the graph of a periodic function per unit. It is equal to the reciprocal of the period of the function or $\frac{|B|}{2\pi}$, where B is the coefficient of the argument.

For example, the frequency of the function $y = 2 \sin (4\pi) + 5$ is $\frac{|4|}{2\pi}$, or $\frac{2}{\pi}$.

Transforming a periodic function by subtracting a C-value from the argument of the function results in horizontal translations of the function. The transformed function can be written as $y = \sin(x - C)$ or $y = \cos(x - C)$. For periodic functions, horizontal translations are called **phase shifts**.

Transforming a periodic function by adding a D-value to the function results in vertical translations of the function. The transformed function can be written as $y = \sin(x) + D$ or $y = \cos(x) + D$.

All the transformations can be represented by the equations $y = A \sin(Bx - C) + D$ or $y = A \cos(Bx - C)$.

For example, consider the function $y = 2 \sin (4x - 2) + 3$.

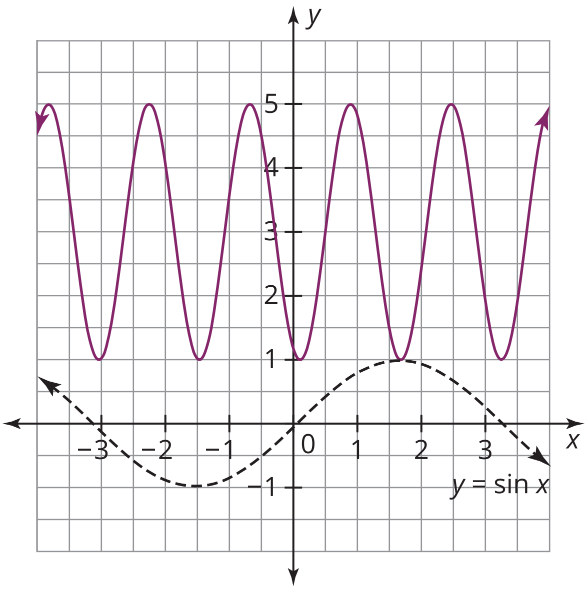
$A = 2, B = 4, C = 2$ and $D = 3$

Amplitude $= A = 2$

Period $= \frac{1}{|B|} \cdot 2\pi = \frac{1}{|4|} \cdot 2\pi$
$= \frac{1}{2}\pi = \frac{\pi}{2}$

Phase Shift $= \frac{C}{B} = \frac{2}{4} = \frac{1}{2}$

Vertical Shift $= D = 3$



As you can see in the graph, the transformed function has a larger amplitude, smaller period, and is translated.

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LESSON 5

Farmer's Tan

The **tangent function** is the ratio of the sine to the cosine, or $\tan(x) = \frac{\sin x}{\cos x}$. To evaluate a tangent function for a given angle, use the unit circle to determine the sine and cosine of the angle. Then, calculate the ratio.

For example, to evaluate the tangent of $\frac{2}{3}\pi$ radians, use the relationship between the tangent function and the sine and cosine functions.

$$\sin\left(\frac{2}{3}\pi\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$$

$$\begin{aligned}\tan\left(\frac{2}{3}\pi\right) &= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} \\ &= \frac{\sqrt{3}}{2} \cdot (-2) \\ &= -\sqrt{3}\end{aligned}$$

So the tangent of $\frac{2}{3}\pi$ radians is $-\sqrt{3}$.

The characteristics of the basic tangent functions are shown in the table.

	$y = \tan x$
y-intercept	(0, 0)
Domain	$\Re$, $x \neq \frac{\pi}{2} + \pi n$ for all integers n
Range	$(-\infty, \infty)$
Period	π
Minimum Output Value	n/a
Maximum Output Value	n/a
Amplitude	undefined
Midline	$y = 0$

The tangent function has asymptotes whenever $\cos x = 0$.

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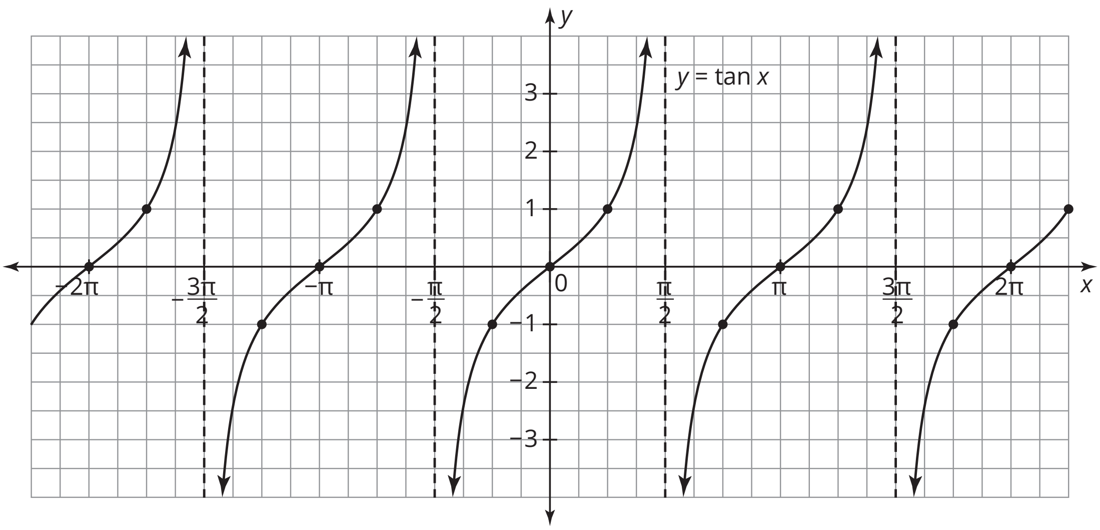

Transformations to the *A*-, *B*-, *C*-, and *D*-values of the tangent function affect the graph of the function in the same way as transformations to the *A*-, *B*-, *C*-, and *D*-values of sine and cosine functions.

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TOPIC 2

Trigonometric Equations

A sextant is an instrument for calculating latitude and longitude. It is based on trigonometry: the earth is a giant circle, and the angular position of the sun (or a given star) above the horizon defines the necessary triangle.

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Module 4: Investigating Periodic Functions

TOPIC 2: TRIGONOMETRIC EQUATIONS

Now that students are equipped with an understanding of trigonometric functions and their key characteristics, they are able to model periodic phenomena that occur in the real-world. They are introduced to strategies for solving trigonometric equations, and then use their knowledge of the unit circle, radian measures, and the graphical behaviors of trigonometric functions to solve sine, cosine, and tangent equations. Students apply all that they have learned to model various situations with trigonometric functions, including circular motion. They model real-world problems with sine or cosine functions and interpret the key characteristics in terms of the problem situation.

Where have we been?

Students have had significant experience solving equations. They know how to solve an equation using graphs, the Properties of Equality, square roots, inverse operations, and factoring. In a previous course, students used the inverses of sine, cosine, and tangent to solve simple trigonometric equations. From their work in the previous topic, they understand periodic functions, their key characteristics, and the periodicity identities.

Where are we going?

The ability to solve trigonometric equations is important in modeling any phenomena involving circular motion. Students who pursue mathematics at the post-secondary level will use trigonometric equations extensively in calculus classes, where they will calculate derivatives to determine the velocity and acceleration of particles in motion. This knowledge is used in physics and electrical engineering.

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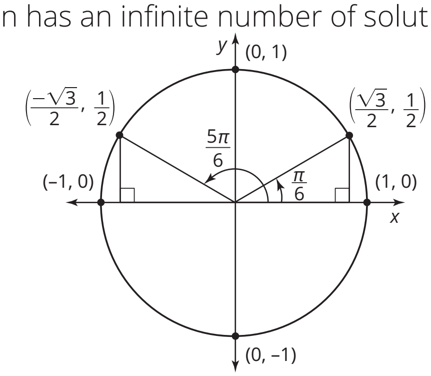
Trigonometric Values in Different Quadrants

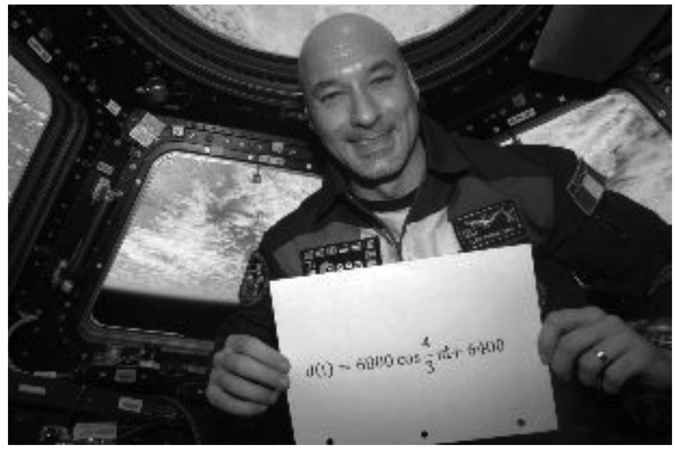
What is the value of x when $\sin x = \frac{1}{2}$?

On the unit circle, you can see that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$. So, $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. When the domain is restricted to $0 \leq x \leq 2\pi$, these are the only 2 solutions to the equation. When there are no domain restrictions, the equation has an infinite number of solutions.



Trig's in Space!

On October 14, 2013, astronaut Luca Parmitano sent us this picture from aboard the International Space Station (ISS). During his mission, Luca became the first Italian astronaut to take part in a spacewalk. The trigonometric function he is holding represents the position of the ISS in orbit over time.



Talking Points

Trigonometric equations can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Let sin $x = \frac{1}{2}$. If $0° < x° < 90°$, what is cos x?

The variable x represents an acute angle measure of a right triangle in the first quadrant. The sine of x (opposite over hypotenuse) is $\frac{1}{2}$. This means that the adjacent side of the right triangle, using the Pythagorean Theorem, is $\sqrt{3}$.

Thus, the cosine of x is $\frac{\sqrt{3}}{2}$.

Key Terms

trigonometric equation

A trigonometric equation is an equation in which the unknown is associated with a trigonometric function. The number of solutions of a trigonometric equation can vary depending on how the domain of the function is restricted.

inverse sine ($\sin^{-1}$)

The inverse sine function is used to determine solutions to sine equations.

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1

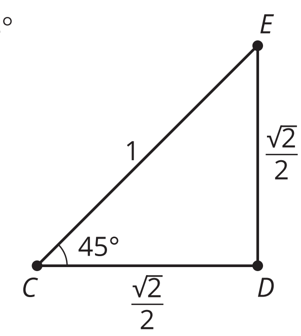
$\text{Sin}^2\ \theta$ Plus $\text{Cos}^2\ \theta$ Equals 1^2

The Pythagorean Identity

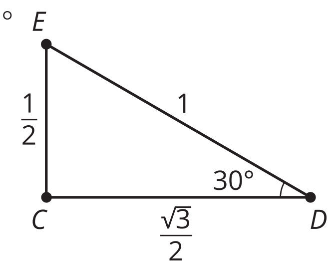
Warm Up

Determine the value for each.

1. sin 45°



2. cos 30°



Learning Goals

- Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
- Use the Pythagorean identity to determine the sine, cosine, or tangent of an angle measure in a given quadrant.

Key Term

- Pythagorean identity

You have learned about right-triangle ratios such as sine, cosine, and tangent and about the Pythagorean Theorem. How is the Pythagorean Theorem related to these trigonometric ratios?

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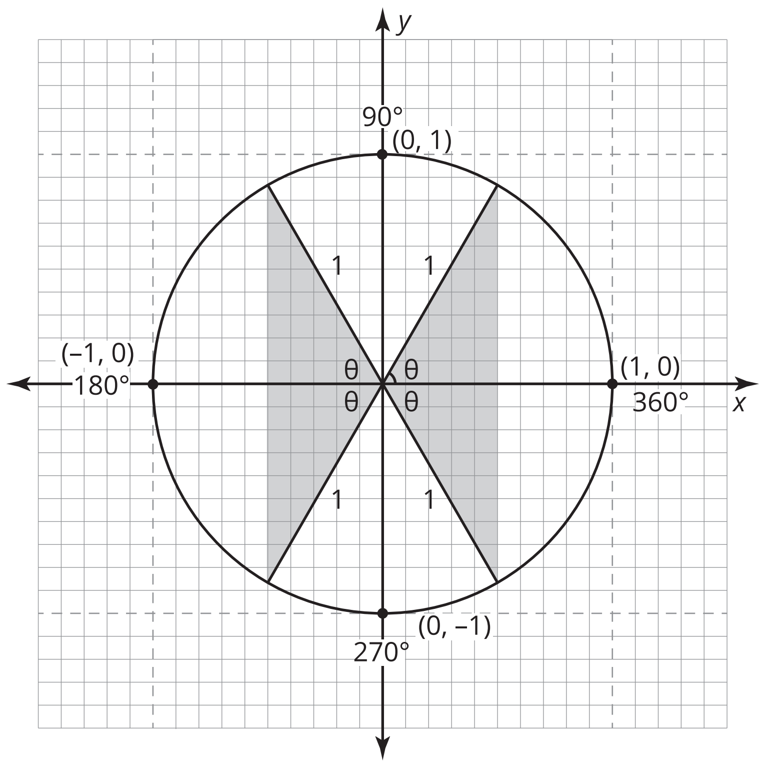
GETTING STARTED

Round the Unit Circle

The trigonometric ratios sine, cosine, and tangent can have different signs, negative or positive, depending on which quadrant of the coordinate plane the angle and right triangle are in.

A unit circle is a circle with a radius of 1.

Consider the unit circle shown.



Think about:

In Quadrant II, the horizontal side of the right triangle has a negative value, and the vertical side has a positive value.

1. Complete the table. Determine the sign of each trigonometric ratio in the given quadrant of the coordinate plane.

	Sign (+/–)			
	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Sine		+		
Cosine		–		
Tangent				

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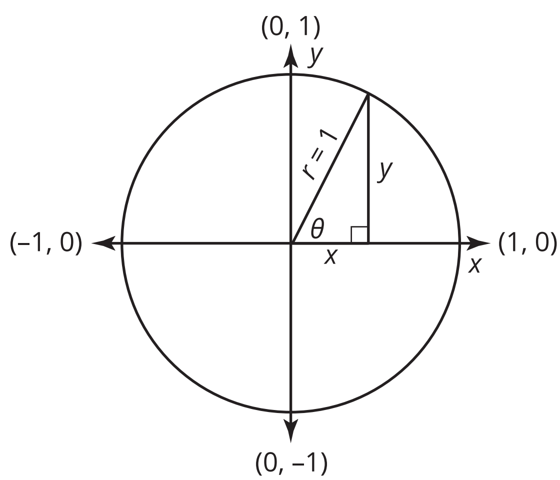
ACTIVITY 1.1 Pythagorean Identity

A **Pythagorean identity** is a trigonometric identity that expresses the Pythagorean Theorem in terms of trigonometric ratios. The basic relationship between the sine and cosine of an angle θ is given by the Pythagorean identity $(\sin\theta)^2 + (\cos\theta)^2 = (1)^2$. The symbol θ represents an angle measure.

This Pythagorean identity can also be written as $\sin^2\theta + \cos^2\theta = 1$.

You can prove this Pythagorean identity using your knowledge of the unit circle and the Pythagorean Theorem.



1. **Demonstrate how the Pythagorean identity follows from the Pythagorean Theorem.**

   a. **Given the unit circle and the angle θ, label the side lengths of the right triangle in terms of $\sin\theta$ and $\cos\theta$.**

   b. **State the Pythagorean Theorem.**

   c. **Use substitution to demonstrate how the Pythagorean identity follows from the Pythagorean Theorem.**

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2. Write the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ in two other forms.

a. Solve for $\sin^2\theta$.

b. Solve for $\cos^2\theta$.

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ACTIVITY 1.2

Determining Sine and Cosine in All Four Quadrants

You can use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ and what you know about solutions in different quadrants to determine values of trigonometric functions.

Worked Example

Determine the exact value of $\cos\theta$ in Quadrant II, given $\sin\theta = \frac{2}{3}$. You can use a Pythagorean identity to determine $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1$$
$$\frac{4}{9} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{5}{9}$$
$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

The solution is $\cos\theta = -\frac{\sqrt{5}}{3}$ because the solution is in Quadrant II $\left(\frac{\pi}{2} \leq \theta \leq \pi\right)$.

Ask yourself:

How does solving this equation relate to solving simple quadratic equations?

1. Given $\sin\theta = \frac{3}{5}$ in Quadrant II, determine $\cos\theta$.

2. Given $\cos\theta = -\frac{12}{13}$ in Quadrant III, determine $\sin\theta$.

3. Given $\cos\theta = \frac{1}{4}$ in Quadrant IV, determine $\sin\theta$.

4. Given $\sin\theta = -\frac{1}{10}$ in Quadrant IV, determine $\cos\theta$.

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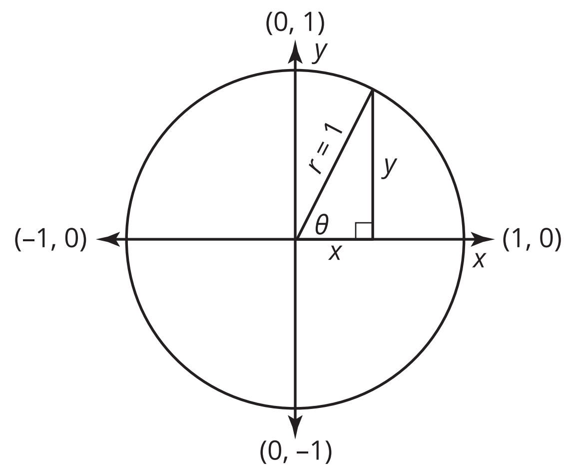
NOTES

TALK the TALK

360 Degrees of Trig

You can also solve for the tangent of an angle in all four quadrants, using what you know about trigonometric ratios.

1. Consider the diagram and how you previously labeled the right triangle. Write tan θ in terms of sine and cosine. Explain your reasoning.



2. Given $\sin\theta = -\frac{1}{4}$ in Quadrant III, determine tan θ.

3. Given $\cos\theta = \frac{9}{10}$ in Quadrant IV, determine tan θ.

4. Given $\sin\theta = \frac{3}{5}$ in Quadrant II, determine tan θ.

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Assignment

Write

Explain in your own words how the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ follows from the Pythagorean Theorem.

Remember

One Pythagorean identity states that $\sin^2 \theta + \cos^2 \theta = 1$. The trigonometric ratios sine, cosine, and tangent can have different signs, negative or positive, depending on which quadrant of the coordinate plane the angle and right triangle are located.

Practice

Use the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ to determine the value of each trigonometric ratio.

1. Given $\sin \theta = \frac{5}{13}$ in Quadrant I, determine $\cos \theta$.
2. Given $\cos \theta = -\frac{7}{25}$ in Quadrant III, determine $\sin \theta$.
3. Given $\sin \theta = -\frac{1}{3}$ in Quadrant IV, determine $\cos \theta$.
4. Given $\cos \theta = -\frac{2}{3}$ in Quadrant II, determine $\sin \theta$.
5. Given $\sin \theta = \frac{1}{6}$ in Quadrant II, determine $\cos \theta$ and $\tan \theta$.

Stretch

1. Rewrite the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ as an identity with $\tan \theta$ and $\sec \theta$. Show your work.
2. Rewrite the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ as an identity with $\cot \theta$ and $\csc \theta$. Show your work.

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Review

1. Given $\tan \theta = -\frac{\sqrt{3}}{3}$. Determine 2 values for θ such that $\theta < 0$ and 2 values for θ such that $\theta > 2\pi$.
2. Determine $\tan\left(\frac{8\pi}{3}\right)$.
3. The function $f(x) = \sin x$ has been vertically stretched by a factor of 4, shifted $\frac{\pi}{2}$ radians to the right, and shifted down 2 units to create the function $v(x)$. Write the function $v(x)$.
4. Solve each logarithmic equation.
   a. $\log(4x - 5) = \log(2x - 1)$
   b. $\log x - \log 2 = 1$

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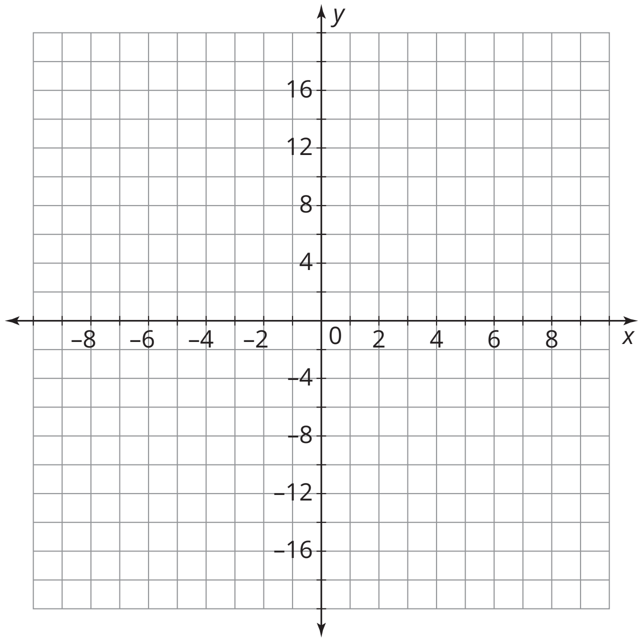
2

Chasing Theta

Solving Trigonometric Equations

Warm Up

Graph the function $f(x) = 2x^2 + 5x - 12$ and determine the zeros.



Learning Goals

- Write and solve trigonometric equations.
- Use periodicity identities to identify multiple solutions to trigonometric equations.
- Solve trigonometric equations using inverse trigonometric functions.
- Solve second-degree trigonometric equations.

Key Terms

- trigonometric equation
- inverse sine ($\sin^{-1}$)
- inverse cosine ($\cos^{-1}$)
- inverse tangent ($\tan^{-1}$)

You have explored trigonometric functions, which take an angle measure as input and output a linear measure. How can you solve a trigonometric equation for the angle measure?

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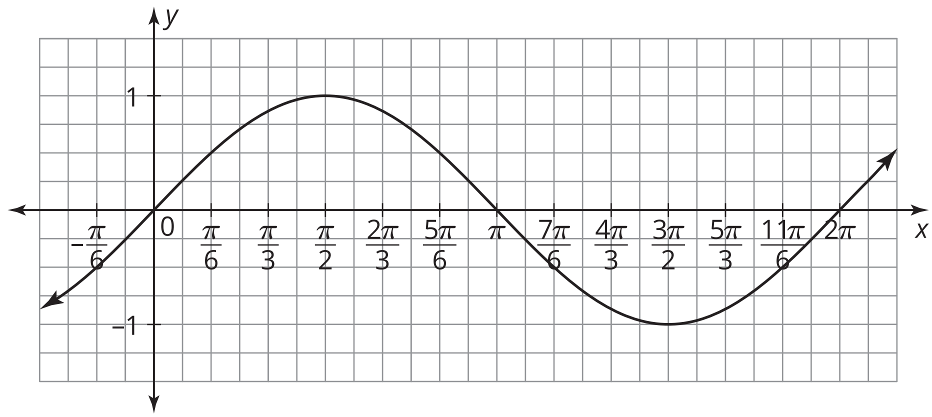
Sine, Sine, Everywhere a Sine

The graph shows the function $f(x) = \sin x$.



1. Draw a horizontal line to approximate the solutions to the equation $\sin x = \frac{1}{2}$. What are the solutions?

2. List the solution(s) of the equation $\sin x = \frac{1}{2}$, given each of the domain restrictions.

a. $0 \le x \le \frac{\pi}{2}$

b. $0 \le x \le 4\pi$

c. $-\pi \le x \le 0$

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ACTIVITY 2.1

Solving Trigonometric Equations

A **trigonometric equation** is an equation in which the unknown is associated with a trigonometric function. The number of solutions of a trigonometric equation can vary depending on how the domain of the function is restricted.

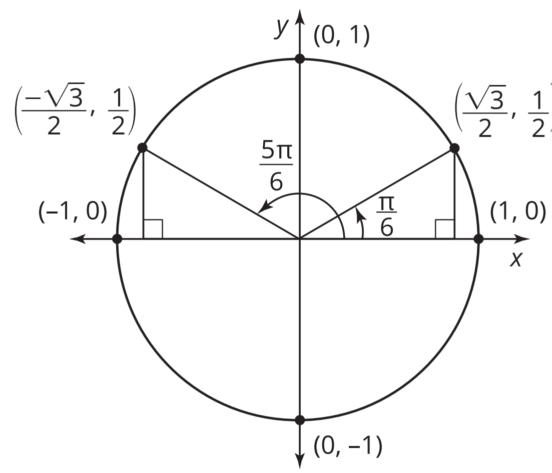
You can solve trigonometric equations using what you already know.

Worked Example

Consider the equation $\sin x = \frac{1}{2}$.

On the unit circle, you can see that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.
So, $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

When the domain is restricted to $0 \le x \le 2\pi$, these are the only 2 solutions to the equation. When there are no domain restrictions, the equation has an infinite number of solutions.



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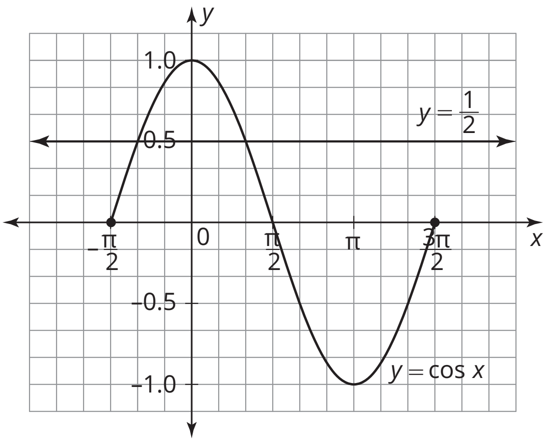

You can also use what you know about the graphs of trigonometric functions to solve trigonometric equations.

Let's consider the equation $\cos x = \frac{1}{2}$.

The equations $y = \cos x$ and $y = \frac{1}{2}$ are graphed on the coordinate plane.

1. Study the graph of $y = \cos x$.

a. Over what domain is the function graphed?

b. Write the solutions to the equation $\cos x = \frac{1}{2}$, given the domain restrictions. Then, plot and label the solutions on the coordinate plane.

Think about:

How can you use reference angles on the unit circle?

2. Write the solution(s) to each equation, given the same domain restrictions in Question 1.

a. $\cos x = 1$

b. $\cos x = 0$

c. $\cos x = -\frac{1}{2}$

d. $\cos x = -1$

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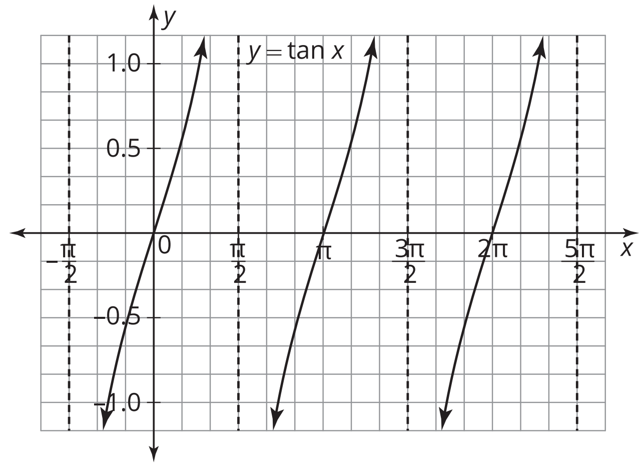
3. Use the graph of $y = \tan x$ over the domain $-\frac{\pi}{2} \leq x \leq \frac{5\pi}{2}$ to solve each equation.

a. $\tan x = 0$



b. $\tan x =$ undefined

You can use what you know about the periods of trigonometric functions to solve trigonometric equations. The periodicity identities you have learned are shown. Adding or subtracting integer multiples of the period for each function ($2\pi n$ or πn) generates solutions to trigonometric equations.

Periodicity Identities	
Sine	$\sin(x + 2\pi n) = \sin x$
Cosine	$\cos(x + 2\pi n) = \cos x$
Tangent	$\tan(x + \pi n) = \tan x$

4. Use a periodicity identity to list 4 solutions to each equation.

a. $\cos x = \frac{\sqrt{2}}{2}$

b. $\tan x = \sqrt{3}$

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ACTIVITY

2.2 Using Inverse Trigonometric Functions to Solve Equations

When a trigonometric equation involves transformations on the basic function, solving the equation requires the same techniques you have used to solve other equations.

Worked Example

Solve $\sqrt{3}\tan x + 5 = 4$ over the domain $0 \le x \le \pi$.

$\sqrt{3}\tan x + 5 = 4$	Subtract 5 from both sides.
$\sqrt{3}\tan x = -1$	Divide both sides by $\sqrt{3}$.
$\tan x = \frac{-1}{\sqrt{3}}$	
$= \frac{-\sqrt{3}}{3}$	Rewrite the radical expression by rationalizing the denominator.
$x = \frac{5\pi}{6}$	Identify the reference angle with a value of $\frac{-\sqrt{3}}{3}$.

1. Identify the solution set of the equation in the worked example over the domain of all real numbers. Show your work.

2. Explain why Fletcher is incorrect.

> **Fletcher**
>
> If $\tan x = \frac{-\sqrt{3}}{3}$ and $\tan(x) = \frac{\sin x}{\cos x}$, then I know that $\sin x = -\sqrt{3}$ and $\cos x = 3$.

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3. **Solve the equation $2 \sin x + \sqrt{3} = 0$ over the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.**

You can use the inverse of each of the trigonometric functions to determine solutions to equations. The **inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$),** and **inverse tangent ($\tan^{-1}$)** functions are used to determine solutions to sine, cosine, and tangent equations, respectively. If you do not recognize the reference angle for the given value of the function, you can use technology.

4. **Sofia and Tyhir each used a graphing calculator to solve the equation $\sin x = \frac{1}{2}$. Explain the differences in their answers.**

Remember:

From a previous worked example, you know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

Sofia

$\sin^{-1}\left(\frac{1}{2}\right) = x$

$x = 0.5235987756$

Tyhir

$\sin^{-1}\left(\frac{1}{2}\right) = x$

$x = 30$

5. **Solve each equation over the domain of all real numbers.**

   a. $-5 + 2\sqrt{3} \cos x = -8$

   b. $5 \sin x + 9 = 3$

   c. $6 \tan x - 4 = -19$

   d. $5 - 8 \cos x = 3$

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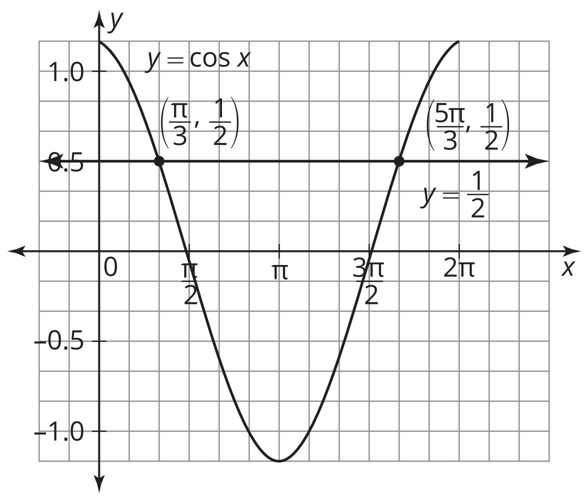
ACTIVITY
2.3

Using the B-Value to Solve Equations

When the B-value is changed from a basic trigonometric function, you must take the change in period into account when determining solutions.

Let's consider the equation $\cos x = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$.

The equations $y = \cos x$ and $y = \frac{1}{2}$ are graphed on the coordinate plane.



The solutions for $\cos x = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$ are $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.

The solutions for $\cos x = \frac{1}{2}$ over the domain for all real numbers are $x = \frac{\pi}{3} + 2\pi n$ or $\frac{5\pi}{3} + 2\pi n$.

1. Now, let's consider the equation $\cos(2x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$.

a. Determine the period of this function.

b. The period of $y = \cos(2x)$ is different than $y = \cos x$. How does your answer to part (a) affect the number of possible solutions for $\cos(2x) = \frac{1}{2}$ over the domain $0 \le x \le 2\pi$?

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You can use what you know about reference angles to determine solutions for x.

Worked Example

Solve $\cos(2x) = \frac{1}{2}$.

You know that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

So, to begin, let $\frac{\pi}{3} = 2x$ and solve for x.

$$2x = \frac{\pi}{3}$$
$$x = \frac{\pi}{6}$$

Because the period of $\cos(2x)$ is π, you know that two of the solutions are $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ for $0 \leq x \leq 2\pi$.

c. Determine the remaining solutions for $\cos(2x) = \frac{1}{2}$ over the domain $0 \leq x \leq 2\pi$ given $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$.

d. Write the solution for $\cos(2x) = \frac{1}{2}$ over the domain for all real numbers.

2. Solve the equation $2\sin(4x) + 1 = -1$ over the set of real numbers.

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ACTIVITY

2.4 Powers of Trigonometric Functions

If an equation that can be written in the form $ax^2 + bx + c = 0$ has x replaced with a trigonometric function, the result is a trigonometric equation in quadratic form. These equations can be solved as you would solve other quadratic equations, by factoring or by using the Quadratic Formula.

Note that $(\sin x)^2$ is usually written as $\sin^2 x$.

Worked Example

You can solve $2\sin^2 x + 5\sin x = 3$ over the domain of all real numbers.

Start with a substitution. This equation involves the sine function, so let $z = \sin x$.

$2z^2 + 5z = 3$	
$2z^2 + 5z - 3 = 0$	Write the equation in general form.
$(2z - 1)(z + 3) = 0$	Factor the quadratic expression.
$2z - 1 = 0$ or $z + 3 = 0$	Set each factor equal to 0.
$z = \frac{1}{2}$ or $z = -3$	Solve each equation.
$\sin x = \frac{1}{2}$ or $\cancel{\sin x = -3}$	Replace z with the sine function.
$x = \frac{\pi}{6}, \frac{5\pi}{6} \ldots + 2\pi n$	Determine the angle measure using $\sin^{-1}$.

1. Explain why $\sin x = -3$ is crossed off in the worked example.

2. Solve $4\sin^2 x - 1 = 0$ over the domain of all real numbers.

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3. Solve each equation over the domain of all real numbers.

a. $2\cos^2 x + \cos x = 1$

b. $2\tan^2 z + 3\tan z - 1 = 0$

c. $6\sin^2 z - 16\sin z - 33 = 0$

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NOTES

TALK the TALK

Problem Solved

1. **Solve $2 \sin x + \sqrt{2} = 0$ over the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.**

2. **Solve $\cos^2 x + 2 \cos x - 5 = -2$ over the domain of all real numbers.**

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Assignment

Write

Describe when you would use an inverse sine, inverse cosine, or inverse tangent function.

Remember

The inverse of each of the trigonometric functions—inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$), and inverse tangent ($\tan^{-1}$)—along with a calculator can be used to determine solutions to equations.

Practice

1. Use a periodicity identity to list 3 solutions for each equation.
   a. $\sin x = -\frac{\sqrt{2}}{2}$
   b. $\cos x = \frac{\sqrt{3}}{2}$
   c. $\tan x = \frac{\sqrt{3}}{3}$
2. Solve each equation over the domain of all real numbers.
   a. $4 + 2 \sin x = 5$
   b. $8 \cos x + 2 = -1$
   c. $5 \tan x - 3 = -11$
   d. $14 - 3 \sin x = 19$
   e. $2 \sin (3x) + 4 = 5$
   f. $4 \cos^2 x - 3 = 0$

Stretch

The average person's blood pressure can be modeled by the periodic function $P(t) = 20 \sin (160\pi t) + 100$, where t represents the time in minutes, and $P(t)$ represents the blood pressure at time t. Determine the amplitude, maximum and minimum values, period, and frequency of the function.

Review

1. Given $\cos \theta = -\frac{5}{13}$ in Quadrant III, use the Pythagorean identity to determine $\sin \theta$.
2. Given $\cos \theta = \frac{2}{9}$ in Quadrant IV, determine $\sin \theta$ and $\tan \theta$.
3. Solve each equation. Round your answer to the thousandths.
   a. $10^{(x+1)} = 7$
   b. $8^{-2a} - 5 = 55$

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3

Wascally Wabbits

Modeling with Periodic Functions

Warm Up

Describe the transformation performed on the graph of the basic function $f(x) = \sin x$ to produce the graph of $g(x)$.

1. $g(x) = \sin\left(\frac{1}{3}x\right)$

2. $g(x) = \sin x - 4$

3. $g(x) = \sin\left(x + \frac{\pi}{6}\right)$

4. $g(x) = -2 \sin x$

Learning Goals

- Model real-world situations with periodic functions.
- Interpret key characteristics of periodic functions in terms of problem situations.

You have explored trigonometric functions and solved trigonometric equations. How can you model real-life situations using trigonometric functions?

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GETTING STARTED

Rabbits, Rabbits Everywhere!

The rabbit population in a national park rises and falls throughout the year. The population is at its approximate minimum of 6000 rabbits in December. As the weather gets warmer and food becomes more available, the population grows to its approximate maximum of 16,000 rabbits in June.

1. Which trigonometric function best models this situation? Justify your answer.

2. Sketch a graph of the function.

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ACTIVITY

3.1

Modeling Population Change

The function to describe the rabbit population is $f(x) = 5000 \sin\left(\frac{\pi}{6}x - \frac{\pi}{2}\right) + 11{,}000$, where x is the time in months and $f(x)$ is the rabbit population.

1. Complete the table to show the rabbit population through one year.

Month	Time (month)	Rabbit Population (rabbits)
December		
January		
February		
March		
April		
May		
June		
July		
August		
September		
October		
November		
December		

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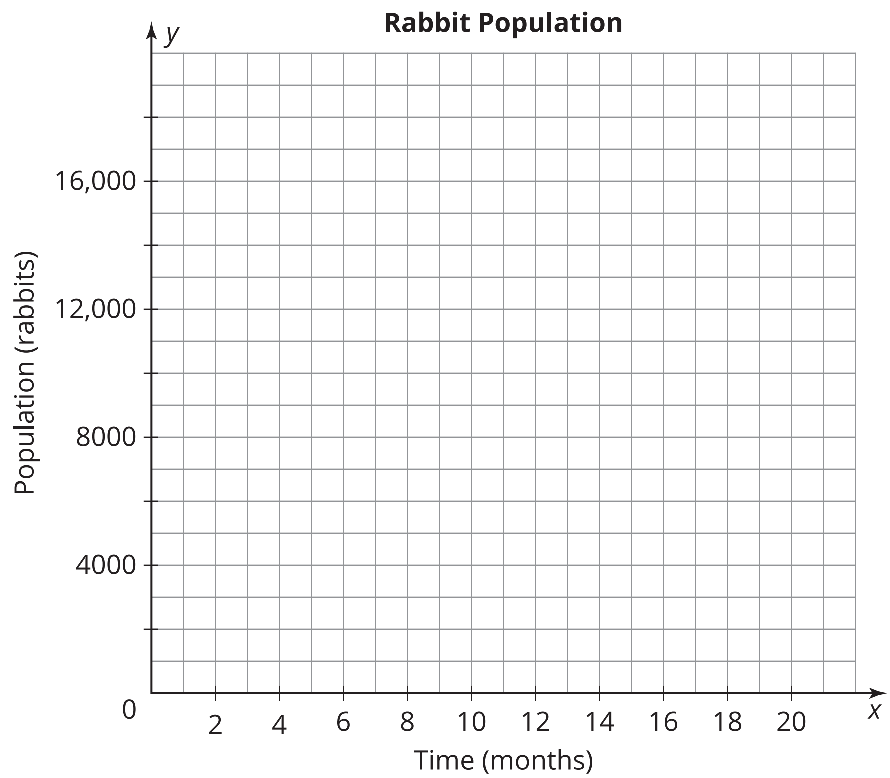
2. Graph the function representing the rabbit population.



3. How has the function been translated vertically from the basic sine function?

4. Determine each characteristic of the function.

a. Amplitude

b. Period

c. Phase shift

5. How is the vertical translation related to the algebraic function? What does it represent in terms of this problem situation?

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6. How is the amplitude related to the algebraic function? What does it represent in terms of this problem situation?

7. How is the period related to the algebraic function? What does it represent in terms of this problem situation?

8. How is the phase shift related to the algebraic function? What does it represent in terms of this problem situation?

9. If the rabbit population cycle occurred over six months instead of one year, how would the graph and equation change?

10. If the rabbit population had a minimum of 4000 and a maximum of 20,000, how would the graph and equation change?

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11. Describe the time(s) in months when the rabbit population is equal to 12,000. Show your work.

ACTIVITY
3.2

Modeling Patterns of Daylight

Patterns of daylight are related to seasonal affective disorder (SAD). The amount of daylight varies in a periodic manner and can be modeled by a sine function. The table shows the number of approximate daylight hours in Chicago, Illinois, which has latitude of 42° N.

Date	Day	Daylight Hours
Dec. 31	0	9.2
Jan. 10	10	9.3
Jan. 20	20	9.6
Jan. 30	30	9.9
Feb. 9	40	10.3
Feb. 19	50	10.7
Mar. 1	60	11.4
Mar. 11	70	11.7
Mar. 21	80	12.2
Mar. 31	90	12.7
Apr. 10	100	13.1
Apr. 20	110	13.6
Apr. 30	120	14.0
May 10	130	14.4
May 20	140	14.7
May 30	150	15.0
June 9	160	15.2
June 19	170	15.2
June 29	180	15.2

Date	Day	Daylight Hours
July 9	190	15.1
July 19	200	14.8
July 29	210	14.5
Aug. 8	220	14.2
Aug. 18	230	13.7
Aug. 28	240	13.3
Sept. 7	250	12.9
Sept. 17	260	12.4
Sept. 27	270	12.0
Oct. 7	280	11.5
Oct. 17	290	11.0
Oct. 27	300	10.6
Nov. 6	310	10.2
Nov. 16	320	9.8
Nov. 26	330	9.5
Dec. 6	340	9.2
Dec. 16	350	9.2
Dec. 26	360	9.1

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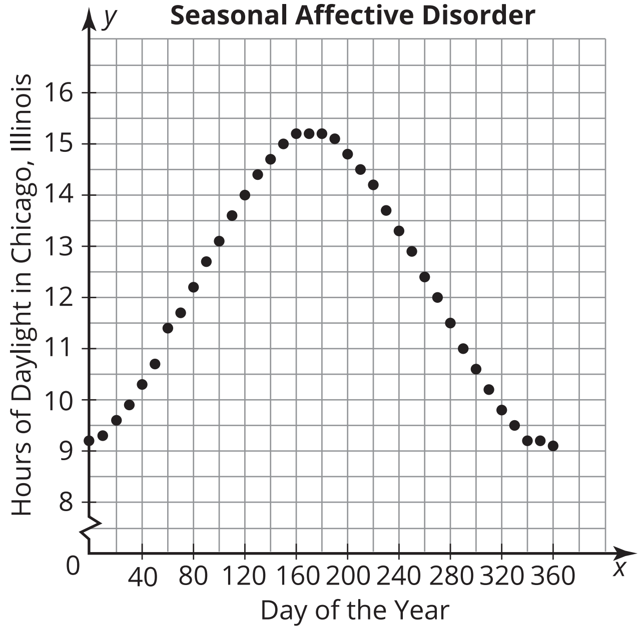
The graph shown models the data in the table.



1. Determine each characteristic.

a. Minimum and maximum values

b. Amplitude

c. Period

d. Phase shift

e. Vertical shift

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2. **To model this situation with a sine function in transformation function form, you need to determine the *A*-, *B*-, *C*-, and *D*-values.**

   a. **Determine the value of *A*. What does it represent in terms of this situation?**

   b. **Determine the value of *B*. Explain your reasoning.**

   c. **Determine the value of *C*.**

   d. **Determine the value of *D*. What does it represent in terms of this situation?**

   e. **Write an algebraic function to model the data for the number of daylight hours in Chicago, Illinois.**

3. **Use technology to perform a sinusoidal regression for this data and write the regression equation. How does it compare to your equation?**

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4. Use the regression equation to describe times of the year when there are exactly 12 hours of daylight. Show your work.

5. Seasonal affective disorder appears to vary according to latitude. The farther a location is from the equator, the more prevalent cases of SAD become. Why might this happen?

6. Anchorage, Alaska, is located at a latitude of 61° N. This is considerably farther north than Chicago. If you created a graph to model the daylight hours in Anchorage, how do you think it would compare to the graph for daylight hours in Chicago? In what ways would it be the same? In what ways would it be different?

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7. In locations like Chicago and Anchorage, SAD is most likely to occur around the month of January. In locations in the southern hemisphere, like Santiago, Chile (latitude 33.5° S), SAD occurs around the month of July. Why does this happen?

NOTES

TALK the TALK

Twansforming Twig Functions

1. Write a sine function to represent the data shown. Explain your reasoning.

x	y
0	200
1	170
2	110
3	80
4	110
5	170
6	200

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Assignment

Write

Describe the types of situations that can be modeled using trigonometric functions.

Remember

The key characteristics of periodic functions, including period, amplitude, midline, and phase shift, are used to model components of real-world situations.

Practice

1. The height of a roller coaster can be modeled by the function $f(x) = 20 \cos\left(\frac{\pi}{60}x\right) + 30$, where x represents the horizontal distance from the start of the ride in meters, and $f(x)$ represents the vertical height of the ride in meters.
   a. Determine the amplitude of the function. What does it represent in terms of this problem situation?
   b. Determine the period of the function. What does it represent in terms of this problem situation?
   c. Determine the vertical shift of the function. What does it represent in terms of this problem situation?
2. The table shows the average monthly high temperature for a town in Tennessee. This data can be modeled with a sine function.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Average High Temperature (°F)	50	53	60	71	80	87	90	89	84	73	59	50

a. Plot the points from the table using the number of the month for your independent variable and the average high temperature for your dependent variable.
b. Determine the amplitude, period, and vertical shift of the function that could be used to model this data. Explain your reasoning.
c. Use technology to perform a sinusoidal regression for the data. Write the regression equation. Is this model a good fit for the data? Explain your reasoning.

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Stretch

1. The data in the tables show the fraction of the Moon illuminated at midnight each day in the month of February, 2018. This data can be modeled with a sine function.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Fraction of Moon Illuminated	0.99	0.96	0.90	0.83	0.74	0.64	0.5 5	0.45	0.35	0.27	0.19	0.12	0.07	0.03

Day	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Fraction of Moon Illuminated	0	0	0.02	0.0 5	0.11	0.18	0.27	0.38	0.49	0.6	0.71	0.81	0.89	0.95

a. Plot the points from the table using the number of the day for the independent variable and the fraction of the Moon illuminated for the dependent variable.

b. Determine the amplitude and period of the function that could be used to model this data. Explain your reasoning.

c. Use technology to perform a sinusoidal regression for the data. Write the regression equation. Is this model a good fit for the data? Explain your reasoning.

Review

1. Use a periodicity identity to list three solutions for the equation $\cos x = -\frac{1}{2}$.

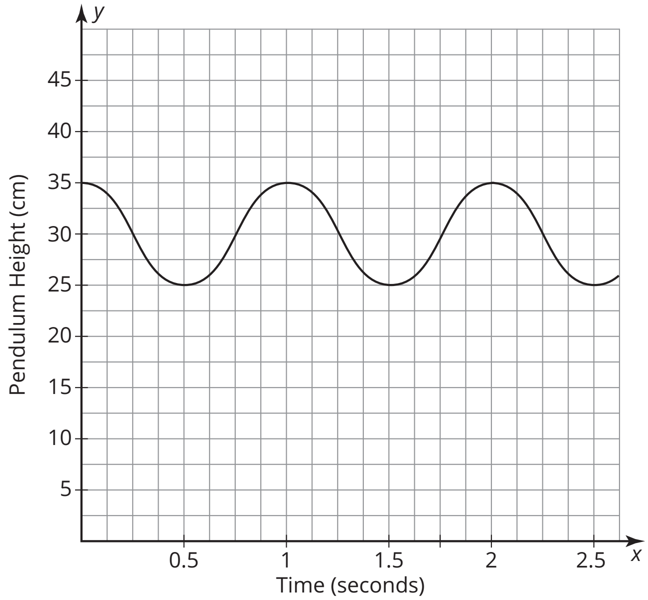
2. Solve the equation over the domain of all real numbers: $5 + 4\cos\theta = 3$.

3. A pendulum clock swings back and forth. At rest, the pendulum is 25 cm above the base. At the highest point of the swing, the pendulum is 35 cm above the base. It takes the pendulum 2 seconds to swing back and forth. The graph shows the height of the pendulum above the base as a function of seconds. Assume the pendulum is released from its highest point.

   a. Determine the amplitude of the function.

   b. Determine the period of the function.

   c. Determine the height of the pendulum at 3.75 seconds.



4. Solve each equation. Round your answers to the thousandths, if necessary.

   a. $\left(\frac{2}{3}\right)^x = 5^{3-x}$

   b. $2\log_5 x = 3\log_5 2$

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4

The Wheel Deal

Modeling Motion with a Trigonometric Function

Warm Up

Describe the transformation of the graph of each function from the basic function $f(\theta) = \sin\theta$.

1. $f(\theta) = 3.5 \sin\theta$

2. $f(\theta) = \sin(\theta + \pi)$

3. $f(\theta) = -\sin\theta + \frac{1}{2}$

Learning Goals

- Interpret characteristics of a graph of a trigonometric function in terms of a problem situation.
- Construct a trigonometric function to model a problem situation.

You have used the unit circle to explore trigonometric functions. You have also explored how the values of the transformed function form affect the shape of the graph of a periodic function. How can you use what you know to build a trigonometric function to model circular motion in real-world problems?

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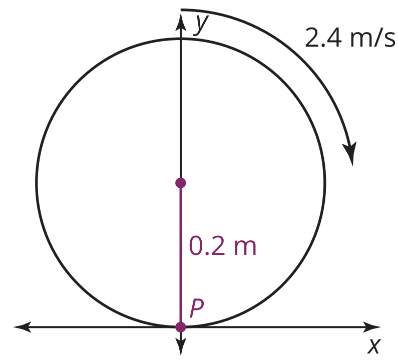
GETTING STARTED

Big Wheel Keeps on Turning

Suppose a wheel with a radius of 0.2 meter rolls clockwise on a street at a rate of 2.4 m/s.

You can build a trigonometric function to model the height, *h*, from the street of a point, *P*, on the wheel as a function of time, *t*, in seconds. As the wheel rolls, the position of point *P* will move along the circle.

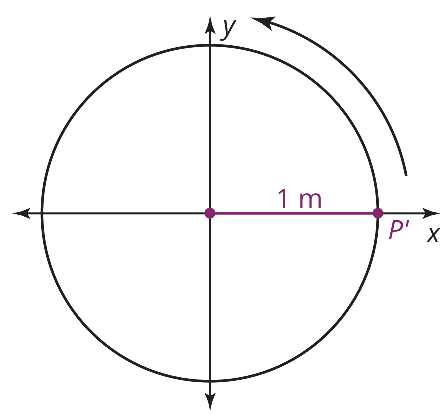
Figure 1



In order to build this trigonometric function, let's first think about point *P* from Figure 1 in standard position on a unit circle as point *P'* in Figure 2, moving counterclockwise.

- Point *P'* is located where a terminal ray in standard position intersects the circle at 0 radians.
- The point is moving counterclockwise instead of clockwise.
- The wheel is rotating in place and has a radius of 1 meter.
- The *x*-axis represents the ground.

Figure 2



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1. Which trigonometric function models the height, *h*, of point *P'* for each angle measure, *θ*, in radians?

ACTIVITY 4.1

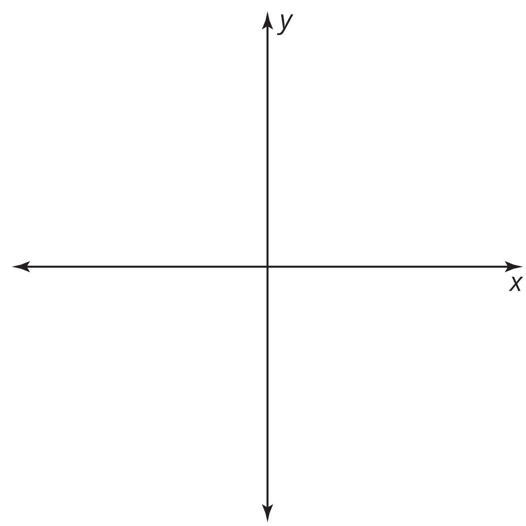
Modeling Motion with a Trigonometric Function

Let's consider each piece of information in the original problem situation from the Getting Started and how you can use transformations to build an equation to model Figure 1.

1. Use the given information to sketch each figure and write each corresponding equation. Describe the transformation.

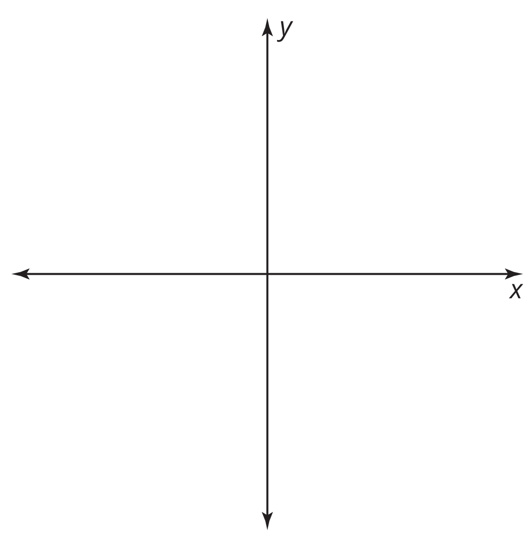
a. To sketch Figure 3, consider Figure 2 but the radius is 0.2 meter. Label point *P*′ on your graph.

Figure 3



b. To sketch Figure 4, consider Figure 3 but the wheel rests on the ground. Label point *P*′ on your graph.

Figure 4



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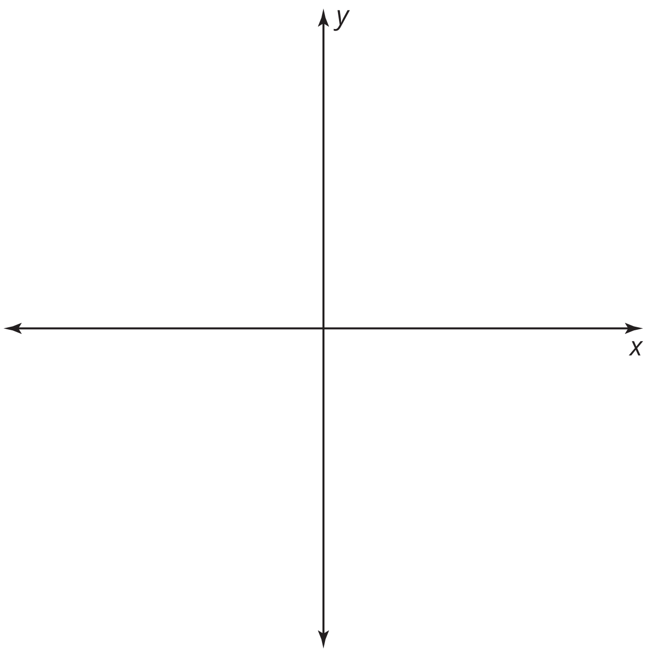
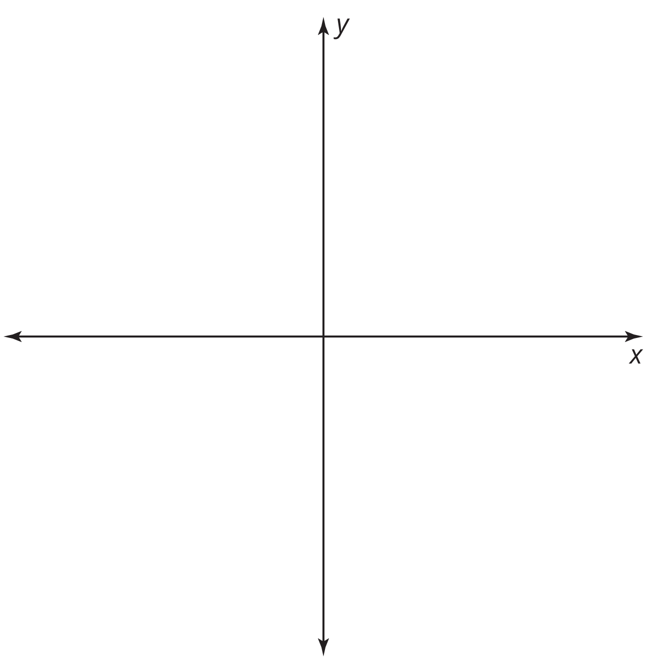
c. **To sketch Figure 5, consider Figure 4 but translate point *P′* to the original starting position, point *P*, in Figure 1. Label point *P* on your graph.**

Figure 5



d. **To sketch Figure 6, consider Figure 5 but the wheel turns clockwise. Label point *P* on your graph.**

Figure 6



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You have just written an equation that models the height of point P on the wheel with a radius of 0.2 meter in terms of θ.

Now let's consider the relationship between time and θ to write an equation for the height of point P on the wheel in terms of time.

2. Write an equation for the height of point *P* on the wheel in terms of time *t*.

a. Determine the relationship between time, *t*, and *θ*.

Think about:

Use the relationship for distance in terms of rate and time to write distance as a function of θ.

b. Write the final equation in terms of time *t*.

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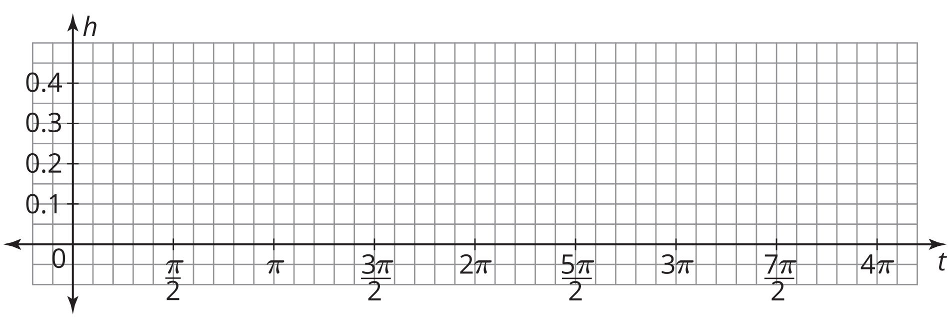
3. Sketch a graph of your function from Question 2. Label the axes.



4. Determine the height of the point at 1 second.

5. Rewrite your function as a cosine function. Explain your reasoning.

6. What are the advantages of rewriting your function as a cosine function?

7. At what time(s) is the height of the point at 0.2 meter?

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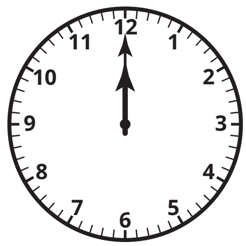
NOTES

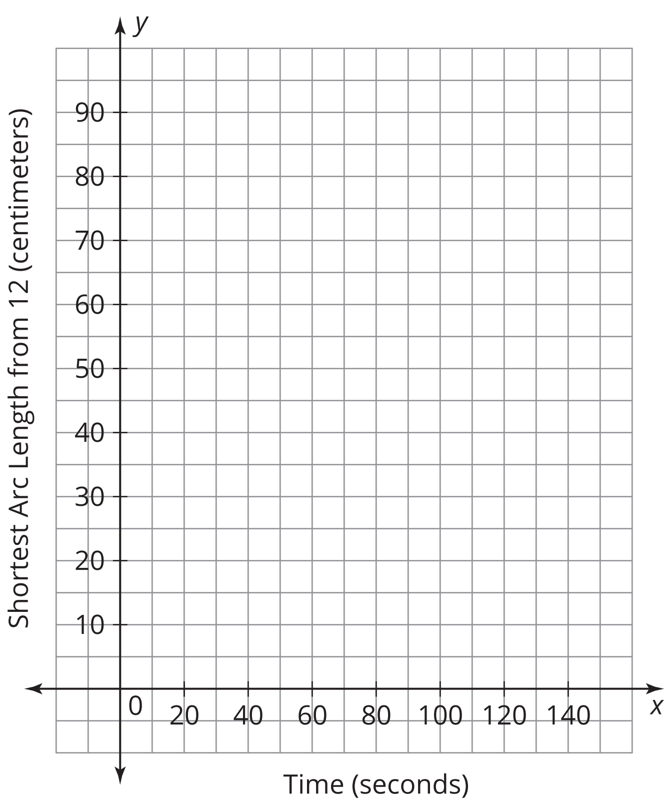
TALK the TALK

Time Stops for No One!

Consider the second hand on the face of a clock. The length of the second hand and radius of the clock face are each 30 centimeters. Suppose the second hand begins its movement at exactly 12:00 midnight.



1. Complete the table to describe the time in seconds and the shortest arc length between the tip of the second hand and its starting position at 12:00, in centimeters. For each complete revolution, suppose that the distance resets to 0. Then create a graph.



Time (seconds)	Shortest Arc Length from 12 (centimeters)
0	
10	
20	
30	
40	
50	
60	0
70	
80	
90	
100	
110	
120	

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NOTES

2. What is the amplitude of this function?

3. What is the vertical shift of this function? Why is a vertical shift necessary?

4. Consider the relationship between time and θ.

a. Write a proportion with ratios for length and angle measure. Solve for d.

b. Use the Distance Formula to express θ in terms of time.

5. Choose the trigonometric function that best models this situation.

$f(x) = -30\pi \cos\left(\frac{\pi}{30}x\right) + 30\pi$

$f(x) = -30\pi \cos\left(\frac{\pi}{6}x\right) + 15\pi$

$f(x) = -15\pi \cos\left(\frac{\pi}{30}x\right) + 15\pi$

$f(x) = -15\pi \cos\left(\frac{\pi}{6}x\right) + 15\pi$

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Assignment

Write

Describe how you can model the motion of points on a circle by using transformations of a trigonometric function.

Remember

Transformations of periodic functions can be used to map function behavior to the behavior of periodic phenomena, such as amplitude, period, frequency, phase shift, and midline.

Practice

1. Angela rode the Ferris wheel at Navy Pier in Chicago. The Ferris wheel has a diameter of 140 feet. She was curious about how long it would take her to get from the lowest point to the highest point of the ride. She began timing her ride while she was at the bottom of the wheel and noticed that it took her 3 minutes and 45 seconds to get to the top. At the highest point, Angela was 150 feet off the ground. The vertical height, h, of a person riding the Ferris Wheel can be modeled as a trigonometric function of time, t, in seconds. The Ferris wheel moves in a clockwise direction.
   a. Determine Angela's vertical height when she is at the lowest point of the ride.
   b. Determine the amount of time it takes for Angela to complete one revolution on the Ferris wheel. Write your answer in seconds.
   c. Sketch a graph of Angela's height in feet on the Ferris wheel as a function of time in seconds.
   d. Determine the amplitude of the function. Explain your reasoning.
   e. Calculate the period and value of B of the function. Explain your reasoning.
   f. Determine the values of C and D of the function if a cosine function is used to model the problem situation. Explain how you determined your answers.
   g. Write a cosine function to model Angela's height on the Ferris wheel as a function of time.
   h. Explain how Angela could write a sine function to model the height of the Ferris wheel as a function of time.

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Stretch

The hour hand of a large clock on a wall of a train station measures 18 inches in length. At noon, the tip of the hour hand is 40 inches from the ceiling. Let y equal the distance from the tip of the hour hand to the ceiling x hours after noon. Determine a trigonometric equation that best models the motion of the hour hand and sketch the graph.

Review

1. The tide at a pier can be modeled by the equation $h(t) = 2\cos\left(\frac{\pi}{6}t\right) + 7$, where t represents the number of hours past noon and $h(t)$ represents the height of the tide in feet.
   a. Determine the amplitude of the function. What does it represent in terms of this problem situation?
   b. Determine the period of the function. What does it represent in terms of this problem situation?
   c. Determine the vertical shift of the function. What does it represent in terms of this problem situation?
2. A satellite in a medium Earth orbit completes one orbit every 12 hours. The satellite follows a circular path with its center at the center of the earth. The satellite is at an altitude of 12,552 miles. The radius of the earth is 3959 miles.
   a. Determine the angle of rotation, in radians, that corresponds to a 5-hour time period.
   b. Determine the distance traveled by the satellite in a 5-hour time period.
3. Multiply the rational expressions.
   a. $\frac{x^2 + 6x + 9}{x - 3} \cdot \frac{x^2 + 3x - 18}{x^2 - 9}$
   b. $\frac{x^3 - 8}{x^4 - 9x^2} \cdot \frac{x^5 - 6x^4 + 9x^3}{x^2 - 4x + 4}$

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5

Springs Eternal

The Damping Function

Warm Up

Solve for x.

1. $8^x = 262{,}144$
2. $\left(\frac{3}{5}\right)^x = \frac{81}{625}$
3. $0.9^x = 0.5$

Learning Goals

- Choose a trigonometric function to model a periodic phenomenon.
- Determine the graphical attributes (amplitude, midline, frequency) of a periodic function from a description of a problem situation.
- Build a function that is a combination of a trigonometric function and an exponential function.

Key Term

- damping function

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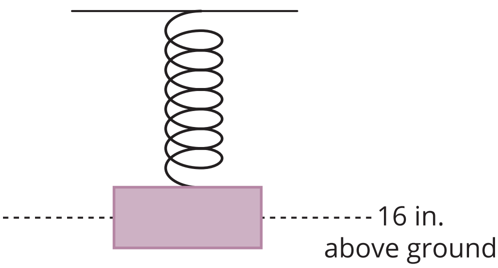
You know how to model a real-world situation that displays periodic tendencies with a trigonometric function. You also know how to model a situation that increases or decreases at a constant ratio using an exponential function. How can you combine these two types of functions to model a real-world situation that is both periodic and decreasing?

GETTING STARTED

Bouncing Up and Down



An object suspended from a spring is pulled 5 inches below its resting position and released, causing the object to bounce up and down once every second. At rest, the object's height above the ground is 16 inches.

Suppose that the object bounces up 5 inches above its resting height and then back down to 5 inches below its resting height without stopping on every bounce. Let's build a periodic function to model the bouncing of the object on the spring over time.

1. Determine the independent and dependent quantities for this situation.

2. Sketch and label the graph of the function to model the bouncing object over time *h*(*t*), given what you know about the height of the object. Represent at least two bounces of the object on the graph.

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ACTIVITY

5.1

Using a Graph to Write a Periodic Function

Consider the problem situation from the Getting Started and the graph you created.

1. Use your graph to determine each characteristic of the periodic function that will model this situation. Explain your reasoning.

a. Determine the equation of the midline of the graph.

b. Determine the minimum, maximum, and amplitude of the function.

Think about:

What characteristics of the graphed function correspond to the *A*-, *B*-, *C*-, and *D*-values of the transformed function?

2. Does your sketch model a sine curve or a cosine curve? Explain your reasoning.

3. Write the values of *A, C,* and *D* for the function *h*(*t*). Explain how you determined each value.

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Remember:

The period of a sine or cosine function is $\frac{2\pi}{|B|}$.

4. Determine the period of the function *h*(*t*). Then write the *B*-value of the function. Show your work.

5. Write the equation for the function *h*(*t*) to represent the height of the object over time.

Think about:

What are the cosine values on the unit circle?

6. Explain why the sign of the *B*-value in this function can be either positive or negative.

7. Solve an equation to determine when the object on the spring is at its minimum height. Show your work.

Ask yourself:

Do your solutions represent every time the object is at the midline?

8. Solve an equation to determine when the object on the spring is at a height of 16 inches. Show your work.

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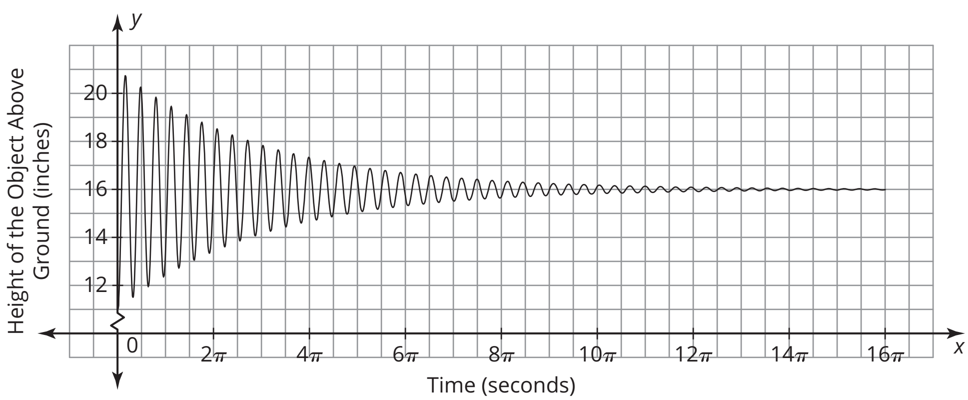
ACTIVITY **5.2**

Damping Functions

An object connected to a string and bouncing up and down the same amount forever is not realistic. Starting from when the object is released, the energy produced will eventually fade away. The object will bounce closer and closer to the midline until it once again comes to rest.

Let's consider the same situation from the Getting Started. A more realistic model of the object's motion is shown.



Recall the function that models the situation from the Getting Started is $h(t) = -5\cos(2\pi t) + 16$.

1. How do you think you can adjust the function $h(t)$ to create the shape of the graph shown? What is changing in each period of this function?

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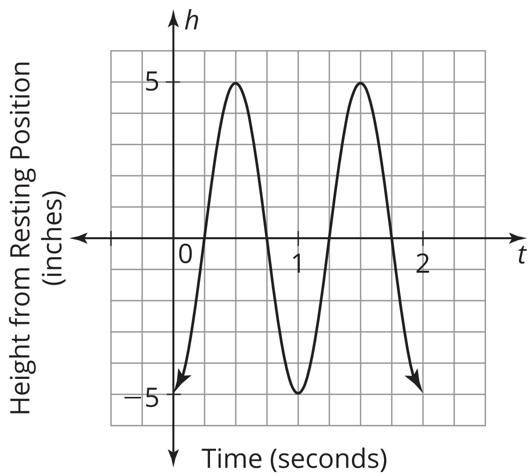

A graph of the function $k(t) = h(t) - 16$ is shown. This is the function relative to its resting position. Suppose that the distance the object bounces from its resting position decreases at a rate of 10% each second.

2. At $t = 0$, the object is at -5 inches from its resting position.

a. Determine the object's new height at $t = 1$ second and $t = 2$ seconds.

b. Write an equation to describe the object's new height, n, over time, t. Explain your reasoning.

c. Does your equation correctly describe the object's new height at $t = \frac{1}{2}$ second? At $t = 1\frac{1}{2}$ seconds? If not, what equation would be correct?

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3. **Explain why Kent is correct.**

Kent

The equation $b(t) = |A| \cdot 0.9^t$ describes the change in the object's height over time, because $|A|$ represents the amplitude of the function.

4. **Write the complete function that represents the height of the object on the spring over time.**

5. **After how many seconds is the maximum height of the object on the spring equal to 18 inches? Explain how you determined your solution.**

The function that you multiply to the periodic function to decrease its amplitude over time is called a **damping function**. A damping function can be linear, quadratic, exponential, and on and on!

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6. **Write a function $g(t)$ to model the height of an object connected to a spring with decreased amplitude over time given the conditions:**
   - **At rest, the object's height is 10 inches above the ground.**
   - **The object bounces up and down once every 2 seconds.**
   - **At $t = 0$, the object's height is 14 inches.**
   - **The distance the object bounces from its resting position decreases at a rate of 15% each second.**

7. **How would the exponent of the *A*-value in the function you wrote in Question 6 change if the rate of decrease for the amplitude is per bounce and not per second?**
   Explain your reasoning.

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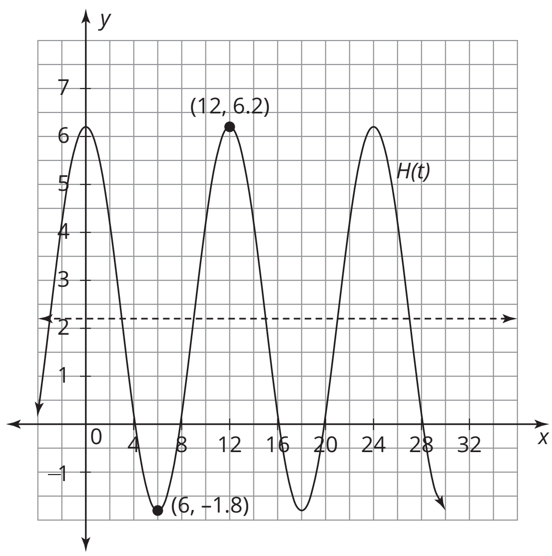
NOTES

TALK the TALK

The Turning of the Tides

A trigonometric function can be used to model the changes in high and low tides at particular locations. The gravitational force of both the Moon and Sun affect the height of the tide. The graphed function models the high and low tides, where $H(t)$ represents the height of the tide in feet over time.



1. What is the amplitude of the function? Explain how you can use the graph to determine this value.

2. What is the vertical shift of the function? How is it determined?

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NOTES

3. **Choose the trigonometric function that models this situation.**

$$H(t) = 4\cos\left(\frac{\pi}{6}(t + 3)\right) + 2.2$$

$$H(t) = 4\sin\left(\frac{\pi}{6}(t + 3)\right) + 2.2$$

$$H(t) = 4\cos(12(t + 3)) + 2.2$$

$$H(t) = 4\sin(12(t + 3)) + 2.2$$

Interpret the different characteristics of the periodic function with respect to this problem situation.

4. **What is the phase shift? How is it related to the problem situation?**

5. **What is the period of the function? How is it related to the problem situation?**

6. **What is the high tide at midnight?**

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Assignment

Write

Describe a real-world example of a damping function.

Remember

A trigonometric function and an exponential function can be combined to model a periodic function whose amplitude decreases over time. The function that is multiplied to the periodic function is called a damping function.

Practice

1. Jordan is swinging on a rope swing that swings over a creek. When he jumps on the swing, he is 20 feet away from the center of the creek. He then swings out to 20 feet past the center of the creek to the other side. As he swings, he pumps his legs to keep his swinging motion constant. Amelia times Jordan as he swings. Jordan's distance in feet from the center of the creek, d, can be modeled with a trigonometric function of the time he swings, t, in seconds. It takes Jordan 2 seconds to swing from one side of the creek to the other.
   a. Sketch the graph of a function that could be used to model this problem situation.
   b. Write the equation of the cosine function, $d(t)$, that can be used to model the distance Jordan is from the center of the creek as a function of time.
   c. Use the equation from part (b) to determine Jordan's distance from the center of the creek at 9.5 seconds. Round your answer to the nearest foot.
   d. Use the equation from part (b) to determine when Jordan is 6 feet from the center of the creek. Round your answer to the nearest tenth of a second.
2. Amelia is swinging on a rope swing over a creek. When she jumps on the swing, she is 20 feet away from the center of the creek. She then swings out past the center of the creek toward the other side. She decides that she will not pump her legs to keep the swing moving and will just let it swing until it stops. Jordan times Amelia as she swings. Suppose Amelia's distance on each side of the creek decreases at a rate of 20% on each swing. It takes her 2 seconds to swing toward the other side of the creek on her first swing.
   a. Determine the distance Amelia swings past the center of the creek on her first trip over the creek on her initial swing.
   b. Determine the distance Amelia swings past the center of the creek on her second trip over the creek.
   c. Write an equation to represent Amelia's distance, d, in terms of the time, t, after each trip across the creek. Hint: It takes 2 seconds for Amelia to swing from one side of the creek to the other.
   d. Let the function $d(t) = -20 \cos\left(\frac{\pi}{2}t\right)$ represent Amelia's distance from the center of the creek if she was swinging at a constant rate back and forth. Use this function to write a new function that represents Amelia's actual distance from the center of the creek given that her distance decreases by 20% each time she swings back over the creek.
   e. Determine Amelia's distance from the center of the creek after 10 seconds. Round your answer to the nearest foot.

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Stretch

1. Lian is swinging on a rope swing over a creek. As she swings, she pumps her legs to keep her swinging motion constant. The table shows her distances from the center of the creek from the moment she jumps on the swing until 8 seconds have passed.

Time (seconds)	0	1	2	3	4	5	6	7	8
Distance (feet)	−15	0	15	0	−15	0	15	0	−15

   a. Sketch the graph of a function that could be used to model this problem situation.
   b. Write the equation of the cosine function, $d(t)$, that can be used to model the distance Lian is from the center of the creek as a function of time.

When Lian is at the spot where she first jumped on the swing, she decides to stop pumping her legs and just let it swing until it stops. The table shows her distances from the center of the creek from the moment she stops pumping her legs until 8 seconds have passed.

Time (seconds)	0	1	2	3	4	5	6	7	8
Distance (feet)	−15	0	11.25	0	−8.4375	0	6.328125	0	−4.74609375

c. Sketch the graph of a function that could be used to model this problem situation.
d. Write an equation to represent Lian's distance, d, in terms of the time, t, after each trip across the creek.
e. Use your function from part (b) to write a new function to represent Lian's actual distance from the center of the creek in terms of the time, t.

Review

1. A person is riding a Ferris wheel. The graph shows the person's height from the ground in feet as a function of time in seconds. The time starts when the rider boards the ride.
   a. Determine the amplitude of the function. Explain your reasoning.
   b. Calculate the period and value of B of the function. Explain your reasoning.
   c. Determine the values of C and D of the function if a cosine function is used to model the problem situation. Explain how you determined your answers.
   d. Write a trigonometric function to model the height of the rider from the ground as a function of time.
2. Add the rational expressions.
   a. $\frac{6}{x-1} + \frac{x}{4}$
   b. $\frac{3}{x-1} + \frac{4}{x+2}$

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Trigonometric Equations Summary

KEY TERMS

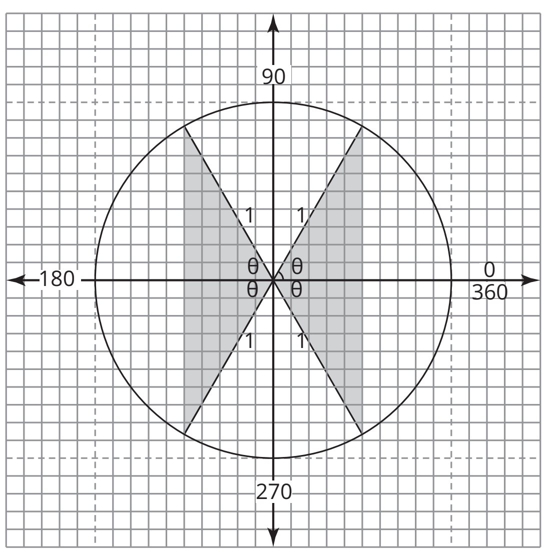
- Pythagorean identity
- trigonometric equation
- inverse sine ($\sin^{-1}$)
- inverse cosine ($\cos^{-1}$)
- inverse tangent ($\tan^{-1}$)
- damping function

LESSON 1

$\text{Sin}^2\ \theta$ Plus $\text{Cos}^2\ \theta$ Equals 1^2

The trigonometric ratios sine, cosine, and tangent can have different signs. They can be negative or positive, depending on what quadrant of the coordinate plane the angle and right triangle are in.

Consider the unit circle shown.



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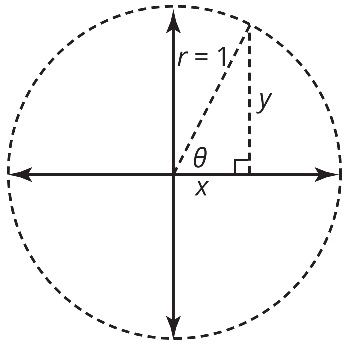
	Sign (+/−)			
	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−

A **Pythagorean identity** is a trigonometric identity that expresses the Pythagorean Theorem in terms of trigonometric ratios. The Pythagorean identity states that $(\sin\theta)^2 + (\cos\theta)^2 = (1)^2$. The symbol θ represents an angle measure.



You can use the Pythagorean identity and what you know about solutions in different quadrants to determine values of trigonometric functions.

For example, to determine the exact value of $\cos\theta$ in Quadrant II, given $\sin\theta = \frac{2}{3}$, use a Pythagorean identity to determine $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1$$
$$\frac{4}{9} + \cos^2\theta = 1$$
$$\cos^2\theta = \frac{5}{9}$$
$$\cos\theta = \pm\frac{\sqrt{5}}{3}$$

The solution is $\cos\theta = -\frac{\sqrt{5}}{3}$ because the solution is in Quadrant II.

You can also solve for the tangent of an angle in all four quadrants using what you know about trigonometric ratios.

$$\tan\theta = \frac{y}{x}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Using sine, cosine, and tangent you can complete a unit circle.

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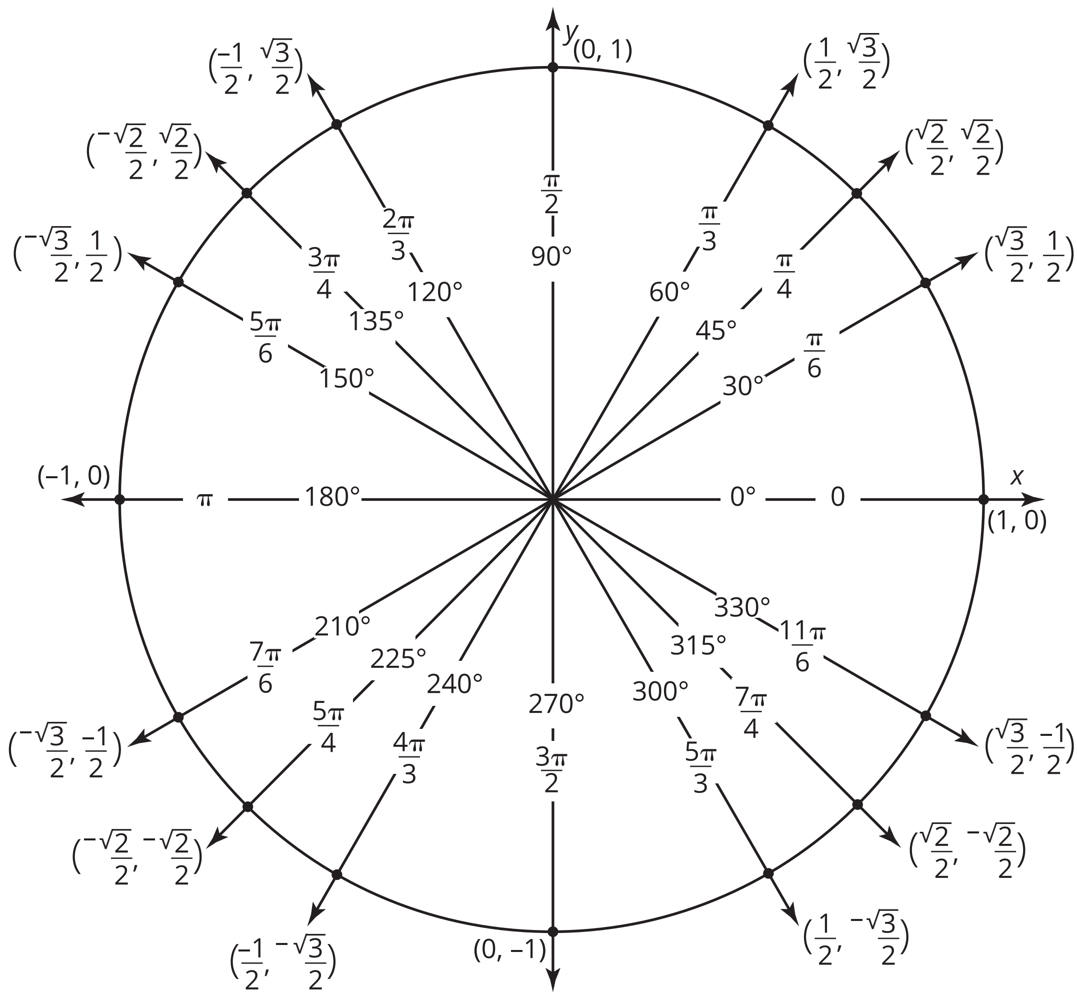

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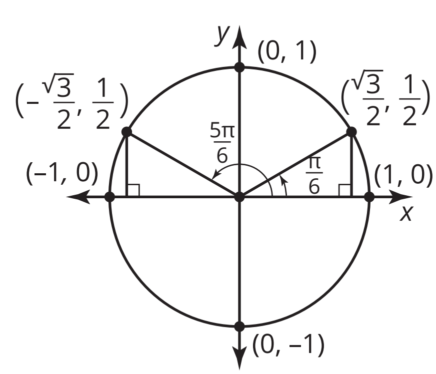
LESSON
2

Chasing Theta

A **trigonometric equation** is an equation in which the unknown is associated with a trigonometric function. Solutions to trigonometric equations can be determined using graphs, knowledge about periods and transformation, and inverses of the trigonometric functions. It is important to be aware of domain restrictions when solving trigonometric equations

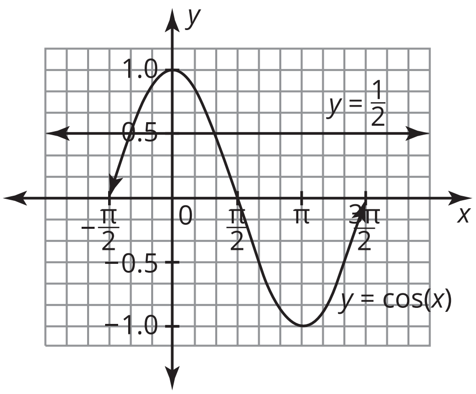


For example, consider the equation $\sin x = \frac{1}{2}$. On the unit circle, you can see that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$. So, $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

When the domain is restricted to $0 \le x \le 2\pi$, these are the only 2 solutions to the equation. When there are no domain restrictions, the equation has an infinite number of solutions.

You can use what you know about the graphs of trigonometric functions to solve trigonometric equations.

For example, consider the equation $\cos x = \frac{1}{2}$. The equations $y = \cos x$ and $y = \frac{1}{2}$ are graphed on the coordinate plane. If we restrict the domain $-\frac{\pi}{2} \le x \le \frac{2\pi}{2}$, you can see that they intersect at two places, when $x = -\frac{\pi}{3}$ or $\frac{\pi}{3}$.



You can also use what you know about the periods of trigonometric functions to solve trigonometric equations. The periodicity identities you have learned are shown. Adding or subtracting integer multiples, *n*, of the period for each function generates solutions to trigonometric equations.

Periodicity Identities	
Sine	$\sin(x + 2\pi n) = \sin x$
Cosine	$\cos(x + 2\pi n) = \cos x$
Tangent	$\tan(x + \pi n) = \tan x$

The **inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$)**, and **inverse tangent ($\tan^{-1}$)** functions are used to determine solutions to sine, cosine, and tangent equations, respectively.

For example, consider the solution steps to the equation $2 \cos x + 5 = 6$, solved over the domain of real numbers.

$$2 \cos x + 5 = 6$$
$$2 \cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \cos^{-1}\left(\frac{1}{2}\right)$$
$$x = \frac{\pi}{3} + 2\pi n \text{ or } \frac{5\pi}{3} + 2\pi n$$

Therefore, the values of x are $\frac{\pi}{3} + 2\pi$n or $\frac{5\pi}{3} + 2\pi$n for integer values of *n*.

When the *B*-value is changed from a basic trigonometric function, you must take the change in period into account when determining solutions. You can use what you know about reference angles to determine solutions for *x*.

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For example, to solve $\cos(2x) = \frac{1}{2}$, you know that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. So, to begin, let $\frac{\pi}{3} = 2x$ and solve for x.

$$2x = \frac{\pi}{3}$$
$$x = \frac{\pi}{6}$$

Because the period of $\cos(2x)$ is π, $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ for $0 \le x \le 2\pi$.

If an equation that can be written in the form $ax^2 + bx + c = 0$ has x replaced with a trigonometric function, the result is a trigonometric equation in quadratic form. These equations can be solved in the same manner that other quadratic equations are solved, by factoring or by using the Quadratic Formula.

For example, consider the solution steps to the equation $2\sin^2 x + 5\sin x = 3$, solved over the domain of all real numbers.

Start with a substitution. This equation involves the sine function, so let $z = \sin x$.

$$2z^2 + 5z = 3$$
$$2z^2 + 5z - 3 = 0$$
$$(2z - 1)(z + 3) = 0$$
$$2z - 1 = 0 \text{ or } z + 3 = 0$$
$$2z = 1$$
$$z = \frac{1}{2} \qquad \text{or } z = -3$$
$$\sin x = \frac{1}{2} \qquad \text{or } \sin x = -3$$
$$\sin x = -3 \text{ is not possible so } \sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6} \ldots + 2\pi n$$

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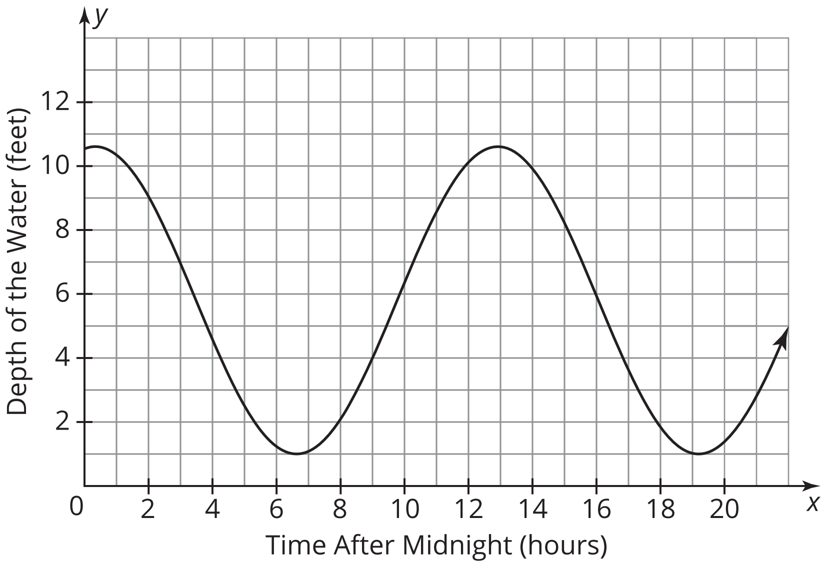
LESSON 3

Wascally Wabbits

Periodic functions can be used to model real-world situations. Analyzing the functions in the context in which they are given can provide deeper understanding about the functions and the situation.

For example, consider the problem situation.

As the tide changes over the course of the day, the depth of the water near the coast also changes. The change in the depth of the water in feet near the coast in Harbortown can be modeled by the function $d(t) = 4.8 \sin(0.5t + 1.4) + 5.8$, where t represents the number of hours after midnight. Graph the function. Then, identify and describe the amplitude, period, and vertical shift in terms of the situation.



The amplitude is approximately $\frac{|10.6 - 1.0|}{2}$ or 4.8. This represents the average change in depth of the water in feet over the course of the day. The period is $\frac{2\pi}{|0.5|}$ or 4π, which is approximately 13 hours. This represents the time it takes for the depth to return to its starting point. The vertical shift is 5.8, and it represents the average depth of the water over the course of the day.

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LESSON 4

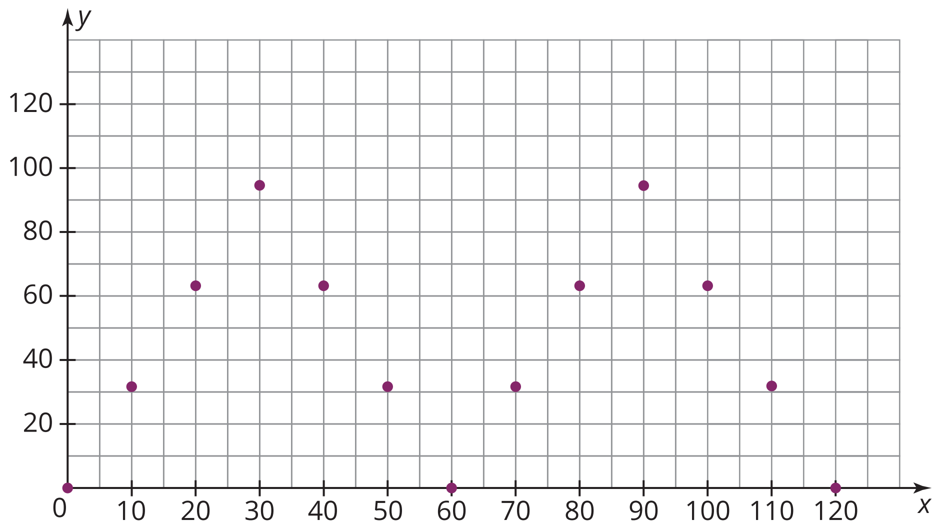
The Wheel Deal

You can use what you know about the unit circle and periodic functions to build a trigonometric function to model circular motion in real-world problems.

For example, consider the second hand on the face of a clock. The length of the second hand and radius of the clock face are each 30 centimeters. Suppose the second hand begins its movement at exactly 12:00 midnight.

Circumference $= 2\pi r = 2\pi(30) = 60\pi$

Time (sec.)	Distance (cm.)	Time (sec.)	Distance (cm.)
0	0	70	10π
10	10π	80	20π
20	20π	90	30π
30	30π	100	20π
40	20π	110	10π
50	10π	120	0
60	0		



From the graph, you can see that the domain is all positive real numbers from 0 to 120 and the range is all positive real numbers from 0 to 30π.

The data can be modeled by the periodic function $y = -15\pi \cos\left(\frac{\pi}{30}x\right) + 5\pi$

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LESSON 5

Springs Eternal

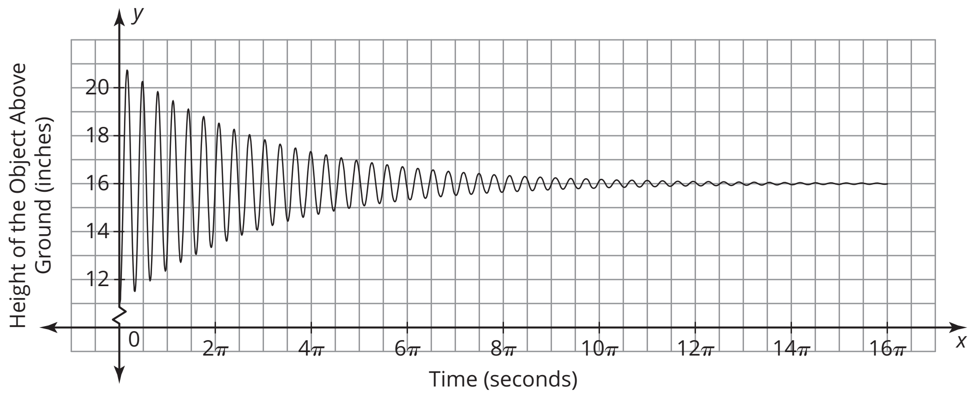
A **damping function** is a function in which the amplitude changes as the input changes. In a damping function, A is not a constant.

For example, consider the problem situation.

An object suspended from a spring is pulled 5 inches below its resting position and released, causing the object to bounce up and down once every second. At rest, the object's height above the ground is 16 inches. Suppose that the object bounces up 5 inches above its resting height and then back down to 5 inches below its resting height without stopping on every bounce.

The periodic function that models the bouncing of the object on the spring over time is $f(x) = 5\cos(2\pi x) + 16$.

An object connected to a string and bouncing up and down the same amount forever is not realistic. Starting from when the object is released, the energy produced will eventually fade away. The object will bounce closer and closer to the midline until it once again comes to rest.



An equation that models the decrease in height of each bounce by 10% is $g(x) = 5 \cdot (0.9)^x$. Combine this exponential equation with the periodic equation to achieve an equation that more accurately represents the decreasing height of each consecutive bounce.

$$h(x) = 5 \cdot (0.9)^x \cdot 5\cos(2\pi x) + 16$$

The function that you multiply to the periodic function to decrease its amplitude over time is the damping function. A damping function can be linear, quadratic, exponential, or so on.

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MODULE 5

RELATING DATA & DECISIONS

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The lessons in this module build off of your knowledge of how to use center, spread, and shape to describe and compare data sets. You will explore the characteristics of normal distributions and learn how to apply standard deviations to fit certain data sets to a normal curve. You will use normal distributions to determine the percent of data that is represented within various areas of the curve. You will then consider how to draw reasonable conclusions based on statistics. You will analyze data using standard deviation, margin of error, and confidence intervals.

TOPIC 1

Interpreting Data in Normal Distributions

Normal distributions are sometimes called "bell curves" because their shape is similar to the silhouette of a bell: wide at the bottom and curving up symmetrically to a central peak.

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Module 5: Relating Data and Decisions

TOPIC 1: INTERPRETING DATA IN NORMAL DISTRIBUTIONS

This topic begins by using student knowledge of relative frequency histograms to introduce normal distributions. Students build their knowledge of normal distributions using the Empirical Rule for Normal Distributions. They are introduced to *z*-scores and use a *z*-score table and technology to determine the percent of data in given intervals that are bounded by non-integer multiples of the standard deviation from the mean. Finally, students integrate their knowledge of probability and normal distributions to analyze scenarios and make decisions.

Where have we been?

Since middle school, students have created and analyzed data in a variety of distributions, and they have compared different displays. They know that mean and standard deviation are two ways to quantify center and spread, and they know how to calculate each. Students may be familiar with percentiles from growth charts and standardized tests, and this topic makes sense of these values and provides a process to calculate them.

Where are we going?

Students are inundated with data in school, on the news, and through social media. They need to recognize the structure of data sets and which measures of center and spread can be used to interpret them. Developing the ability to analyze data helps students to interpret test scores, make good consumer decisions, and draw their own sound conclusions using data.

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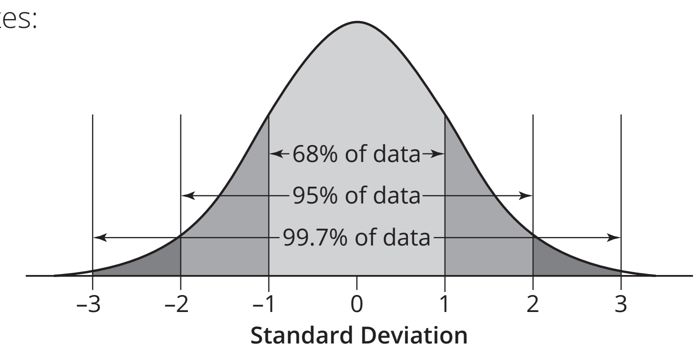
The Empirical Rule for Normal Distributions

The Empirical Rule for Normal Distributions states:

- Approximately 68% of the data in a normal distribution for a population is within 1 standard deviation of the mean.
- Approximately 95% of the data in a normal distribution for a population is within 2 standard deviations of the mean.
- Approximately 99.7% of the data in a normal distribution for a population is within 3 standard deviations of the mean.



The Empirical Rule applies most accurately to population data rather than sample data. However, the Empirical Rule is often applied to data in large samples.

Black Monday

On October 19, 1987, stock markets around the world fell into sharp decline. In the United States, the Dow Jones Industrial Average dropped 508 points—a 22% loss in value. Black Monday, as the day came to be called, represented at the time the largest one-day decline in the stock market ever.

According to some economic models, the crash that occurred on Black Monday represented an event that was 20 standard deviations away from the normal behavior of the market. Mathematically, the odds of a Black Monday-type event occurring were 1 in 10^{50}.

Talking Points

Normal distributions can be an important topic to know about for college admissions tests.

Here is an example of a sample question:

Suppose SAT scores are approximately normally distributed. In recent years, the SAT mathematics scores have averaged around 480 with a standard deviation of 100. What percent of students will have a math score less than 550?

Calculate the *z*-score for a math score of 550: $z = \frac{550 - 480}{100} = 0.7$. Then consult a *z*-score table for 0.7.

Approximately 75.8% of students will score less than 550.

Key Terms

normal distribution

A normal curve is a bell-shaped curve that is symmetric about the mean of the data. A normal curve models a theoretical data set that is said to have a normal distribution.

standard deviation

The standard deviation of data is a measure of how spread out the data are from the mean.

***z*-scores**

A *z*-score is the number of standard deviations that a data point is from the mean of a normal distribution.

percentile

A percentile is a data value for which a certain percentage of the data is below the data value in a normal distribution.

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Recharge It!

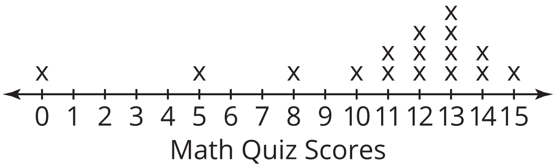
Normal Distributions

1

Warm Up

Describe the data distribution in each representation as skewed right, skewed left, or symmetric.

1.



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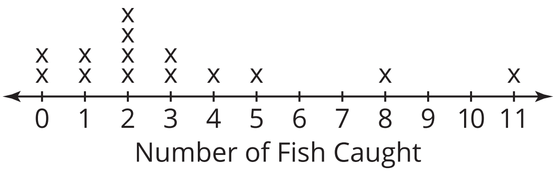

Learning Goals

- Differentiate between discrete data and continuous data.
- Draw distributions for continuous data.
- Recognize the difference between normal distributions and non-normal distributions.
- Recognize and interpret properties of a normal curve and a normal distribution.
- Describe the effect of changing the mean and standard deviation on a normal curve.

Key Terms

- discrete data
- continuous data
- sample
- population
- normal curve
- normal distribution
- mean (μ)
- standard deviation (σ)

You have constructed histograms from data sets. What does the shape of the histogram tell you about the data set?

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GETTING STARTED

Low Battery

Discrete data are data whose possible values are countable. The scores of baseball games are examples of discrete data, because a team's score must be a whole number.

Continuous data are data which can include any numeric value within a range. Heights of students, times required to complete a test, and distances between cities are examples of continuous data.

Recall that a discrete graph is a graph of isolated points and a continuous graph is a graph of points that are connected by a line or smooth curve on the graph. Data can also be discrete or continuous.

Suppose that two cell phone companies, E-Phone and Unlimited, claim that two of their comparable cell phone models have a mean battery life of 10 hours.

1. **Are the durations of the cell phone batteries examples of discrete data or continuous data? Explain your reasoning.**

2. **If the mean battery life is 10 hours, does that indicate that all of E-Phone's phones and all of Unlimited's phones have a 10-hour battery life? Explain your reasoning.**

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ACTIVITY 1.1

Normal Distributions

One way to display continuous data is to use a relative frequency table. The relative frequency tables shown display the battery lives of a *sample* of 100 E-Phone cell phones and 100 Unlimited cell phones.

A **sample** is a subset of data selected from a *population*. A **population** represents all the possible data that are of interest in a study or survey.

The battery lives are divided into intervals. Each interval includes the first value but does not include the second value. For example, the interval 8.0–8.5 includes phones with battery lives greater than or equal to 8 hours and less than 8.5 hours.

Remember:

A relative frequency is the ratio of occurrences within an interval to the total number of occurrences. You can represent this ratio as a decimal or percent.

1. Complete the tables by calculating the relative frequency of phones in each interval. Explain how you determined the relative frequencies.

E-Phone		
Battery Life (hours)	Number of Phones	Relative Frequency
8.0–8.5	1	
8.5–9.0	2	
9.0–9.5	17	
9.5–10.0	30	
10.0–10.5	32	
10.5–11.0	15	
11.0–11.5	3	
11.5–12.0	0	

Unlimited		
Battery Life (hours)	Number of Phones	Relative Frequency
8.0–8.5	0	
8.5–9.0	1	
9.0–9.5	14	
9.5–10.0	37	
10.0–10.5	36	
10.5–11.0	11	
11.0–11.5	0	
11.5–12.0	1	

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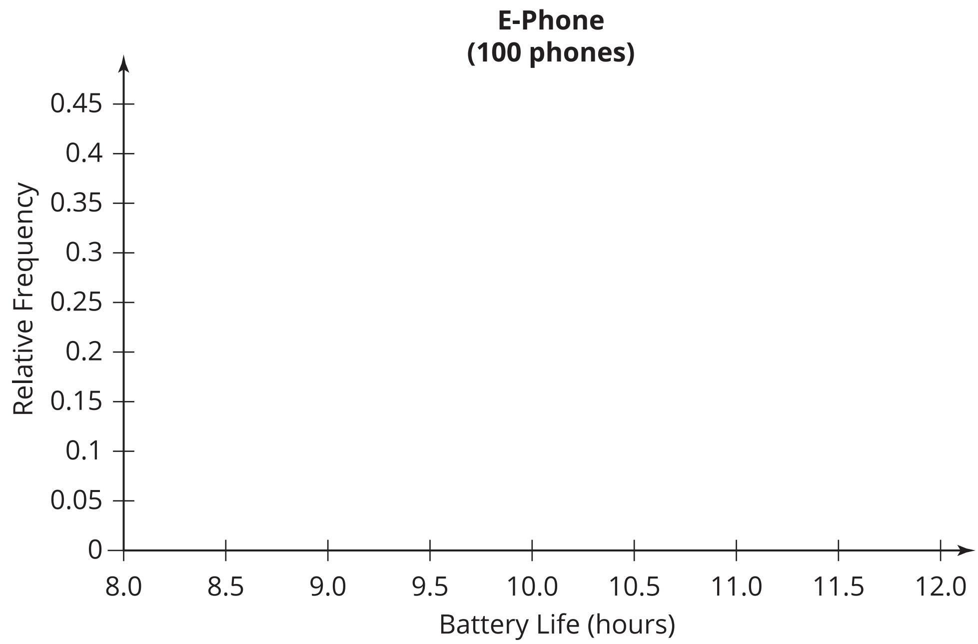
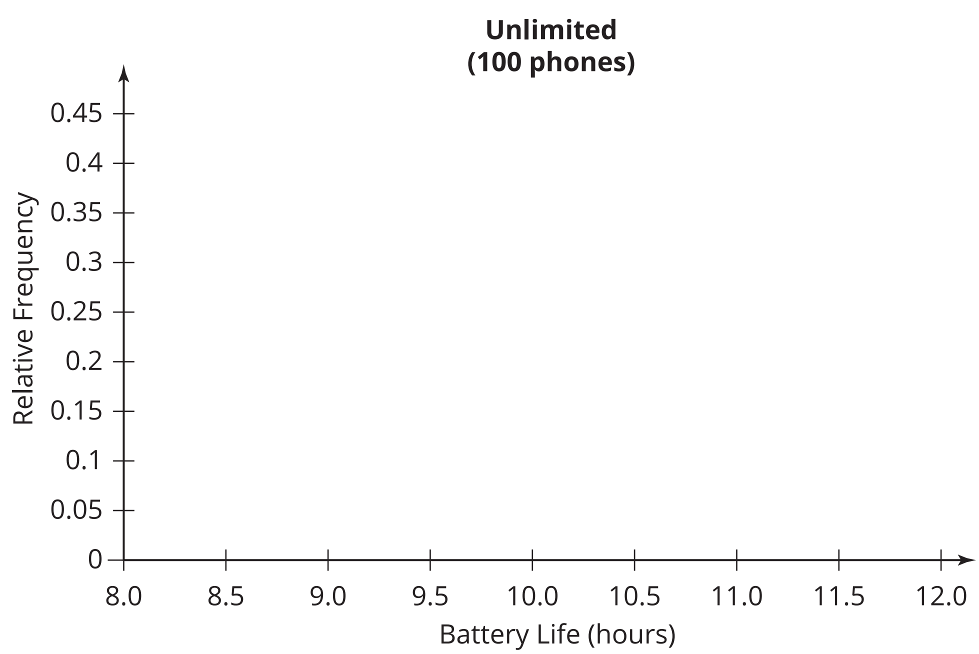
For continuous data, a relative frequency histogram displays continuous intervals on the horizontal axis and relative frequency on the vertical axis.

2. Create a relative frequency histogram to represent the battery lives of the 100 cell phones in each sample.





Remember:

The shape of a data distribution can reveal information about the data. Data can be widely spread out or packed closely together. Data distributions can also be skewed or symmetric.

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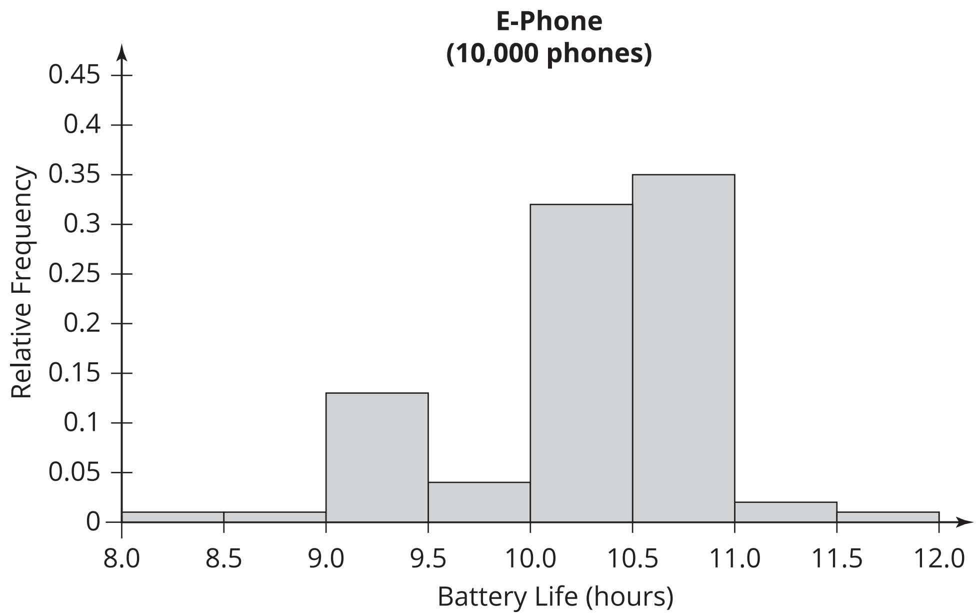
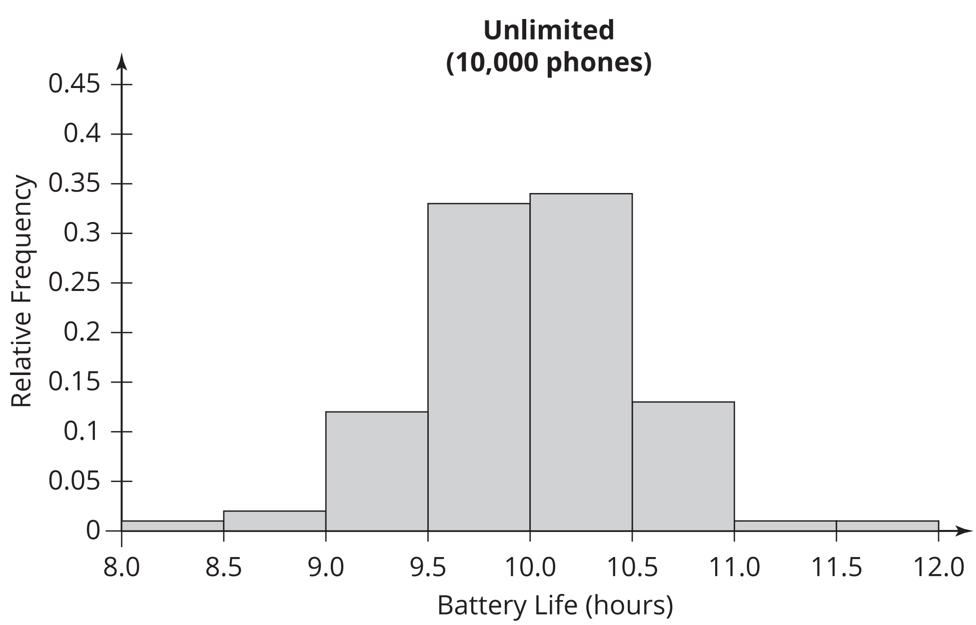
3. **Describe the shape and spread of each histogram. What might these characteristics reveal about the data for each company?**

4. **The relative frequency histograms shown represent samples of 10,000 phones from each of the two companies. Compare the histograms created from a sample of 10,000 cell phones to the histograms created from a sample of 100 cell phones. How does increasing the sample size change the appearance of the data distributions?**





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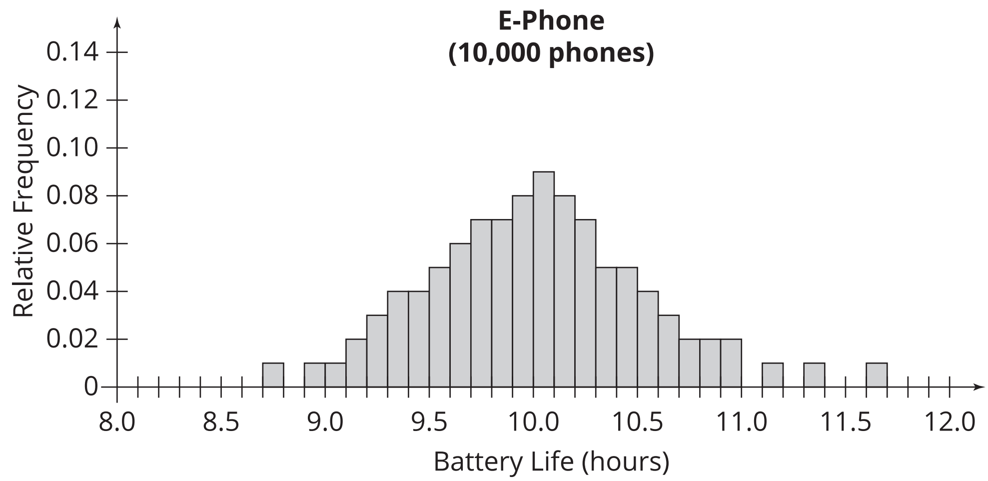
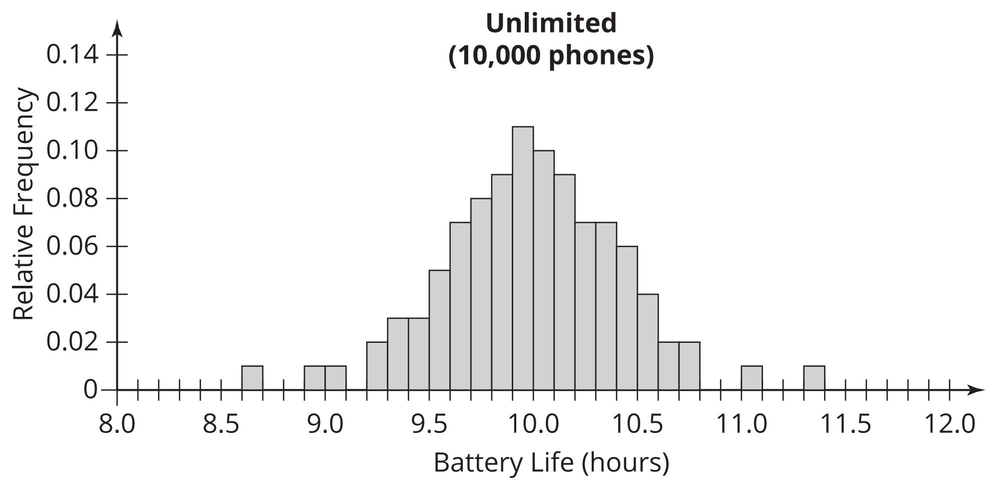
5. **The histograms shown represent the same samples of 10,000 phones, but now the data have been divided into intervals of 0.1 hour instead of 0.5 hour. Compare these histograms with the histograms from the previous question. How does decreasing the interval size change the appearance of the data distributions?**





6. **Explain why the scale of the *y*-axis changed when the interval size decreased.**

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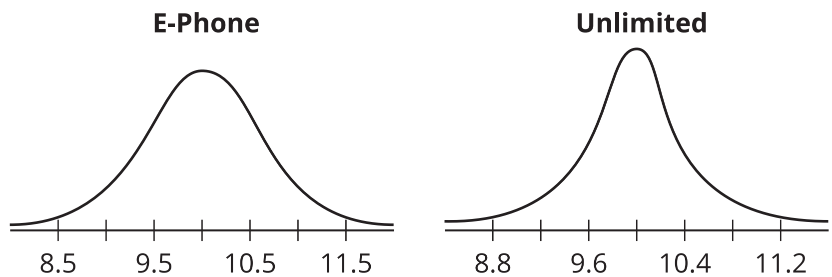
In this problem situation, as the sample size continues to increase and the interval size continues to decrease, the shape of each relative frequency histogram will likely start to resemble a *normal curve*. A **normal curve** is a bell-shaped curve that is symmetric about the mean of the data. A normal curve models a theoretical data set that is said to have a **normal distribution**.

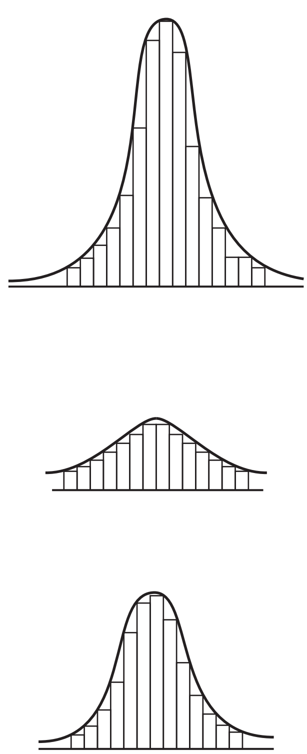
The vertical axis for a graph of a normal curve represents relative frequency, but normal curves are often displayed without a vertical axis.

The normal curves for the E-Phone and Unlimited cell phone battery lives are shown. In order to display normal curves for each data set, different intervals were used on the horizontal axis in each graph.

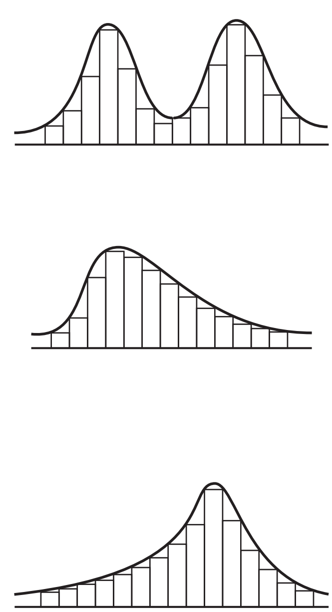


Although normal curves can be narrow or wide, all normal curves are symmetric about the mean of the data.

Normal Distributions



Not Normal Distributions



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ACTIVITY
1.2

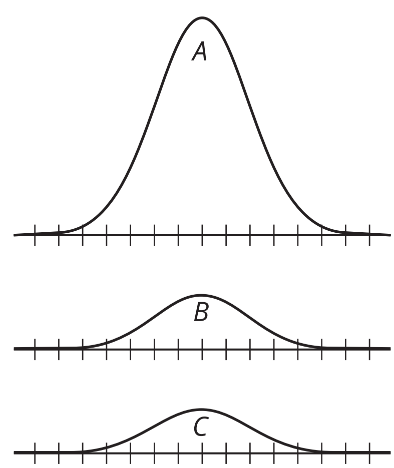
Standard Deviation of a Curve

The **Greek letter** μ is pronounced "mu". The symbol $\overline{x}$ is expressed as "*x* bar."

You already know a lot about the mean. With normal curves, the **mean** of a population is represented with the symbol μ. The mean of a sample is represented with the symbol $\overline{x}$. The **standard deviation** of data is a measure of how spread out the data are from the mean. The symbol used for the standard deviation of a population is the sigma symbol (σ). The standard deviation of a sample is represented with the variable *s*. When interpreting the standard deviation of data:

- A lower standard deviation represents data that are more tightly clustered near the mean.
- A higher standard deviation represents data that are more spread out from the mean.

1. Normal curves *A*, *B*, and *C* represent the battery lives of a population of cell phones of comparable models from three different companies.



The normal curves represent distributions with standard deviations of $\sigma = 0.1$, $\sigma = 0.4$, and $\sigma = 0.5$. Match each standard deviation value with one of the normal curves and explain your reasoning.

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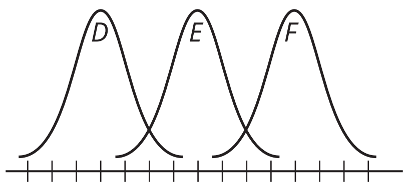
2. **Normal curves *D*, *E*, and *F* represent the battery lives of cell phones from three different companies.**



a. **Identify and compare the mean of each company.**

b. **Compare the standard deviation of each distribution. What do you notice?**

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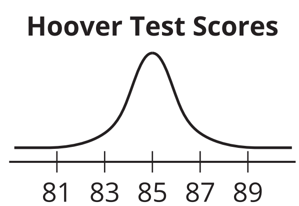
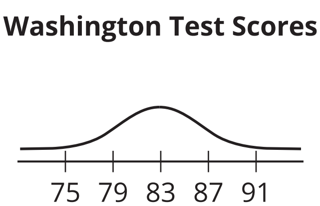
NOTES

TALK the TALK

Throw Me a Curve

The standardized test scores at Hoover Elementary School and Washington Elementary School are represented by the curves shown.





1. **Describe the shape of each curve.**

2. **Identify and compare the mean of each school.**

3. **Compare the standard deviation of each school. What do you notice?**

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Assignment

Write

Write the term that best completes each statement.

1. A normal curve models a theoretical data set that is said to have a ______________.
2. ______________ are data which can take any numeric value within a range.
3. A bell shaped curve that is symmetric about the mean of a data set is a ______________.
4. Data whose possible values are countable and often finite are ______________.
5. The ______________ of a population is often represented with the symbol μ.
6. The ______________ of data is a measure of how spread out the data are and its often used symbol is σ.

Remember

The shape of a distribution can change as the number of data points increase and the size of the intervals decrease. A relative frequency histogram will likely start to resemble a normal curve. The smaller the standard deviation, the closer most of the data lie to the mean.

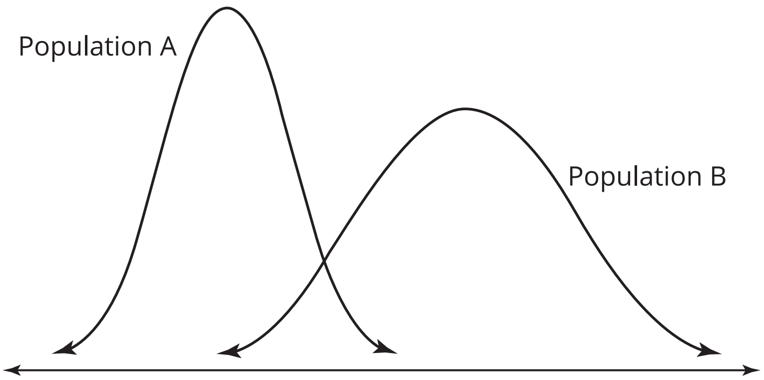
Practice

1. Two hundred runners completed the annual Burgoo Festival 5K race.
   a. The table displays the race times for the 200 runners. Complete the table to determine the relative frequency for each interval of race times.
   b. Create a relative frequency histogram to represent the race times of the 200 runners.
   c. Does the distribution of the race time data appear to be a normal distribution? Explain your reasoning.

Race Time (minutes)	Number of Runners	Relative Frequency
14 – 18	7	
18 – 22	28	
22 – 26	65	
26 – 30	71	
30 – 34	24	
34 – 38	5	

2. Wildlife biologists recorded the weights of grizzly bears in two different populations. The normal curves represent the weights of the bears in Population A and the weights of bears in Population B.
   a. Which population has the greater mean weight? Explain your reasoning.
   b. Which population has the greater standard deviation? Explain your reasoning.
   c. Explain what the difference in the standard deviations means in terms of this problem situation.

Weight of Bears



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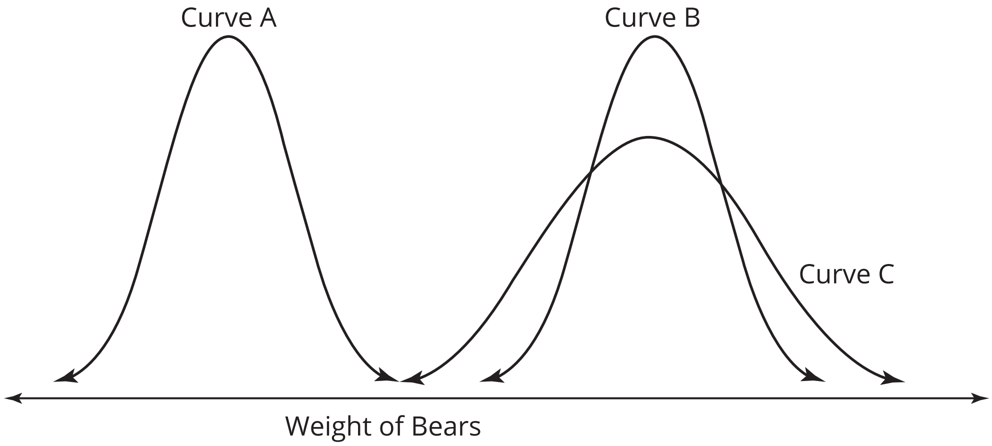
d. Two years after the original data was recorded, the biologists recorded the weights of the bears in Population A again. The mean weight had increased by 5 pounds, but the standard deviation remained the same. Explain what the difference in the new data and the original data means in terms of this problem situation.

e. Two years after the original data was recorded, the biologists recorded the weights of the bears in Population B again. The mean weight was the same, but the standard deviation had decreased. Explain what the difference in the new data and the original data means in terms of this problem situation.

Stretch

1. Consider the three normal curves shown. Determine a plausible data set for each curve. Explain your reasoning.



Review

1. The equation $d(t) = 9 \cos (\frac{\pi}{4}t)$ can be used to model the distance the pendulum of a clock is in inches from its center position as a function of time. The pendulum is released from its rightmost position. Assume that the right of center is a positive distance and the left of center is a negative distance.
   a. Determine the pendulum's distance from the center at 11 seconds.
   b. Determine when the pendulum is 4 inches to the left of center.
2. Use a periodicity identity to list 3 solutions for the equation $\tan x = -\sqrt{3}$.
3. Solve the equation $3 - 2 \tan \theta = -2$ over the domain of all real numbers.
4. Write an equation of a sine curve with amplitude 4, period 3, and phase shift 2.
5. Write an equation of a cosine curve with amplitude $\frac{1}{2}$, period $\frac{\pi}{3}$, and phase shift $-\frac{3}{2}$.

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2

The Form of Norm

The Empirical Rule for Normal Distributions

Warm Up

1. Describe the similarities and differences between two normal curves that have the same mean but different standard deviations.
2. Describe the similarities and differences between two normal curves that have the same standard deviation but different means.

Learning Goals

- Recognize the connection between relative frequency histograms, normal curves, and the Empirical Rule for Normal Distributions.
- Use the Empirical Rule for Normal Distributions to determine the percent of data in a given interval.

Key Terms

- standard normal distribution
- Empirical Rule for Normal Distributions

You have learned to recognize data sets that are normally distributed. How can you determine the percent of data in a given interval?

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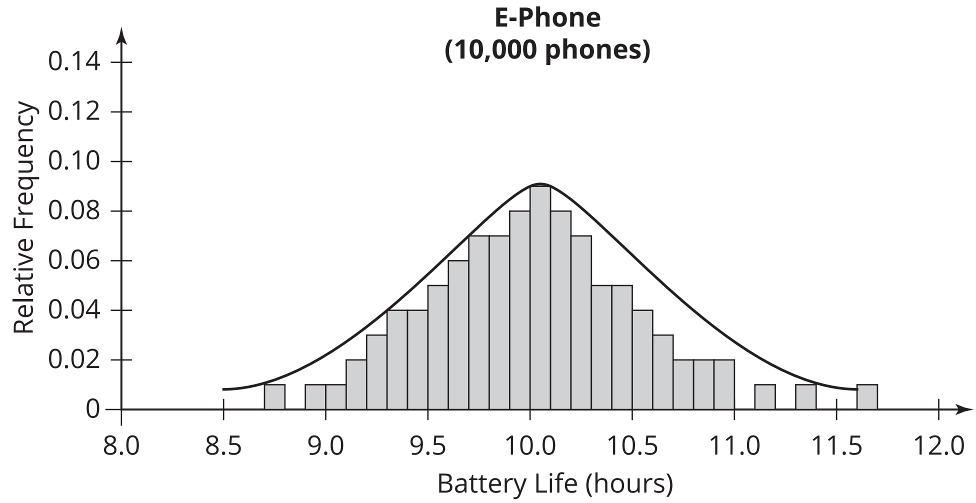
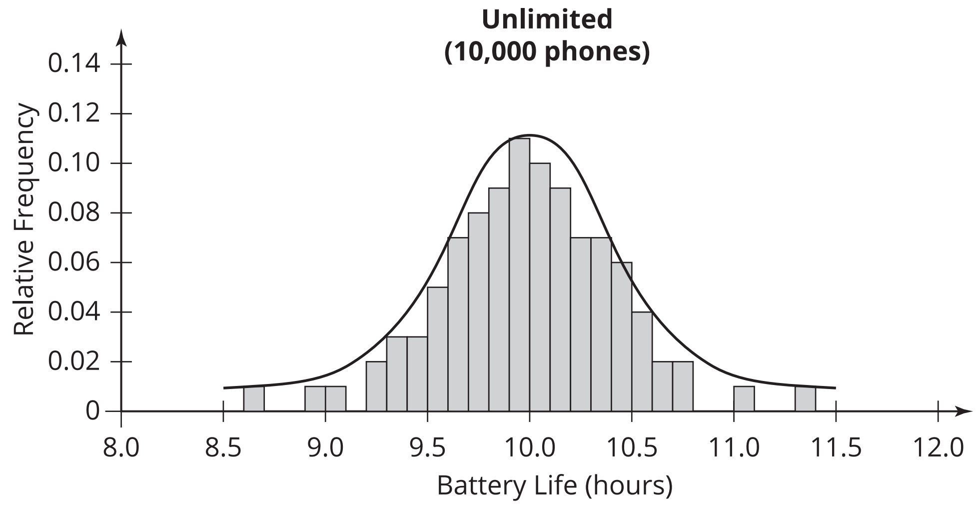
Under Normal Circumstances

The relative frequency histograms for the battery lives of E-Phone and Unlimited cell phones are shown. The normal curves for each data set are mapped on top of the histogram.





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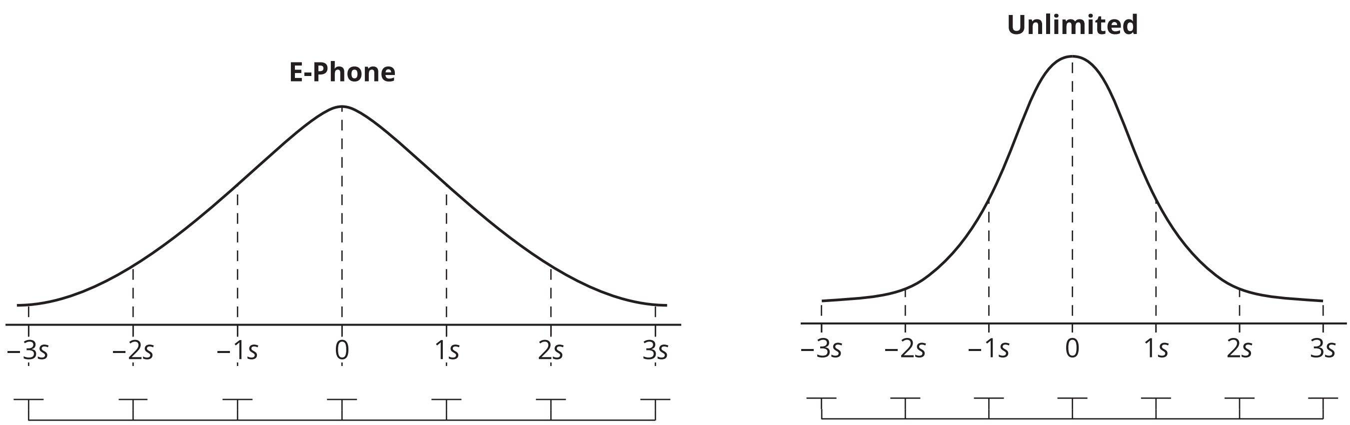
Normal curves can be graphed with units of standard deviation on the horizontal axis. The normal curve for the E-Phone sample has a standard deviation of 0.5 hour ($s = 0.5$), and the normal curve for the Unlimited sample has a standard deviation of 0.4 hour ($s = 0.4$). The mean of each sample is $\overline{x} = 10.0$ hours.

Notice that different symbols are used to represent the mean and standard deviation of a sample as opposed to a population.

1. The normal curve for the battery life of each cell phone is given. For each standard deviation shown, label the corresponding battery life in hours.



2. Interpret the meaning of $s = 0$ for each graph.

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ACTIVITY
2.1 Standard Normal Distribution

A relative frequency histogram not only gives the normal curve its shape, it also provides information regarding the percent of data that lie between each standard deviation of the normal curve.

1. Use the histograms from the Getting Started to estimate the percent of data between each standard deviation. Write each percent in the appropriate space on the graphs of the normal curves in the previous activity. Explain your reasoning.

"Within one standard deviation" means between $-1s$ and $1s$, or between -1σ and 1σ.

2. Compare the percents in each standard deviation interval for E-Phone with the percents in each standard deviation interval for Unlimited. What do you notice?

3. Use your results to answer each question. Explain your reasoning.

a. Estimate the percent of data within 1 standard deviation of the mean.

b. Estimate the percent of data within 2 standard deviations of the mean.

c. Estimate the percent of data within 3 standard deviations of the mean.

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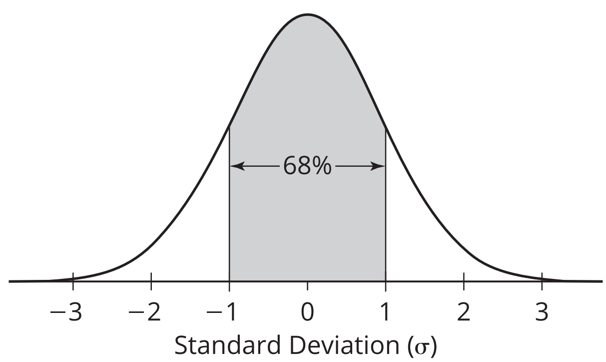
The **standard normal distribution** is a normal distribution with a mean value of 0 and a standard deviation of 1σ or $1s$. In a standard normal distribution, 0 represents the mean. Positive integers represent standard deviations greater than the mean. Negative integers represent standard deviations less than the mean.

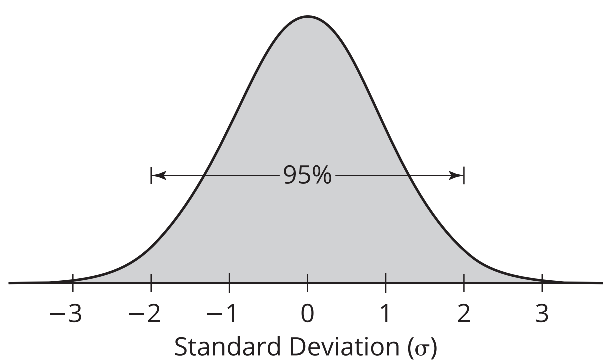
The **Empirical Rule for Normal Distributions** states:

The Empirical Rule for Normal Distributions is often summarized using a standard normal distribution curve because it can be generalized for any normal distribution curve.

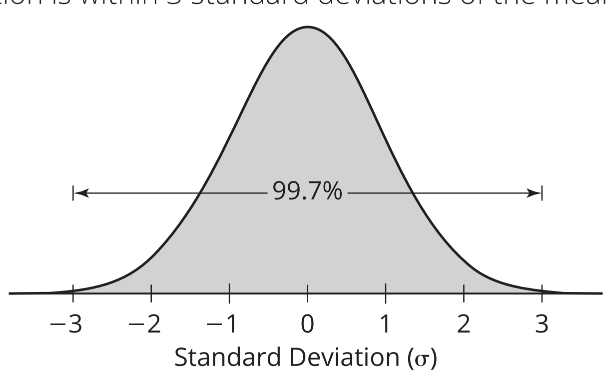
- Approximately 68% of the data in a normal distribution for a population is within 1 standard deviation of the mean.



- Approximately 95% of the data in a normal distribution for a population is within 2 standard deviations of the mean.



- Approximately 99.7% of the data in a normal distribution for a population is within 3 standard deviations of the mean.



The Empirical Rule applies most accurately to population data rather than sample data. However, the Empirical Rule is often applied to data in large samples.

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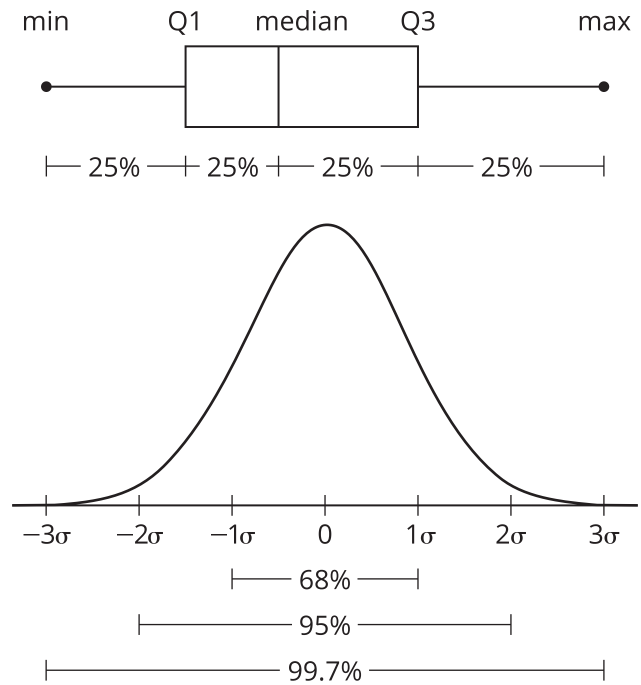
Recall that a box-and-whisker plot is a graph that organizes, summarizes, and displays data based on quartiles that each contains 25% of the data values.



4. Compare the box-and-whisker plot with the standard normal distribution curve. What similarities and/or differences do you notice?

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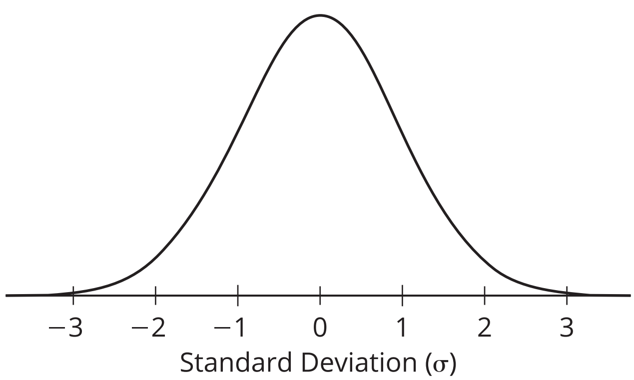
ACTIVITY 2.2

Analyzing Intervals of a Normal Distribution

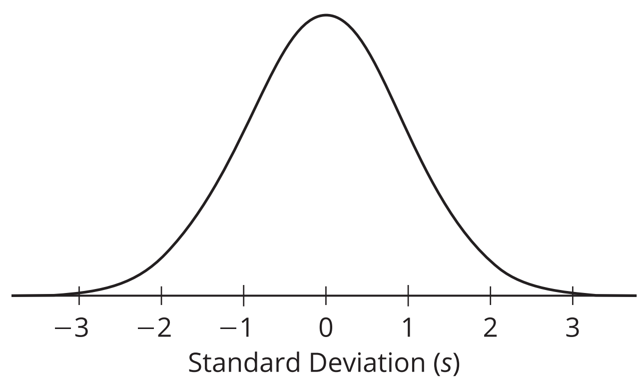
You can use the Empirical Rule for Normal Distributions to estimate the percent of data within specific intervals of a normal distribution.

1. State whether the distribution shown represents population data or sample data. Then determine each percent, shade the corresponding region under each normal curve, and explain your reasoning.

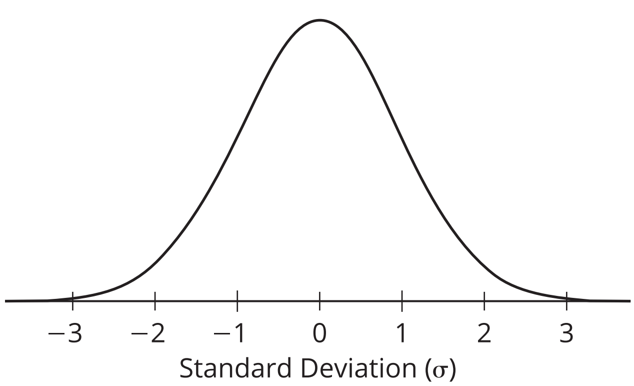
a. What percent of the data is greater than the mean?



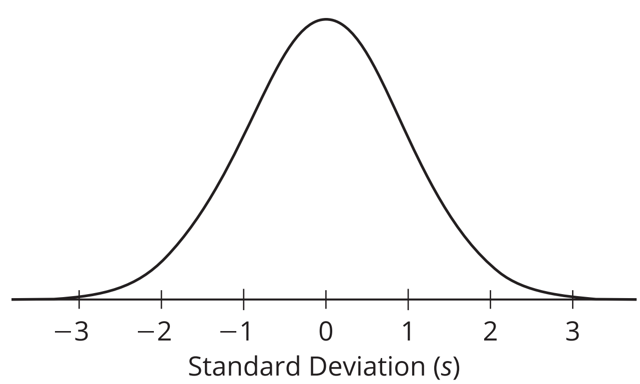
b. What percent of the data is between the mean and 2 standard deviations below the mean?



c. What percent of the data is between 1 and 2 standard deviations above the mean?



d. What percent of the data is more than 2 standard deviations below the mean?



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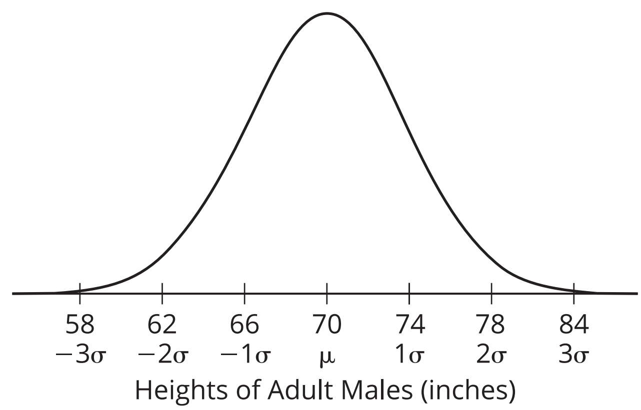
Remember:

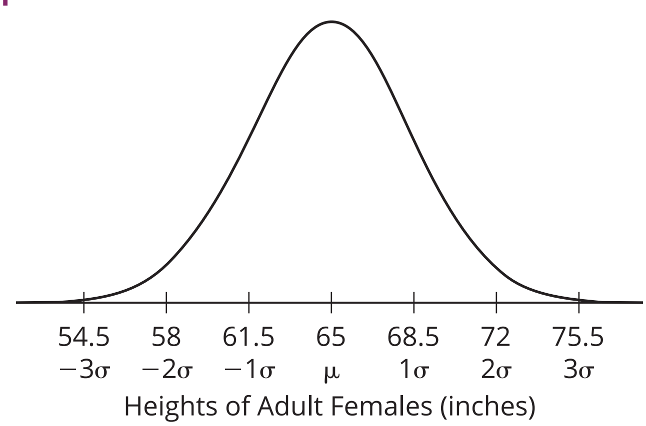
1σ represents a data value that is one standard deviation greater than the population mean and -1σ represents a data value that is one standard deviation less than the population mean.

2. Consider each normal curve. State whether the distribution shown represents population data or sample data. Then determine each percent, shade the corresponding region under each normal curve, and explain your reasoning.

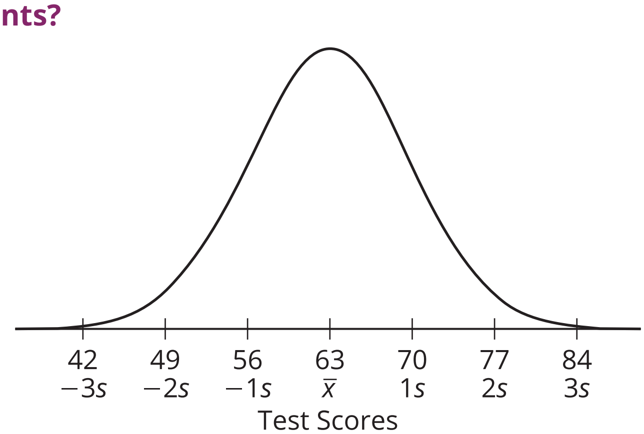
a. What percent of adult males have a height between 62 inches and 74 inches?



b. What percent of adult females are taller than 68.5 inches?



c. What percent of history test scores are between 63 points and 70 points?



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ACTIVITY
2.3

Area Under the Curve

There are actually many different normal distributions. Each is dependent upon only two parameters, the mean and standard deviation.

Similarities between all normal distributions:

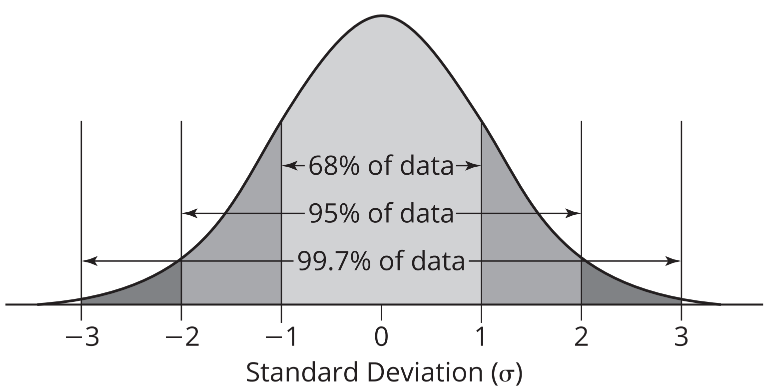
- The graph is bell-shaped and symmetric about the mean.
- 68% of the data lies within −1 and +1 standard deviation of the mean.
- 95% of the data lies within −2 and +2 standard deviations of the mean.
- 99.7% of the data lies within −3 and +3 standard deviations of the mean.

Differences between normal distributions:

- Means can be different.
- Standard deviation can be different.

Characteristics of the standard normal curve:

- Symmetric about the *y*-axis (mean).
- Concave downward between $x = 1$ and $x = -1$ and concave upward everywhere else.
- Asymptotic to the *x*-axis from above.
- The mean is equal to 0. $\mu = 0$
- The standard deviation is equal to 1. $\sigma = 1$
- The area under the curve equals 1 square unit.



The total area under the standard normal curve is 1 square unit. This corresponds to the sum of all the relative frequencies of a histogram of a normally distributed set of data.

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1. **What is the area under the curve of a standard normal distribution between 2 standard deviations below the mean to 1 standard deviation above the mean?**

2. **How is the area under the curve of a standard normal distribution related to the percent of the data within a specific interval of the normal curve?**

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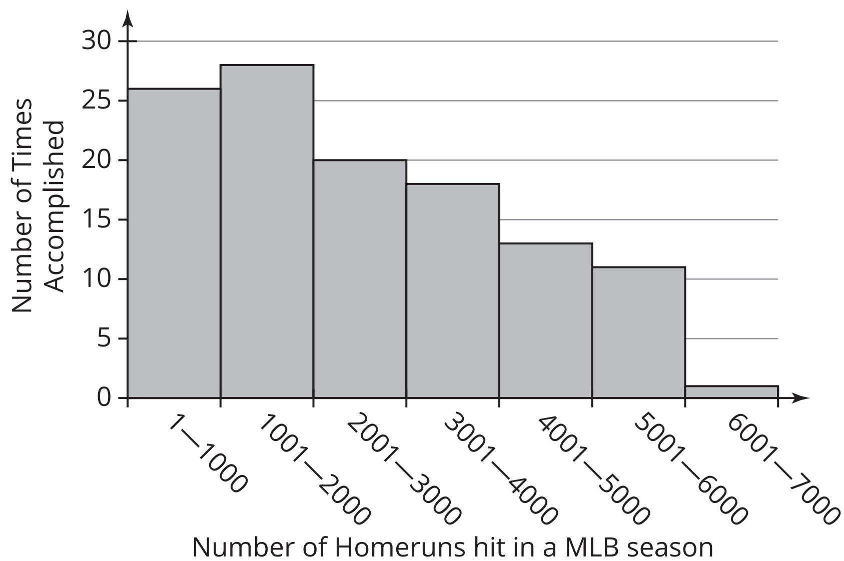
ACTIVITY
2.4 Data Distributions

Consider each situation. Determine whether the data generated from the context has a normal distribution. Then state whether the Empirical Rule for Normal Distributions can be applied to the data. Explain your reasoning.

Remember:

The Empirical Rule applies only to normally distributed data sets.

1. The histogram displays the number of homeruns hit in major league baseball each year since 1901.



2. James rolled a single number cube 100 times. His results are shown.

15 ones, 16 twos, 17 threes, 17 fours, 18 fives, and 17 sixes.

3. Juanita is a researcher who is studying a type of bacteria that doubles in number every minute.

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NOTES

TALK the TALK

Sing for Your Supper

Most days Yesenia eats dinner at 6:00 PM. Her dinner time can be modeled using a normal distribution curve with a standard deviation of 15 minutes.

1. Identify the mean in this situation.

2. What is the meaning of one standard deviation in this situation?

3. Sketch a standard normal curve for this situation that includes 3 standard deviations above and below the mean.

4. What percent of the data lies between 6:15 PM and 6:30 PM? What does this mean in the context of the problem

5. What is the area under the standard normal curve between 6:15 PM and 6:30 PM? How did you determine your answer?

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Assignment

Write

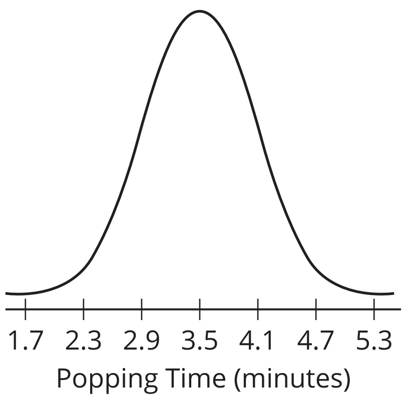
Describe the percent of data within 1, 2, and 3 standard deviations of the mean in a normal distribution according to the Empirical Rule.

Remember

The standard normal distribution is a normal distribution with a mean value of 0 and a standard deviation of 1. The Empirical Rule defines the percent of data within 1, 2, and 3 standard deviations of the mean in a normal distribution.

Practice

1. A researcher recorded the birth weights of a sample of newborn babies. The average birth weight was 7.2 pounds and the standard deviation was 0.9 pound. Assume that the birth weights follow a normal distribution.
   a. Sketch and label a normal curve. Include 3 standard deviations above and below the mean.
   b. Determine the percent of newborns that weigh between 2 standard deviations below the mean and 2 standard deviations above the mean.
   c. Determine the percent of newborns that weigh less than the mean.
   d. Determine the percent of newborns that weigh between 1 standard deviation above the mean and 3 standard deviations above the mean.
   e. Approximately what percent of newborns weighed between 4.5 pounds and 9.9 pounds? Weighed more than 9 pounds? Weighed less than 6.3 pounds?



2. The time to cook a bag of microwave popcorn is normally distributed with a mean of 3.5 minutes and a standard deviation of 0.6 minute. Suppose that you randomly select a microwave popcorn bag from the sample. Use the given information and the distribution to answer each question. Explain your reasoning. Determine the percent of microwave popcorn bags that will properly cook within each timespan.
   a. Between 2.9 minutes and 4.1 minutes
   b. More than 3.5 minutes
   c. Less than 2.3 minutes
   d. More than 2.9 minutes?
   e. Between 4.7 minutes and 5.3 minutes?

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Stretch

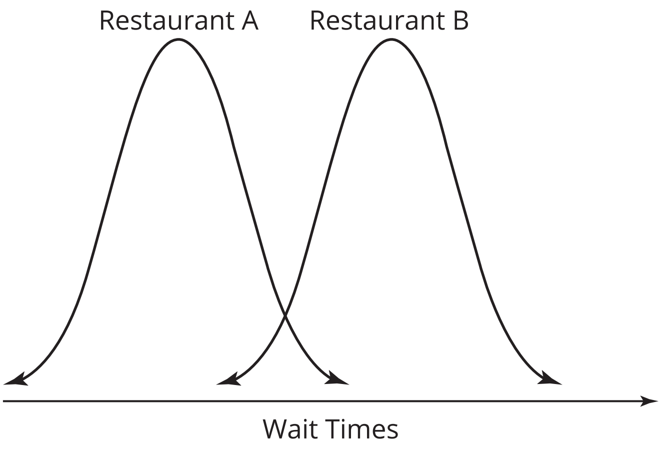
1. A department store has a manager training program for qualified applicants. Before applying to the manager training program, applicants are given a basic mathematics test. The scores are normally distributed, with a mean of 40 and a standard deviation of 5.
   a. If an applicant is told that 50% of all the applicants scored worse than they did on the test, what was their score? Explain your reasoning.
   b. If an applicant is told that only 16% of all the applicants scored better than they did on the test, what was their score? Explain your reasoning.
   c. If an applicant is told that only 2.5% of all the applicants scored worse than they did on the test, what was their score? Explain your reasoning.

Review

1. An annual photography competition received three hundred entries. The table shows the distribution of ages of the photographers who entered the competition.
   a. Create a relative frequency histogram to represent the ages of the 250 photographers.
   b. Does the distribution of the race time data appear to be a normal distribution? Explain your reasoning.

Ages	Number of Relative Photographers	Relative Frequency
15 – 24	13	0.043
24 – 33	61	0.203
33 – 42	87	0.290
42 – 51	55	0.183
51 – 60	48	0.160
60 – 69	24	0.080
69 – 78	12	0.040

2. A chain restaurant records the times that diners spend waiting for a table in 2 different restaurants. The normal curves represent the wait times at Restaurant A and the wait times at Restaurant B.
   a. Compare the mean wait times of the two restaurants. Explain your reasoning.
   b. Compare the standard deviation of the two restaurants. Explain your reasoning.



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3

Above, Below, and Between the Lines

z-scores and Percentiles

Warm Up

The battery life of a laptop computer has a mean of 16 hours with a standard deviation of 2.5 hours. Determine the interval of battery life for each given percent of laptop computers.

1. 68%

2. 95%

3. 99.7%

Learning Goals

- Use a *z*-score table or technology to calculate the percent of data below any given data value, above any given data value, and between any two given data values in a normal distribution.
- Use a *z*-score table or technology to determine the data value that represents a given percentile.

Key Terms

- *z*-score
- percentile

You have learned to determine the percent of data in an interval defined by integer value standard deviations using the Empirical Rule. How can you determine the percent of data within any interval in a normal distribution?

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GETTING STARTED

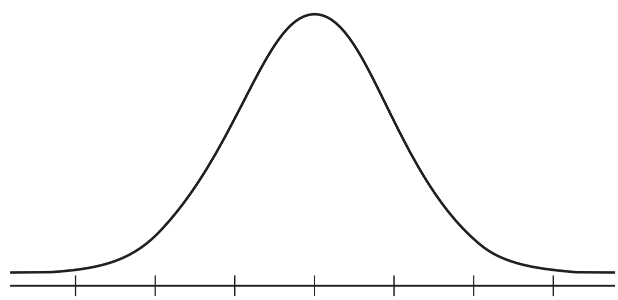
More Bang for the Buck

Ask yourself:

What does 1*s* equal in this situation?

The fuel efficiency of a sample of hybrid cars is normally distributed with a mean of 54 miles per gallon (mpg) and a standard deviation of 6 miles per gallon.

1. Use the mean and standard deviation to label the intervals on the horizontal axis of the normal curve in miles per gallon.



2. Determine the percent of hybrid cars that get less than 60 miles per gallon. Explain your reasoning.

3. Determine the percent of hybrid cars that get less than 66 miles per gallon. Explain your reasoning.

4. Determine the percent of hybrid cars that get less than 72 miles per gallon. Explain your reasoning.

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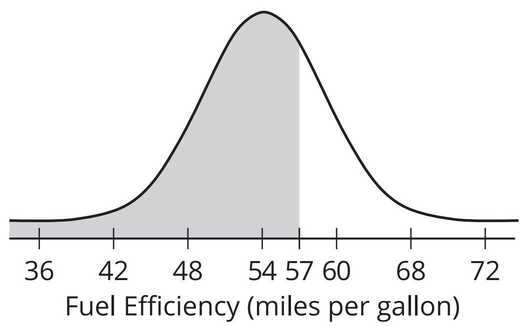
Determining *Z*-Scores

When data values are aligned with integer multiples of the standard deviation from the mean, you can use the Empirical Rule for Normal Distributions to calculate the percent of data values less than that value. But what if a data value does not align with the standard deviations?

1. Let's consider the fuel efficiency of hybrid cars again. The mean is 54 miles per gallon and 1 standard deviation is 6 miles per gallon. What percent of cars get less than 57 miles per gallon?



a. How many standard deviations from the mean is 57 miles per gallon? Explain how you determined your answer.

b. DMitrius incorrectly estimated the percent of hybrid cars that get less than 57 miles per gallon. Explain why DMitrius' reasoning is incorrect.

DMitrius

Approximately 67% of hybrid cars get less than 57 miles per gallon.

$$50\% + \frac{1}{2}(34\%) \approx 67\%$$

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You can alter any normally distributed set of data to make it a standard normal distribution by converting the data values into z-scores.

The number you calculated in Question 1, part (a) is a z-score. A ***z*-score** is a number that describes a specific data value's distance from the mean in terms of standard deviation units.

For a population, a z-score is determined by the equation $z = \frac{x - \mu}{\sigma}$, where x represents a value from the data.

You can use a z-score table to determine the percent of data less than a given data value with a corresponding z-score. A z-score table is located at the end of this lesson.

Think about:

The formula $z = \frac{x - \mu}{\sigma}$ when $x = 57$ results in a z-score of 0.5.

Worked Example

The percent of hybrid cars that get less than 57 miles per gallon has a z-score of 0.5.

z	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199
0.1	0.5398	0.5433	0.5478	0.5517	0.5557	0.5596
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088

Note that the table represents z-scores only to the hundredths place.

To use a z-score table, locate the row that represents the ones and tenths place of the z-score. For a z-score of 0.5, this is the row labeled 0.5. Then, locate the column that represents the hundredths place of the z-score. For a z-score of 0.5, this is the column labeled 0.00.

Next, locate the cell that is the intersection of the row and column. For a z-score of 0.5, the corresponding cell reads 0.6915. Rewrite this decimal value as a percent.

This means that 69.15% of hybrid cars get less than 57 miles per gallon.

2. What would a negative *z*-score indicate? Explain your reasoning.

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Technology can help you determine the percent of data below a certain *z*-score. There are apps that allow you to enter a *z*-score and will calculate the area to the left of the *z*-score. Many calculators also have this feature.

3. Use technology to determine the approximate percent of hybrid cars that get less than 57 miles per gallon.

4. How does your answer using technology compare to your answer using the *z*-score table?

5. Irena calculated the percent of hybrid cars that get less than 56 miles per gallon using a *z*-score table and technology. Explain why Irena received different results.

z	0.0	0.01	0.02	0.03
0.0	0.5000	0.5040	0.5080	0.5120
0.1	0.5398	0.5438	0.5478	0.5517
0.2	0.5793	0.5832	0.5871	0.5910
0.3	0.6179	0.6217	0.6255	0.6293

Irena
normal cdf (−1E99, 56, 54, 6)

0.6305585963

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6. Calculate the percent of hybrid cars that get less than 50 miles per gallon.

7. Calculate the percent of hybrid cars that get between 50 and 60 miles per gallon. Explain your reasoning.

ACTIVITY
3.2

Calculating Percentiles

You may have heard someone say, "My baby's weight is in the 90th percentile" or, "My student scored in the 80th percentile in math." What do these phrases mean?

A **percentile** is a data value for which a certain percent of the data is below the data value in a normal distribution. For example, 90% of the data in a set is below the value at the 90th percentile, and 80% of the data is below the value at the 80th percentile.

1. The number of text messages teens send and receive every day can be represented as a normal distribution with a mean of 100 text messages per day and a standard deviation of 25 texts per day.

a. Calculate the 50th percentile for this data set. Explain your reasoning.

b. Would a teen in the 90th percentile send and receive more or fewer than 100 text messages per day? Explain your reasoning.

c. Would a teen in the 10th percentile send and receive more or fewer than 100 text messages per day? Explain your reasoning.

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2. Use the *z*-score table located at the end of the lesson to determine the 90th percentile for teen text messages.

a. Determine the percent value in the *z*-score table that is closest to 90%. Explain what information the *z*-score provides.

b. Calculate the 90th percentile. Show your work.

You can also use technology to calculate a percentile. To calculate a percentile, use a calculator or other application that determines the inverse of the function that determined the percent of data below a *z*-score. In the texting situation, the percentile is 0.90, the mean is 100, and the standard deviation is 25.

3. Determine the total number of text messages that represents the 20th percentile.

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NOTES

TALK the TALK

Unplugged

The battery life of a certain laptop computer has a mean of 16 hours with a standard deviation of 2.5 hours.

1. **Is the *z*-score associated with 17 hours positive or negative? Explain your reasoning.**

2. **Is the *z*-score associated with 15 hours positive or negative? Explain your reasoning.**

3. **What is the *z*-score associated with 17 hours? 15 hours?**

4. **What percent of laptops have a battery life that is less than 17 hours?**

5. **What percent of laptops have a battery life that will last more than 17 hours?**

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z-Score Table

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
−3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
−3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
−3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
−3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
−3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
−2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
−2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
−2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
−2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
−2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
−2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
−2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
−2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
−2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
−2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
−1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
−1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
−1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
−1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
−1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
−1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
−1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
−1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
−1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
−1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
−0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
−0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
−0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
−0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
−0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
−0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
−0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
−0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
−0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
−0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

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z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	09983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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Assignment

Write

Describe how to determine a percent from a *z*-score table.

Remember

A *z*-score is a number that describes a specific data value's distance from the mean in terms of standard deviation units. A percentile is a data value for which a certain percent of the data is below the data value.

Practice

The birth weights of African lions are normally distributed. The average birth weight of an African lion is 3.6 pounds with a standard deviation of 0.4 pound.

1. What percent of newborn African lions weigh less than 3 pounds? Weigh more than 3.8 pounds? Weigh between 2.7 and 3.7 pounds?
2. What is the birth weight of a lion cub in the 80th percentile? In the 10th percentile? In the 97th percentile?
3. A lioness gives birth to two cubs. One cub is in the 47th percentile and the other is in the 62nd percentile. Determine the difference in the cubs' weights.

Stretch

The birth weights of African lions are normally distributed. The average birth weight of an African lion is 3.6 pounds with a standard deviation of 0.4 pound.

1. There is concern in the biology community about underweight and overweight newborn African lions, specifically those less than 2.6 pounds and those more than 4.6 pounds at birth. What is the percent of newborn lions in those ranges?
2. A more recent study of the birth weights of African lions suggest that the average birth weight is 3.6 pounds with a standard deviation of 0.5 pound. Without doing any calculations, determine whether the percent of newborn lions weigh less than 2.6 pounds or more than 4.6 pounds will increase or decrease. Explain your reasoning.
3. Determine the percent of newborn lions that weigh less than 2.6 pounds or more than 4.6 pounds in the population with a mean of 3.6 pounds and a standard deviation of 0.5 pound. Did it match your result from part b)?

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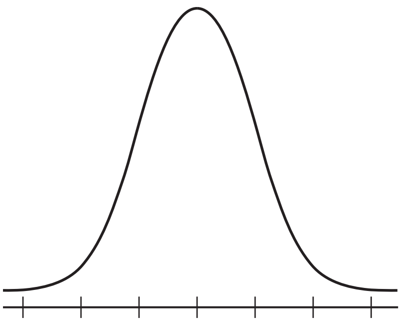
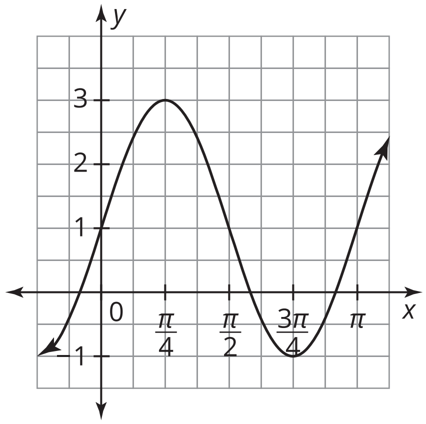
Review

1. A researcher recorded the resting heart rates of a sample of men aged 40-45. The average resting heart rate was 72 beats per minute and the standard deviation was 3.5 beats per minute. The resting heart rates follow a normal distribution.
   a. Sketch and label a normal curve. Include 3 standard deviations above and below the mean.
   b. Determine the percent of men who have resting heart rates between 2 standard deviations below the mean and 1 standard deviation above the mean.
   c. Approximately what percent of men had resting heart rates less than 65 beats per minute?

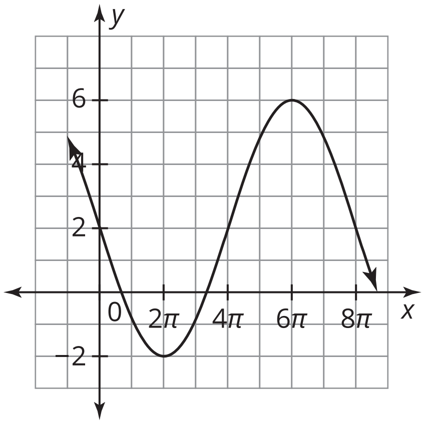


2. The equation $d(t) = 11 \cos\left(\frac{8\pi}{5}t\right)$ can be used to model the distance the pendulum of a clock is in inches from its center position as a function of time. The pendulum is released from its rightmost position. Assume that the right of center is a positive distance and the left of center is a negative distance.
   a. Determine the pendulum's distance from the center at 3.5 seconds.
   b. Determine when the pendulum is 7 inches to the left of center.
3. Write a sine function for each graph.

a.



b.



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4

Toh-May-Toh, Toh-Mah-Toh

Normal Distributions and Probability

Warm Up

The mean battery life of a video camera is 10 hours with a standard deviation of 1.5 hours. Use technology to determine the percent of video camera batteries with a battery life greater than each specified battery life.

1. 10 hours
2. 12 hours

Learning Goals

- Interpret a normal curve in terms of probability.
- Use normal distributions to determine probabilities.
- Use normal distributions and probabilities to make decisions.

You have learned to determine the percent of data within a given interval on a normally distributed set of data. How can you interpret those percents as the probability of a data value occurring in a particular interval?

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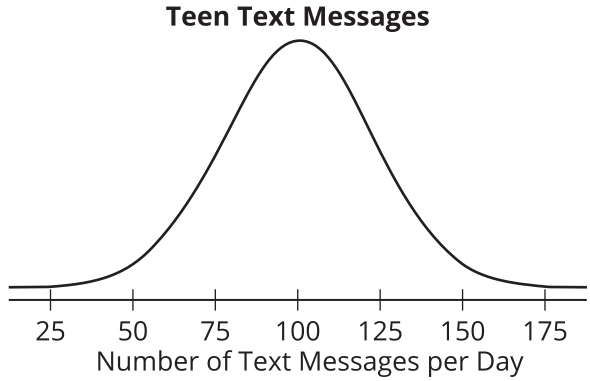
GETTING STARTED

R U Ready 4 This? :D

So far, you have explored the percent of data values that fall within specified intervals. However, you can also interpret a normal distribution in terms of probabilities.

Based on a survey, the number of text messages that teens send and receive every day is a normal distribution with a mean of 100 text messages and a standard deviation of 25 text messages.



Consider that a teen is randomly selected from the survey results.

1. **Determine the probability that the randomly selected teen sends and receives the specified number of text messages per day.**

    a. **Between 100 and 125 text messages per day**

    b. **Fewer than 75 text messages per day**

    c. **More than 140 text messages per day**

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ACTIVITY

4.1

Comparing Delivery Times

You have collected data on the delivery times for two local pizza shops, Antonio's Pizza and Wood Fire Pizza. Based on your data, Antonio's Pizza has a mean delivery time of 30 minutes and a standard deviation of 3 minutes. Wood Fired Pizza has a mean delivery time of 25 minutes and a standard deviation of 8 minutes.

1. **What factors could influence the delivery time of an order from either pizza shop?**

2. **What can you conclude based only on the mean and standard deviation for each pizza shop?**

3. **A friend of yours is planning a party. She needs the pizza for the party delivered in 35 minutes or less or the party will be a complete disaster! Which pizza shop has a greater probability of delivering the order within 35 minutes? Explain your reasoning.**

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ACTIVITY

4.2

Comparing Tomatoes

Brad and Toby both plan to enter the county tomato growing competition. Each person who enters the competition must submit a basket of tomatoes. The judges randomly select a tomato from each contestant's basket. According to the rules of the competition, a "golden" tomato has a diameter between 4 inches and 4.5 inches.

The diameters of tomatoes in Brad's basket are normally distributed with a mean diameter of 3.6 inches and a standard deviation of 1 inch. The diameters of tomatoes in Toby's basket are also normally distributed with a mean diameter of 3.8 inches and a standard deviation of 0.2 inches.

1. When the judges randomly select a tomato from Brad's and Toby's basket, whose is more likely to result in a "golden" tomato?

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NOTES

TALK the TALK

Ready for Your Close-Up?

The mean battery life of a certain video camera is 10 hours with a standard deviation of 1.5 hours.

1. **Use a *z*-score table to determine the percent of video cameras with a battery life greater than 13 hours.**

2. **Verify your answer to Question 1 using technology.**

3. **What is the probability that a randomly selected video camera has a battery life between 8 and 9 hours?**

4. **Use the Empirical Rule for Normal Distributions to determine the probability that a randomly selected video camera will have a battery life between 8.5 and 11.5 hours.**

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Assignment

Write

Explain how percents of data from normal distributions can be used to determine probabilities.

Remember

You can use the probabilities interpreted from a normal distribution to compare data sets and make decisions based on the comparisons.

Practice

1. Marine biologists in Florida are studying the tiger shark to determine the factors that are contributing to their diminishing population. Adult tiger sharks along the Atlantic Coast of Florida have a mean length of 11.5 feet with a standard deviation of 0.9 foot. Adult tiger sharks along the Gulf Coast of Florida have a mean length of 11.9 feet with a standard deviation of 0.6 foot. Tiger shark lengths are normally distributed.
   a. The biologists need to capture 1 more adult tiger shark at least 13 feet in length for their study. Along which coast will the researchers have a higher probability of capturing the shark they need? Explain your reasoning.
   b. The biologists need to capture 1 more adult tiger shark less than 10.5 feet in length in order to complete their study. Along which coast will the researchers have a higher probability of capturing the shark they need? Explain your reasoning.
2. Chicken eggs are sold according to their size. The chart shows how eggs are classified by their size.

Egg Size	Weight (ounces)
Extra-Large	$2.25 < m \leq 2.5$
Large	$2.0 < m \leq 2.25$
Medium	$1.75 < m \leq 2.0$

The eggs produced at Jen's Hen Farm and Rick's Chick Farm are normally distributed. The eggs produced at Jen's Hen Farm have a mean weight of 2.11 ounces and a standard deviation of 0.08 ounce. The eggs produced at Rick's Chick Farm have a mean weight of 2.15 ounces and a standard deviation of 0.07 ounce. Each farm produces a total of 100,000 eggs per month.
   a. If Jen and Rick each randomly select an egg from their farm, who is more likely to select an egg classified as large?
   b. If Jen and Rick each randomly select an egg from their farm, who is more likely to select an egg classified as medium?
   c. Jen and Rick sell their extra-large eggs for 90 cents per dozen. Estimate the amount of money they each make per month from the sale of extra-large eggs.

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Stretch

1. Consider the shark problem from the Practice section.
   a. What is the probability that a biologist catches an adult tiger shark along the Atlantic Coast of Florida that has a mean length of exactly 11.6 feet? Explain your reasoning.
   b. A fisherman claims he caught an Adult tiger shark along the Gulf Coast of Florida that was more than 25 feet long. Andre claims it is impossible because the calculator displays the probability as 0. Is Andre correct? Explain your reasoning.

Review

1. Women's heights are normally distributed. The average height of a woman is 65 inches with a standard deviation of 2.5 inch.
   a. What percent of women are taller than 66 inches?
   b. What percent of women are between 63 and 66.5 inches tall?
   c. Determine the height of a woman in the 90th percentile.
2. The outside temperature over a day can be modeled by the equation $T(t) = 16\cos\left(\frac{\pi}{12}t\right) + 80$, where t represents the number of hours past midnight, and $T(t)$ represents the temperature in degrees Fahrenheit. Determine each characteristic and describe what it means in terms of this problem situation.
   a. Amplitude
   b. Period
   c. Vertical shift
3. Solve the equation $-4 + 6\cos x = -7$ for $0 \leq x \leq 2\pi$.

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Interpreting Data in Normal Distributions Summary

KEY TERMS

- discrete data
- continuous data
- population
- sample
- normal curve
- normal distribution
- mean (μ)
- standard deviation (σ)
- standard normal distribution
- Empirical Rule for Normal Distributions
- z-scores
- percentile

LESSON 1

Recharge It!

Discrete data are data whose possible values are countable and often finite. The scores of baseball games are examples of discrete data, because a team's score must be a whole number. **Continuous data** are data which can take any numeric value within a range. Heights of students, times required to complete a test, and distances between cities are examples of continuous data.

For continuous data, a relative frequency histogram displays continuous intervals on the horizontal axis and relative frequency on the vertical axis .

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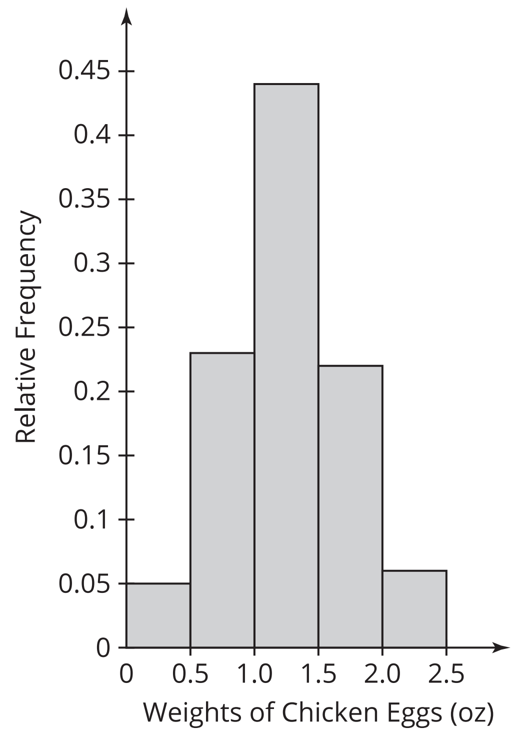
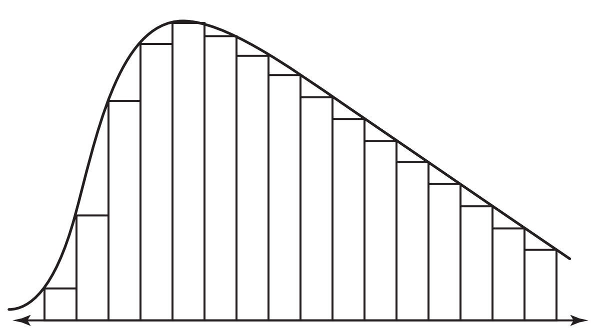
For example, consider the relative frequency table and the relative frequency histogram shown which both display the same data set.

Weights of Chicken Eggs (ounces)	Relative Frequency
0.0–0.5	0.05
0.5–1.0	0.23
1.0–1.5	0.44
1.5–2.0	0.22
2.0–2.4	0.06



A **population** represents all the possible data that are of interest in a study or survey. A **sample** is a subset of data selected from a population. As the sample size continues to increase and the interval size continues to decrease, the shape of each relative frequency histogram will likely start to resemble a normal curve. A **normal curve** is a bell-shaped curve that is symmetric about the mean of the data. A normal curve models a theoretical data set that is said to have a **normal distribution**. A non-normal distribution is neither bell-shaped nor symmetric.

For example, the graph shown does not represent a normal distribution. It is neither bell-shaped nor symmetric; it is skewed right.



Although normal curves can be narrow or wide, all normal curves are symmetric about the mean of the data. The **mean** is the average of a set of data; the sum of the data divided by the size of the data set. With normal curves, the mean of a population is represented with the symbol μ. The mean of a sample is represented with the symbol $\overline{x}$.

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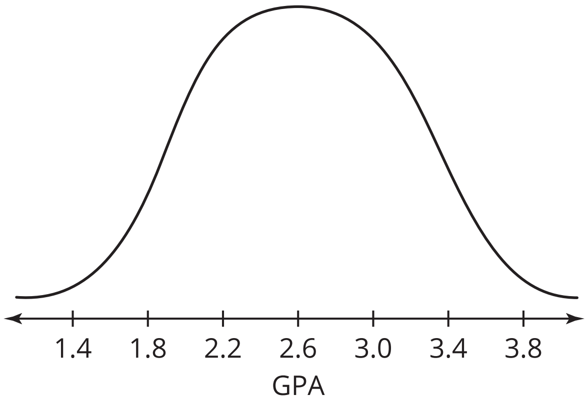
The **standard deviation** of data is a measure of how spread out the data are from the mean. The standard deviation of a sample is represented by the letter s. The symbol used for the standard deviation of a population is the sigma symbol (σ). When interpreting the standard deviation of data:

- A lower standard deviation represents data that are more tightly clustered near the mean.
- A higher standard deviation represents data that are more spread out from the mean.

For example, in the normal curve shown, the mean is 2.6 and the standard deviation is 0.4.



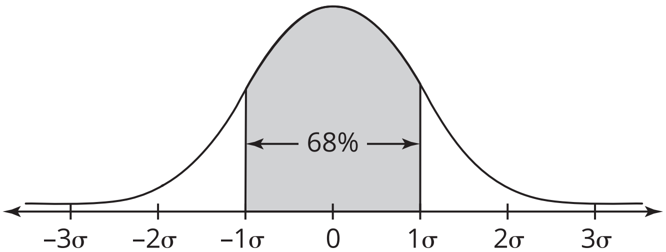
LESSON 2

The Form of Norm

The **standard normal distribution** is a normal distribution with a mean value of zero and a standard deviation of 1σ or -1σ. Positive integers represent standard deviations greater than the mean. Negative integers represent standard deviations less than the mean.

The **Empirical Rule for Normal Distributions** states:

- Approximately 68% of the data in a normal distribution for a population is within 1 standard deviation of the mean.



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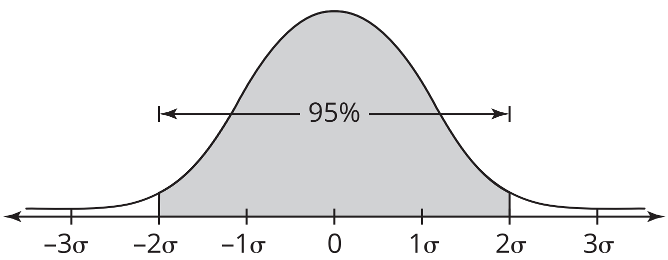
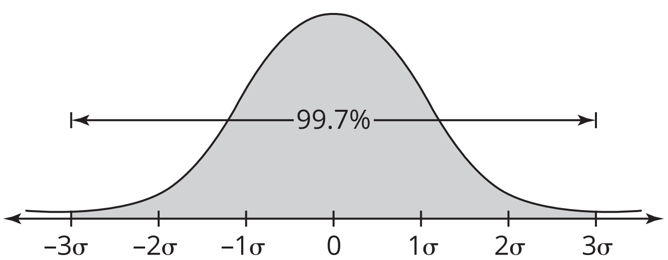
- Approximately 95% of the data in a normal distribution for a population is within 2 standard deviations of the mean.



- Approximately 99.7% of the data in a normal distribution for a population is within 3 standard deviations of the mean.



The Empirical Rule for Normal Distributions applies most accurately to population data rather than sample data. However, the Empirical Rule is often applied to data in large samples.

The percent of data for any normal distribution can be determined using the Empirical Rule for Normal Distributions.

For example, consider a situation in which you must determine the percent of commute times less than 36 minutes for a certain city, given that the commute times are normally distributed, and the mean commute is 41 minutes with a standard deviation of 2.5 minutes.

A commute time of 36 minutes is 2 standard deviations below the mean. The Empirical Rule for Normal Distributions states that 50% of the data is below the mean and that 47.5% of the data is within 2 standard deviations below the mean. So, 50% – 47.5% or 2.5% of the data is below 2 standard deviations below the mean. Approximately 2.5% of commute times are less than 36 minutes.

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Similarities between all normal distributions:

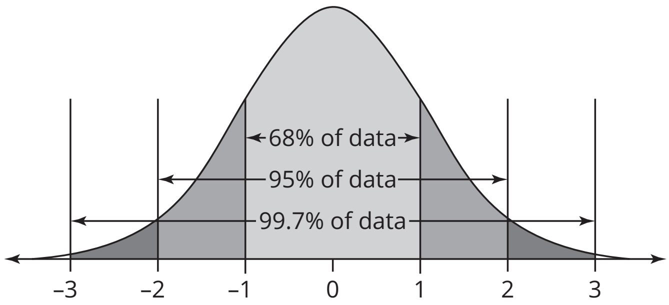
- Bell-shaped
- Symmetric about the mean
- 68% of the data within −1 and −1 standard deviation of the mean.
- 95% of the data within −2 and −2 standard deviations of the mean.
- 99.7% of the data within −3 and +3 standard deviations of the mean.

Differences between normal distributions:

- Means can be different.
- Standard deviation can be different.

Characteristics of the Standard Normal Curve:

- Symmetric about the *y*-axis (mean).
- Concave downward between $x = 1$ and $x = -1$ and concave upward everywhere else.
- Asymptotic to the *x*-axis from above.
- The mean is equal to 0. $\mu = 0$
- The standard deviation is equal to 1. $\sigma = 1$
- The area under the curve equals 1.



The total area under the curve of a normal distribution in standard form is 1. This corresponds to the sum of all the relative frequencies of a histogram of a normally distributed set of data.

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LESSON 3

Above, Below, and Between the Lines

Data points can be converted into z-scores which represent the number of standard deviations the data value is from the mean. A z-score is positive if it is above the mean and negative if it is below the mean. A z-score table can then be used to determine the percent of data you are looking for based on the z-scores.

For example, you can calculate the percent of adult men taller than 70 inches, given that adult men's heights are normally distributed and the mean height is 69.3 inches with a standard deviation of 2.8 inches.

$$\begin{aligned} z &= \frac{70 - 69.3}{2.8} \\ &= \frac{0.7}{2.8} \\ &= 0.25 \end{aligned}$$

In the z-score table, the cell that is the intersection of the row that represents the ones (0.2) and the column that represents the hundredths (0.05) reads 0.5987.

z	**0.0**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199
0.1	0.5398	0.5433	0.5478	0.5517	0.5557	0.5596
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088

So, about 59.87% of adult men are shorter than 70 inches, so 100 − 59.87, or 40.13% of adult men are taller than 70 inches.

Technology can help you determine the percent of data below a certain z-score. There are apps that allow you to enter a z-score and get the area to the left of the z-score. Many calculators also have this feature.

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A **percentile** is the data value for which a certain percent of the data is below the data value in a normal distribution. For example, 90% of the data in a set is below the value at the 90th percentile, and 80% of the data is below the value at the 80th percentile.

A z-score can be used to determine the data value. First, the percent value in the table closest to the percent you are looking for should be determined. Then the z-score for the percentile can be determined from the table. This can be converted back to the original data value using the formula for a z-score and solving for x.

For example, you can determine the 80th percentile for SAT scores, given that SAT scores are normally distributed and the mean is 1500 with a standard deviation of 280.

The percent value in the z-score table that is closest to 80% is 0.7995. The z-score for this percent value is 0.84.

$$0.84 = \frac{x - 1500}{280}$$

$$235.2 = x - 1500$$

$$1735.2 = x$$

The SAT score that represents the 80th percentile is approximately 1735.

You can also use technology to calculate a percentile. To calculate a percentile, use a calculator or other application that determines the inverse of the normal cumulative density function.

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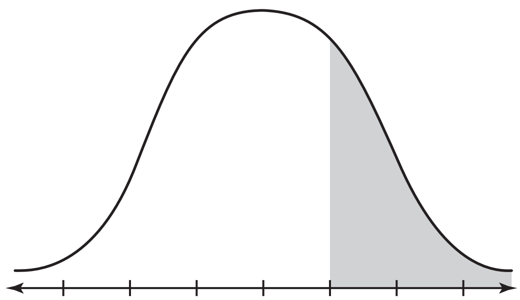
LESSON 4

Toh-May-Toh, Toh-Mah-Toh

The percent of data values that fall within specified intervals on a normal distribution can also be interpreted as probabilities.

For example, you can calculate the probability that a randomly selected annual precipitation amount in a city is more than 340 inches, given that the amounts are normally distributed with a mean of 320 inches and a standard deviation of 20 inches.



The mean is 320 and one standard deviation above the mean is 340. You know that 34% of the data is between the mean and one standard deviation above the mean. You also know that 50% of the data is above the mean. So, 50 − 34, or 16% of the data is more than one standard deviation above the mean.

The probability that the randomly selected precipitation amount in a city is more than 340 inches is 16%.

A normal distribution can be used to determine probabilities. The percent of data between specified intervals represents probabilities. A *z*-score table or technology can be used to determine the percents.

For example, you can determine the probability that a randomly selected student will score between a 74 and an 80 on an exam, if the exam scores are normally distributed and the mean is 82 with a standard deviation of 2.8.

A score of 74 has a *z*-score of approximately −2.85 and a score of 80 has a *z*-score of approximately −0.71. Using a *z*-score table, you can determine that the probability of scoring below 74 is 0.22% and the probability of scoring below 80 is 23.89%.

The probability that a randomly selected student will score between a 74 and an 80 is approximately 23.89% − 0.22%, or 23.67%.

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TOPIC 2

Making Inferences and Justifying Conclusions

Collecting data is an essential part of statistical reasoning. The method depends on the type of data to be collected, and may be anything from linear measurement to chemical tests to door-to-door surveys.

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Module 5: Relating Data and Decisions

TOPIC 2: MAKING INFERENCES AND JUSTIFYING CONCLUSIONS

Students focus on methods of collecting data to analyze a question or characteristic of interest, specific sampling methods, and the significance of randomization. They then determine how to draw samples that most closely represent a population and contrast a simple random sample with 3 types of biased samples—convenience, subjective, and volunteer. Students use data from samples to estimate population means and proportions, and determine whether results are statistically significant. Throughout the topic, they work on a culminating project.

Where have we been?

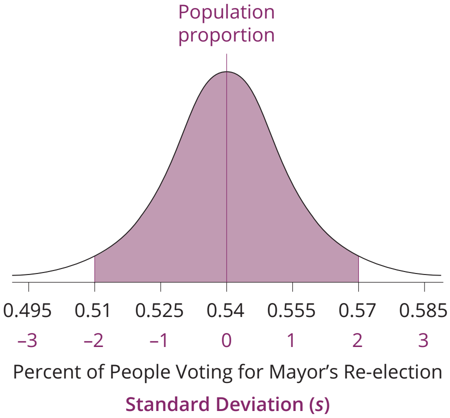
Students are very familiar with measures of center and spread and have used them to compare data sets. In the previous topic, they dealt with frequency distributions and how they relate to the normal curve. Students learned the Empirical Rule for Normal Distributions and now relate this rule to margin of error, confidence interval, and statistical significance.

Where are we going?

As students consider all of the data that they have access to in this technological age, including advertisements, news reports, propaganda, and political polling, they need tools to analyze the data and determine whether the information is meaningful, significant, and/or useful in decision-making. Most importantly, any student pursuing a research-based career will use and build upon these skills extensively.

Statistical Significance

A result that is very unlikely to have occurred by chance, typically more than 2 standard deviations from the mean, or outside a 95% confidence interval, is considered statistically significant.



- There is a 95% confidence interval, the percent of values that lie between $-2s$ and $2s$.
- The confidence interval is 51% to 57%, the values that correspond to $-2s$ and $2s$.
- The margin of error is ±3%, 0.57 − 0.54 or 0.54 − 0.51.

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The Placebo Effect

Have you taken medicine to treat an illness? Imagine that the medicine you took was not really medicine, but just a sugar pill. In medical studies, people who have unknowingly taken a sugar pill—called a placebo—have reported that the pill has had an effect similar to medicine, even though there was no medicine in the pill at all. This is an example of what is called the placebo effect.

Researchers must always be on the lookout for placebo effects. They may be to blame for successful or unsuccessful outcomes to experiments.

Talking Points

Confidence intervals and margin of error can be important topics to know about for college admissions tests.

Here is an example of a sample question:

Researchers estimate that a population of chimpanzees is 1336. They determine that at a 95% confidence level, their margin of error for the population of chimpanzees is 10.1. If the actual population is within the confidence interval, what is the lowest possible population of chimpanzees?

This involves simple subtraction. If the actual population was within the confidence interval, then the lowest population could be is 1336 − 10.1, or 1325.9. Since this must be a whole number, round up to 1326.

Key Terms

confounding

Confounding occurs when there are other possible reasons, called confounds, for the results to have occurred that were not identified prior to the study.

convenience sample

A convenience sample is a sample whose data is based on what is convenient for the person choosing the sample.

confidence interval

An estimated range of values that will likely include the population proportion or population mean is called a confidence interval.

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1

Data, Data, Everywhere

Sample Surveys, Observational Studies, and Experiments

Warm Up

Explain how the situation may lead to an invalid conclusion.

1. A student wants to determine whether eating carrots is related to improved vision. She asks a local eye doctor to administer a voluntary survey to his patients. The survey contains questions about the amount of carrots they regularly eat and their vision.

Learning Goals

- Identify characteristics of sample surveys, observational studies, and experiments.
- Differentiate between sample surveys, observational studies, and experiments.
- Identify possible confounds in the design of experiments.

Key Terms

- characteristic of interest
- sample survey
- random sample
- biased sample
- observational study
- experiment
- treatment
- experimental unit
- confounding

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You have graphed data and used data to determine probabilities. How can you collect data and consider whether the process is biased or unbiased?

GETTING STARTED

Survey Says

You can use data to help answer questions about the world. The specific question that you are trying to answer or the specific information that you are trying to gather is called a **characteristic of interest**.

For example, you can use data to help determine which prescribed medication is most effective, which television show is the most popular with teenagers, or how often doctors wash their hands.

One way of collecting data is to use a *sample survey*. A **sample survey** poses one or more questions of interest to obtain sample data from a population.

Remember:

A population represents all the possible data that are of interest in a survey, and a sample is a subset of data that is selected from the population.

A researcher wants to design a sample survey to determine the amount of time that U.S. teenagers between the ages of 16 to 18 spend online each day.

1. Identify the characteristic of interest in the sample survey.

2. Identify the population that the researcher is trying to measure with a sample survey.

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3. **Huck and Patch were discussing the population of the survey.**

Huck	Patch
The population is all 16- to 18-year-olds in the United States.	The population is all teenagers in the United States.

Who is correct? Explain your reasoning.

4. **Write a survey question or questions that the researcher could use to collect data from the participants in the survey.**

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ACTIVITY
1.1

Collecting a Representative Sample

When sample data are collected to describe a characteristic of interest, it is important that such a sample be as representative of the population as possible. One way to collect a representative sample is to use a *random sample*. A **random sample** is a sample that is selected from the population in such a way that every member of the population has the same chance of being selected. A **biased sample** is a sample that is collected in a way that makes it unrepresentative of the population.

1. Consider the sample survey from the Getting Started. Joanie and Richie were discussing strategies that the researcher could use to select a representative sample of 16- to 18-year-olds.

Joanie
The sample should include 16- to 18-year-olds from various states.

Richie
Include 16- to 18-year-olds who live in various types of communities, like urban, suburban, and rural.

List some additional strategies the researcher should consider when selecting the sample.

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2. Cherese suggests that the researcher could post the survey online and then distribute the link to the survey to students after school on Friday as they are leaving the building. Does this method result in a representative or a sample? Explain your reasoning.

ACTIVITY
1.2

Observational Studies and Experiments

NOTES

In an **observational study**, data are gathered about a characteristic of the population by simply observing and describing events in their natural settings. Recording the number of children who use the swings at a local park is an example of a simple observational study.

The results of an observational study state that approximately 70% of in-house day care centers in one U.S. state show as much as 2.5 hours of television to the children per day. The observational study examined 132 day care centers in one state.

1. **Identify the population, the sample, and the characteristic of interest in the observational study.**

2. **List some similarities and differences between an observational study and a sample survey.**

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An **experiment** gathers data on the effect of one or more **treatments**, or experimental conditions, on the characteristic of interest. Members of a sample, also known as **experimental units**, are randomly assigned to a treatment group.

A placebo treatment is a treatment that is assumed to have no real effect on the characteristic of interest.

Researchers conducting an experiment to test the effectiveness of a new asthma medication collected data from a sample of 200 asthma patients. One hundred of the patients received a placebo treatment along with an inhaler. The other one hundred patients received the new medication along with an inhaler. Monthly blood and breathing tests were performed on all 200 patients to determine whether the new medication was effective.

3. **Identify the population, the sample, and the characteristic of interest in the experiment.**

4. **What are the treatments in the experiment?**

5. **What are some ways the researchers could choose a biased sample for this experiment?**

Confounding occurs when there are other possible reasons, called confounds, for the results to have occurred that were not identified prior to the study.

6. **Suppose one of the treatment groups was given the new medication with an inhaler and the other group was given a placebo with no inhaler. Describe how this design of the experiment introduces a confound.**

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ACTIVITY

1.3

Online Time Study, Part I

You will now design a data collection plan in order to implement an actual sample survey.

To design a sample survey, observational study, or experiment, complete this checklist.

- Determine a characteristic of interest to learn about from a sample survey, observational study, or experiment.
- Identify the population.
- Write a question(s) that can be answered by collecting quantitative data.
- Collect a sample using a method that avoids bias.
- Eliminate elements of the design that may introduce confounding.

You will revisit this Online Time Study in each lesson of this topic.

1. Design a data collection plan to learn how much time students in your school spend online each day.

a. Identify the characteristic of interest and the population.

b. Restate or revise the question(s) you crafted in the Getting Started.

c. Explain how you can gather data from a representative, unbiased sample of students in your school.

d. Address any confounds that might be introduced, and how you will eliminate them.

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NOTES

TALK the TALK

How Do YOU Get Your Data?

In this lesson, you compared three different methods of data collection and identified characteristics of interest, populations, and samples.

1. Classify each scenario as a sample survey, an observational study, or an experiment, and explain your reasoning. Then, identify the characteristic of interest, the population, and the sample.

a. To determine whether there is a link between high-voltage power lines and illnesses in children who live in the county, researchers examined the illness rate for 100 children that live within $\frac{1}{4}$ mile from power lines and the illness rate for 100 children that live more than $\frac{1}{4}$ mile from power lines.

b. Seventy of the school's calculus students are randomly divided into two classes. One class uses a graphing calculator regularly, and the other class never uses graphing calculators. The math department team leader wants to determine whether there is a link between graphing calculator use and a student's calculus grade.

c. A medical researcher wants to learn whether there is a link between the amount of TV children watch each day and childhood obesity in a particular school district. She gathers data from the records of 15 local pediatricians.

d. In a particular school district, a researcher wants to learn whether there is a link between a child's daily amount of physical activity and their overall energy level. During lunch at a school, she distributed a short questionnaire to students in the cafeteria.

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Assignment

Write

Match each definition to its corresponding term.

1. A study that gathers data about a characteristic of the population by simply observing and describing events in their natural settings.
2. A survey that poses one or more questions of interest to a sample of a targeted population.
3. The members of the sample for an experiment.
4. The specific question that you are trying to answer or the specific information you are trying to gather.
5. A situation that occurs when there are other possible reasons for the results to have occurred that were not identified prior to the study.
6. A sample that is not representative of the population.
7. An experimental condition that is used on treatment groups.
8. A process that gathers data on the effect of one or more treatments on the characteristic of interest.
9. A sample that is selected from the population in such a way that every member of the population has the same change of being selected.

a. biased sample
b. sample survey
c. random sample
d. characteristic of interest
e. confounding
f. observational study
g. experiment
h. treatment
i. experimental unit

Remember

To design a sample survey, observational study, or experiment: determine a characteristic of interest; identify the population; write a question(s) that can be answered by collecting quantitative data; collect a sample using a method that avoids bias; and eliminate elements of the design that may introduce confounding.

Practice

1. Determine whether the given method of data collection is a sample survey, an observational study, or an experiment. Explain your reasoning. Then identify the population, the sample, and the characteristic of interest.
   a. A high school principal wants to determine whether students who work in groups in geometry class receive higher grades than students who do not work in groups. He directs 5 of the geometry classes to participate in group work and 5 of the geometry classes to complete their work individually.

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b. You are curious about student interest in your school about doing volunteer work in the community. You ask 120 randomly selected students in your school whether they are interested in doing volunteer work in the community.

c. A researcher wants to know whether female professional athletes are more prone to knee injuries than male professional athletes. She gathers data from 6 different sports organizations that have injury records for all of their male and female professional athletes.

2. You are organizing a survey to learn about the exercise habits of students in your school. You are interested to know the number of hours students spend exercising during an average week.
   a. What is the population of interest?
   b. How could a representative random sample be selected?
   c. What is the characteristic of interest?
   d. Give an example of a question that is unbiased for this survey.
3. A medical researcher wants to determine whether there is a connection between the frequency of migraine headaches in adults and changes in weather. The researcher collects data on 75 adults who experience migraine headaches that live in temperate climates and 75 adults who experience migraine headaches that live in climates with varying extremes.
   a. What is the population of interest?
   b. What is the sample?
   c. How could confounding be avoided or addressed?
4. Explain how each sampling method is biased.
   a. One hundred fish caught in a bass tournament are arranged from largest to smallest. The fish are then clustered into 5 groups so that the 20 largest are in the first group, the next 20 largest are in the second group, and so on. You randomly choose 10 fish from the last group to perform an experiment to analyze the lengths of the fish caught in the tournament.
   b. A scientist is preparing an experiment in which he will analyze the bacteria levels in a lake. He walks to the edge of the lake and fills 40 vials with water to represent the water supply in the lake.
   c. You want to analyze the fitness levels of runners after they run in a marathon by performing a blood test. There are 1000 runners in the marathon. You choose the first 25 runners that finish the marathon to represent the population of runners for your experiment.
   d. You want to perform an experiment to determine the amount of money that Americans feel the government should be spending on public transit. You choose a random sample of 50 bus drivers and interview them to represent the population for your experiment.

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Stretch

1. Suppose a study was done to see whether eating fried foods more than three times a week is related to coronary heart disease. The study sampled 2000 random men with coronary heart disease and found that there did appear to be a relationship. Name five possible confounding variables that could have affected the results.

2. Three groups of students in grades 7 through 9 are studying political polling. They are asked to come up with a way they would sample 50 adults age 18 and over about their preference for a mayoral candidate in an upcoming election. The sample should be as representative of the voting population in the city as possible. The first group said they would open a local phone book and randomly pick 50 phone numbers out of the book. The second group said they would put an ad in the newspaper and ask people to contact them with their preference. The third group said they would randomly select 50 students in the school and ask them to ask a parent or guardian.
   a. Which group's sample was the most convenient to obtain?
   b. Which group's sample was the most reliant on volunteers?
   c. Which group's sample was the most random?
   d. Which of the samples, if any, seem most representative of the population of voter's in the city? Explain your reasoning.

Review

1. Two different apple orchards record the diameter of apples that are in each bushel of apples picked. The mean diameter of apples from Orchard A is 60 millimeters with a standard deviation of 2 millimeters. The mean diameter of apples from Orchard A is 62 millimeters with a standard deviation of 3 millimeters. Each bushel contains 126 apples. Diameters of apples are normally distributed.
   a. Apples that are between 55 and 58 millimeters in diameter are generally considered useful only for apple cider. If a farmer from each orchard randomly selects an apple from a bushel on their orchard, who is more likely to select an apple that it only useful for apple cider? Explain your reasoning.
   b. Apples that are more than 63 millimeters in diameter are sold for a profit of $0.55 per apple. Estimate the amount of profit each orchard will generate from each bushel of apples.
2. A random sample of 250 students in a high school were asked how many hours they spent outdoors over the past weekend.
   a. Complete the table to determine the relative frequency for each interval of the number of hours spent outdoors. Round your answers to the nearest thousandth.
   b. Does the distribution of the time spent outdoors data appear to be a normal distribution? Explain your reasoning.

Hours Outdoors	Number of Students	Relative Frequency
0 – 2	53	
2 – 4	61	
4 – 6	51	
6 – 8	40	
8 – 10	35	
10 – 12	10	

3. Solve each equation for $0 \le x \le 2\pi$.
   a. $2 \sin^2 x - 1 = 0$
   b. $3 \tan^3 x = \tan x$

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2

Ample Sample Examples

Sampling Methods and Randomization

Warm Up

Explain how the situation may introduce confounding.

1. A local bookstore offered all of their customers a $10 gift certificate to complete a brief online survey in order to determine the most popular reading genre.

Learning Goals

- Use a variety of sampling methods to collect data.
- Identify factors of sampling methods that could contribute to gathering biased data.
- Explore, identify, and interpret the role of randomization in sampling.
- Use data from samples to estimate population mean.

Key Terms

- convenience sample
- subjective sample
- volunteer sample
- simple random sample
- stratified random sample
- cluster sample
- cluster
- systematic sample
- parameter
- statistic

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You have considered samples that are and are not representative of a population. What are different sampling methods are used to collect data?

GETTING STARTED

Circle Time

When you use statistics, you often measure the values of a population by focusing on the measurements of a sample of that population. A population does not have to refer to people. It can be any complete group of data—like the areas of 100 circles.

Let's consider the 100 circles and table located at the end of this lesson. The table lists an identification number, the diameter, and the area for each circle. Suppose you want to determine the mean area of all 100 circles. Calculating the areas of all of the circles would be time-consuming. Instead, you can use different samples of this population of circles to estimate the mean area of the entire population.

1. **Without looking at the circles, Mauricia decided to use Circles 0–4 for her sample. Is it likely that the area of 5 circle are representative of all 100 circles? Explain your reasoning.**

2. **Analyze the circles. Select a sample of 5 circles that you think best represents the entire set of circles. Explain your reasoning.**

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ACTIVITY
2.1

Biased Samples

Consider the questions from the Getting Started. The sample of circles Mauricia chose is called a *convenience sample*. A **convenience sample** is a sample whose data is based on what is convenient for the person choosing the sample.

The sample of circles you chose in the Getting Started is called a *subjective sample*. A **subjective sample** is a sample drawn by making a judgment about which data items to select.

Another type of sample is a *volunteer sample*. A **volunteer sample** is a sample whose data consists of those who volunteer to be part of a sample.

Okay, circles can't really volunteer to be in a sample. But people can!

1. **Olivia and Ricky discussed whether a convenience sample or a subjective sample is more likely to be representative of the population of circle areas.**

Olivia
I think a subjective sample is more likely to be representative of the 100 circles than the convenience sample.

Ricky
The subjective sample and the convenience sample are equally likely to be representative of the 100 circles.

Who is correct? Explain your reasoning.

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Think about:

It's the sampling method that leads to the bias. It's not that an individual sample is biased or not.

2. Olivia shared her conclusion about convenience samples, subjective samples, and volunteer samples. Explain why Olivia's statement is correct.

> **Olivia**
>
> Even though one method may be better than another in a specific situation, collecting data using a convenience sample, subjective sample, or volunteer sample will likely result in a biased sample.

3. Explain how convenience samples, subjective samples, and volunteer samples do not include data elements that were equally likely to have been chosen from the population.

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ACTIVITY
2.2 Using a Random Digit Table

A **simple random sample** is a sample composed of data elements that are equally likely to have been chosen from the population.

Using a random digit table is one option for selecting a simple random sample. Remove the random digit table located at the end of the lesson to complete this activity. To use the table, begin at any digit and follow the numbers in a systematic way, such as moving across a row until it ends and then moving to the beginning of the next row.

You should use two digits at a time to choose a sample of 5 circles.

If the same two-digit number comes up more than once, skip it each time it is repeated and go to the next number.

1. **Select a simple random sample of 5 circles using the random digit table. Pick any cell in the table to start. Use the first two digits to represent the first circle of the sample, the next two digits to represent the second circle of the sample, and so on. List the identification numbers of the 5 circles.**

You can also use technology to generate a random list of numbers.

2. **Use technology to generate a random sample of 5 circles.**

3. **Calculate the mean of the areas of the circles included in each of your simple random samples.**

4. **Compare your simple random sample with your classmates' samples. What do you notice?**

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ACTIVITY
2.3 Other Types of Random Samples

There are several other types of random samples, including *stratified random samples*, *cluster samples*, and *systematic samples*.

A **stratified random sample** is a random sample obtained by dividing a population into different groups, or strata, according to a characteristic and randomly selecting data from each group.

Worked Example

You can collect a stratified random sample of circles by first dividing the circles into groups.

Define groups of circles based on the lengths of their diameters.

- Small circles: diameter $\leq \frac{1}{4}$ in.
- Medium circles: $\frac{1}{4}$ in. $<$ diameter $< 1\frac{1}{2}$ in.
- Large circles: diameter $\geq 1\frac{1}{2}$ in.

Small Circles (46)	Medium Circles (39)	Large Circles (15)
1, 4, 6, 13, 14, 16, 17, 19, 22, 24, 26, 28, 30, 33, 34, 37, 39, 42, 45, 46, 47, 51, 53, 56, 57, 58, 59, 62, 63, 67, 68, 72, 74, 78, 79, 82, 85, 87, 88, 89, 93, 94, 95, 97, 98, 99	0, 2, 3, 8, 9, 10, 11, 12, 21, 23, 25, 29, 31, 35, 36, 40, 41, 43, 49, 50, 52, 61, 64, 65, 66, 69, 71, 73, 75, 76, 77, 80, 81, 83, 84, 86, 90, 91, 96	5, 7, 15, 18, 20, 27, 32, 38, 44, 48, 54, 55, 60, 70, 92

There are about an equal number of small and medium circles and about a third as many large circles. To maintain this ratio in your stratified random sample, you can choose 3 small circles, 3 medium circles, and 1 large circle.

Select random circles from each group using a random digit table or technology.

Another option is to randomly select 2 large circles, 6 medium circles, and 6 small circles. This keeps the ratios the same.

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NOTES

1. **Collect a stratified random sample of circles. List the sample and explain your method.**

2. **Calculate the mean of the areas of the circles in your stratified random sample.**

A **cluster sample** is a random sample that is obtained by creating *clusters*. Then, one cluster is randomly selected for the sample. Each **cluster** contains the characteristics of a population.

3. **Use the 100 circles located at the end of this lesson to answer each question.**

   a. **Draw 3 horizontal lines and 2 vertical lines so that the page is divided into 12 congruent rectangles. Each rectangle represents a cluster of circles. Number each cluster from 1 to 12.**

   b. **Use technology or the random digit table to randomly select one of the clusters. List the cluster sample.**

   c. **Calculate the mean of the areas of the circles included in your cluster sample.**

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A **systematic sample** is a random sample obtained by selecting every *n*th data value in a population.

4. Select a systematic sample by choosing every 20th circle. First, randomly choose a number from 0 to 20 to start at and then choose every 20th circle after that.

5. Calculate the mean of the areas of the circles included in your systematic sample.

6. Faheem and Calvin shared their thoughts about random sampling.

Faheem
Simple random sampling, stratified random sampling, cluster sampling, and systematic sampling will always produce a representative, unbiased sample.

Calvin
Simple random sampling, stratified random sampling, cluster sampling, and systematic sampling do not guarantee a representative, unbiased sample.

Who is correct? Explain your reasoning.

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The mean of a sample, $\bar{x}$, can be used to estimate the population mean, μ. The population mean is an example of a **parameter**, because it is a value that refers to a population. The sample mean is an example of a **statistic**, because it is a value that refers to a sample.

The population mean for the 100 circles is $\mu = 0.19\pi$ square inches, or approximately 0.59 square inches.

7. Carla collected three simple random samples from the population of 100 circles and calculated the mean of each sample.

Carla

I didn't expect the sample of 5 circles to have a mean closest to the mean of the population. I must have done something wrong when collecting the samples.

Mean of 5 circles $\approx 0.18\pi$ square inches

Mean of 15 circles $\approx 0.15\pi$ square inches

Mean of 30 circles $\approx 0.24\pi$ square inches

Is Carla's statement correct? Explain your reasoning.

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NOTES

How can you apply your new knowledge of sampling to the Online Time Study?

TALK the TALK

Online Time Study, Part II

In the previous lesson, you designed a plan to learn about the amount of time students in your school are online each day. Now, take the next steps to conduct your sample survey.

1. **Which sampling method would be best to select the data? Explain your reasoning.**

2. **Identify the members of two samples of equal size from your population.**

3. **Conduct your survey. Have all members of each of your samples answer your survey question(s). Organize your results.**

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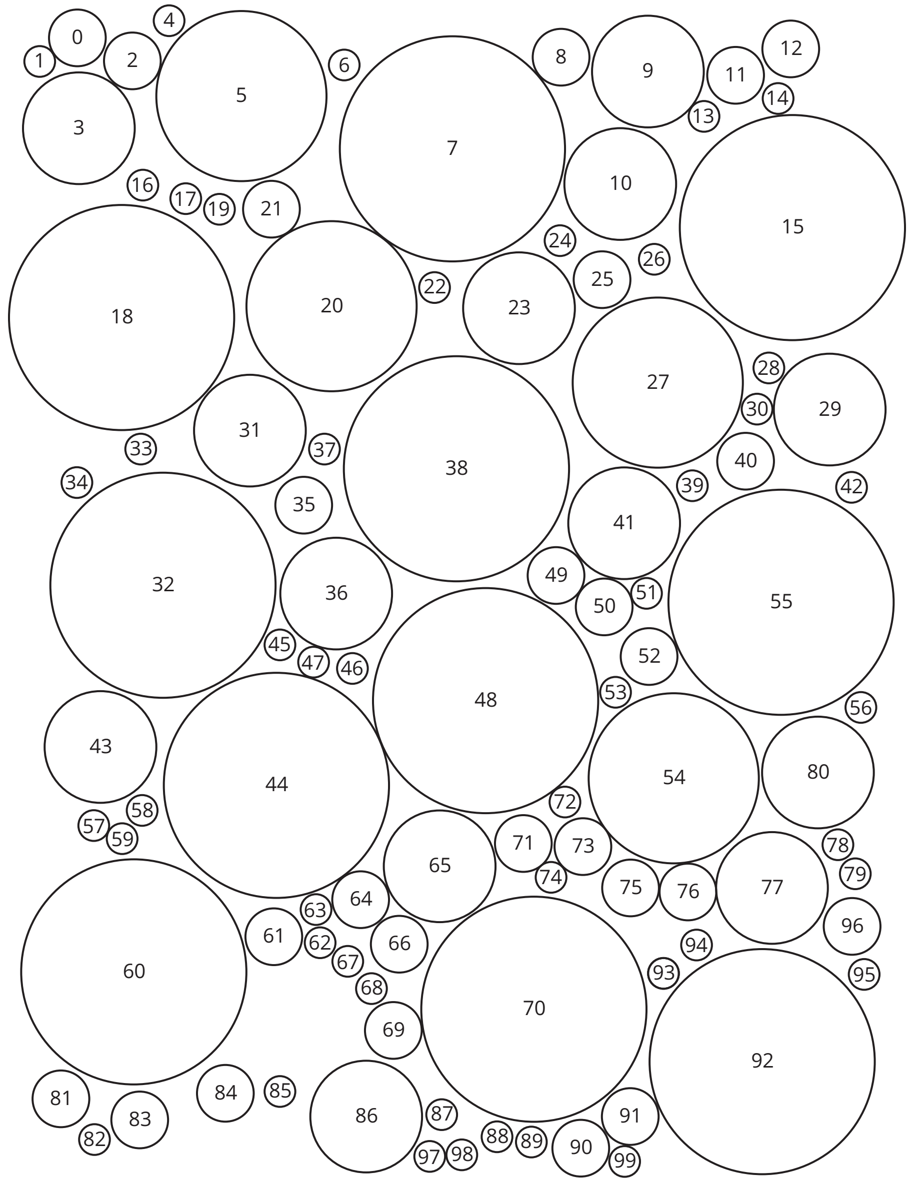
100 Circles



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100 Circles Information

Circle Number	Diameter (in.)	Area (in.^{2})	Circle Number	Diameter (in.)	Area (in.^{2})	Circle Number	Diameter (in.)	Area (in.^{2})
0	$\frac{1}{2}$	$\frac{1}{16}\pi$	**18**	2	π	**36**	1	$\frac{1}{4}\pi$
1	$\frac{1}{4}$	$\frac{1}{64}\pi$	**19**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**37**	$\frac{1}{4}$	$\frac{1}{64}\pi$
2	$\frac{1}{2}$	$\frac{1}{16}\pi$	**20**	$1\frac{1}{2}$	$\frac{9}{16}\pi$	**38**	2	π
3	1	$\frac{1}{4}\pi$	**21**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**39**	$\frac{1}{4}$	$\frac{1}{64}\pi$
4	$\frac{1}{4}$	$\frac{1}{64}\pi$	**22**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**40**	$\frac{1}{2}$	$\frac{1}{16}\pi$
5	$1\frac{1}{2}$	$\frac{9}{16}\pi$	**23**	1	$\frac{1}{4}\pi$	**41**	1	$\frac{1}{4}\pi$
6	$\frac{1}{4}$	$\frac{1}{64}\pi$	**24**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**42**	$\frac{1}{4}$	$\frac{1}{64}\pi$
7	2	π	**25**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**43**	1	$\frac{1}{4}\pi$
8	$\frac{1}{2}$	$\frac{1}{16}\pi$	**26**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**44**	2	π
9	1	$\frac{1}{4}\pi$	**27**	$1\frac{1}{2}$	$\frac{9}{16}\pi$	**45**	$\frac{1}{4}$	$\frac{1}{64}\pi$
10	1	$\frac{1}{4}\pi$	**28**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**46**	$\frac{1}{4}$	$\frac{1}{64}\pi$
11	$\frac{1}{2}$	$\frac{1}{16}\pi$	**29**	1	$\frac{1}{4}\pi$	**47**	$\frac{1}{4}$	$\frac{1}{64}\pi$
12	$\frac{1}{2}$	$\frac{1}{16}\pi$	**30**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**48**	2	π
13	$\frac{1}{4}$	$\frac{1}{64}\pi$	**31**	1	$\frac{1}{4}\pi$	**49**	$\frac{1}{2}$	$\frac{1}{16}\pi$
14	$\frac{1}{4}$	$\frac{1}{64}\pi$	**32**	2	π	**50**	$\frac{1}{2}$	$\frac{1}{16}\pi$
15	2	π	**33**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**51**	$\frac{1}{4}$	$\frac{1}{64}\pi$
16	$\frac{1}{4}$	$\frac{1}{64}\pi$	**34**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**52**	$\frac{1}{2}$	$\frac{1}{16}\pi$
17	$\frac{1}{4}$	$\frac{1}{64}\pi$	**35**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**53**	$\frac{1}{4}$	$\frac{1}{64}\pi$

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100 Circles Information

Circle Number	Diameter (in.)	Area (in.^{2})	Circle Number	Diameter (in.)	Area (in.^{2})	Circle Number	Diameter (in.)	Area (in.^{2})
54	$1\frac{1}{2}$	$\frac{9}{16}\pi$	**72**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**90**	$\frac{1}{2}$	$\frac{1}{16}\pi$
55	2	π	**73**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**91**	$\frac{1}{2}$	$\frac{1}{16}\pi$
56	$\frac{1}{4}$	$\frac{1}{64}\pi$	**74**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**92**	2	π
57	$\frac{1}{4}$	$\frac{1}{64}\pi$	**75**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**93**	$\frac{1}{4}$	$\frac{1}{64}\pi$
58	$\frac{1}{4}$	$\frac{1}{64}\pi$	**76**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**94**	$\frac{1}{4}$	$\frac{1}{64}\pi$
59	$\frac{1}{4}$	$\frac{1}{64}\pi$	**77**	1	$\frac{1}{4}\pi$	**95**	$\frac{1}{4}$	$\frac{1}{64}\pi$
60	2	π	**78**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**96**	$\frac{1}{2}$	$\frac{1}{16}\pi$
61	$\frac{1}{2}$	$\frac{1}{16}\pi$	**79**	$\frac{1}{4}$	$\frac{1}{64}\pi$	**97**	$\frac{1}{4}$	$\frac{1}{64}\pi$
62	$\frac{1}{4}$	$\frac{1}{64}\pi$	**80**	1	$\frac{1}{4}\pi$	**98**	$\frac{1}{4}$	$\frac{1}{64}\pi$
63	$\frac{1}{4}$	$\frac{1}{64}\pi$	**81**	$\frac{1}{2}$	$\frac{1}{16}\pi$	**99**	$\frac{1}{4}$	$\frac{1}{64}\pi$
64	$\frac{1}{2}$	$\frac{1}{16}\pi$	**82**	$\frac{1}{4}$	$\frac{1}{64}\pi$			
65	1	$\frac{1}{4}\pi$	**83**	$\frac{1}{2}$	$\frac{1}{16}\pi$			
66	$\frac{1}{2}$	$\frac{1}{16}\pi$	**84**	$\frac{1}{2}$	$\frac{1}{16}\pi$			
67	$\frac{1}{4}$	$\frac{1}{64}\pi$	**85**	$\frac{1}{4}$	$\frac{1}{64}\pi$			
68	$\frac{1}{4}$	$\frac{1}{64}\pi$	**86**	1	$\frac{1}{4}\pi$			
69	$\frac{1}{2}$	$\frac{1}{16}\pi$	**87**	$\frac{1}{4}$	$\frac{1}{64}\pi$			
70	2	π	**88**	$\frac{1}{4}$	$\frac{1}{64}\pi$			
71	$\frac{1}{2}$	$\frac{1}{16}\pi$	**89**	$\frac{1}{4}$	$\frac{1}{64}\pi$			

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Random Digit Table

Random Digit Table										
Line 1	65285	97198	12138	53010	94601	15838	16805	61004	43516	17020
Line 2	17264	57327	38224	29301	31381	38109	34976	65692	98566	29550
Line 3	95639	99754	31199	92558	68368	04985	51092	37780	40261	14479
Line 4	61555	76404	86210	11808	12841	45147	97438	60022	12645	62000
Line 5	78137	98768	04689	87130	79225	08153	84967	64539	79493	74917
Line 6	62490	99215	84987	28759	19177	14733	24550	28067	68894	38490
Line 7	24216	63444	21283	07044	92729	37284	13211	37485	10415	36457
Line 8	16975	95428	33226	55903	31605	43817	22250	03918	46999	98501
Line 9	59138	39542	71168	57609	91510	77904	74244	50940	31553	62562
Line 10	29478	59652	50414	31966	87912	87154	12944	49862	96566	48825
Line 11	96155	95009	27429	72918	08457	78134	48407	26061	58754	05326
Line 12	29621	66583	62966	12468	20245	14015	04014	35713	03980	03024
Line 13	12639	75291	71020	17265	41598	64074	64629	63293	53307	48766
Line 14	14544	37134	54714	02401	63228	26831	19386	15457	17999	18306
Line 15	83403	88827	09834	11333	68431	31706	26652	04711	34593	22561
Line 16	67642	05204	30697	44806	96989	68403	85621	45556	35434	09532
Line 17	64041	99011	14610	40273	09482	62864	01573	82274	81446	32477
Line 18	17048	94523	97444	59904	16936	39384	97551	09620	63932	03091
Line 19	93039	89416	52795	10631	09728	68202	20963	02477	55494	39563
Line 20	82244	34392	96607	17220	51984	10753	76272	50985	97593	34320

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Assignment

Write

Choose a term from the box that best completes each statement.

convenience sample	subjective sample	volunteer sample
simple random sample	stratified random sample	cluster sample
cluster	systematic sample	parameter
statistic		

1. A professor divided his class into females and males, then randomly selected a sample from each group. The sample the professor obtained is a ______________________________ .
2. The manager at a discount store determines the mean salary of all of the store workers. The mean salary is an example of a ______________________________ because it describes all of the workers.
3. John is asked to select a sample of his favorite foods from the school cafeteria. This sample is an example of a ______________________________ .
4. A quality control specialist tests every 100th tablet that comes off the line. This sample is an example of a ______________________________ .
5. In order to get a set of data of girl's heights, Risa uses the heights of all the girls in her class. This is an example of a ______________________________ .
6. A college randomly selects 100 out of the 600 students who have taken the GRE exam and records their scores. The mean of these test scores is a ______________________________ because it describes a sample.
7. A city manager randomly selects one block in the city and surveys all of the residents of that block. This type of sample is a ______________________________ .
8. An online newspaper asks its readers to answer a question about their satisfaction with the content of the paper. This data collected from the survey results represents a ______________________________ .
9. A theater owner randomly selects 15 different customers to receive free tickets to the next show. This sample is a ______________________________ .
10. A researcher wants to collect data from a state. He divides the state into 16 regions and randomly chooses one of the regions to interview all of its residents. Each of the 16 regions is an example of a ______________________________ .

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Remember

There are many different types of sampling methods to use when collecting data. None of the methods are entirely free from bias, but some methods provide more of an equal chance for a member of the population to be selected.

Practice

Twenty-four professional athletes are participating in a charity golf tournament. Each golfer has been given an identification number from 01 through 24. Golfers 01 through 12 are professional football players and golfers 13 through 24 are professional baseball players. Par for the course is 72. The table shows the golfers' tournament scores after the first round.

Golfer ID Number	Score	Golfer ID Number	Score
01	72	13	79
02	75	14	85
03	69	15	67
04	78	16	75
05	80	17	68
06	68	18	76
07	81	19	68
08	72	20	69
09	74	21	71
10	77	22	76
11	75	23	70
12	77	24	74

1. Create a simple random sample of 6 scores from the table. Explain how you created your sample. Then calculate the average of your sample.
2. Create a stratified random sample of 6 scores from the table. Explain how you created this sample. Then calculate the average of this sample.
3. Create a cluster sample of 6 scores from the table. Explain how you created this sample. Then calculate the average of this sample.
4. Create a systematic sample of 6 scores from the table. Explain how you created this sample. Then calculate the average of this sample.
5. The actual average score is 74. Which of your sample averages was closest to the actual average? Is this what you expected? Explain.

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Stretch

1. Data is collected for each state every year on the number of workplace fatalities. The table shows the data all 50 states plus the District of Columbia in the year 2016.

State	Number Of Fatalities	State	Number Of Fatalities	State	Number Of Fatalities	State	Number of Fatalities
Alabama	100	Indiana	137	Nevada	54	Tennessee	122
Alaska	35	Iowa	76	New Hampshire	22	Texas	145
Arizona	77	Kansas	74	New Jersey	101	Utah	44
Arkansas	68	Kentucky	92	New Mexico	41	Vermont	10
California	376	Louisiana	95	New York	272	Virginia	153
Colorado	81	Maine	18	North Carolina	56	Washington	78
Connecticut	28	Maryland	92	North Dakota	174	West Virginia	47
Delaware	12	Massachusetts	109	Ohio	28	Wisconsin	105
District of Columbia	5	Michigan	162	Oklahoma	164	Wyoming	34
Florida	309	Minnesota	92	Oregon	92		
Georgia	171	Mississippi	71	Pennsylvania	72		
Hawaii	29	Missouri	124	Rhode Island	163		
Idaho	30	Montana	38	South Carolina	9		
Illinois	171	Nebraska	60	South Dakota	96		

a. Calculate the mean number of workplace fatalities for all of the states in the year 2016.

b. Create a simple random sample of 10 scores from the table. Explain how you created your sample.

c. Calculate the mean of your sample and compare it to the mean number of fatalities for all the states. Explain your reasoning.

d. Explain how you can create a stratified random sample of 10 workplace fatalities from the states data.

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2. A company asked a polling firm to conduct several polls for upcoming elections. Determine the requested values for each poll result.
   a. For candidate A, the company polled 500 voters and found that 42.2% favored the candidate. How many voters favored the candidate?
   b. For candidate B, 662 out of 1000 polled voters favored the candidate. What percentage of voters favored the candidate?
   c. For candidate C, 763 voters favored the candidate. This was approximately 38% of the total number of voters polled. Approximately how many voters did the firm poll?
   d. The proportion of voters polled who would vote for Candidate D was 0.653. If 800 voters were polled, approximately how many of the voters polled would vote for Candidate D?

Review

1. A researcher wants to know whether elementary school students enrolled in music classes do better on standardized math tests. He gathers data from 10 different elementary schools that have records for all of their students.
   a. Is the given method of data collection a sample survey, an observational study, or an experiment? Explain your reasoning.
   b. Identify the population, the sample, and the characteristic of interest.
2. You want to perform an experiment to determine the support for a new stadium at the local high school. You choose a random sample of 40 senior citizens and interview them to represent the population for your experiment. Explain how this sampling method is biased.
3. Two produce stands at a farmer's market sell watermelons for $5.99 each. The watermelons sold at Bev's Fresh Picks have a mean weight of 20 pounds and a standard deviation of 1.5 pounds. The watermelons sold at Frank's Produce have a mean weight of 19.5 pounds and a standard deviation of 2 pounds. Assume the watermelon weights are normally distributed.
   a. Fenna is having a picnic and would like a watermelon that is at least 21 pounds. From which stand would Fenna have a higher probability of getting the watermelon she wants? Explain your reasoning.
   b. Grace would like a watermelon that is less than 19 pounds for her pool party. From which stand would Grace have a higher probability of getting the watermelon she wants? Explain your reasoning.
4. Solve each equation.
   a. $2 \cdot 3^{x+2} - 5 = 15$
   b. $\log_4 (x + 3) - \log_4 x = 2$

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3

A Vote of Confidence

Using Confidence Intervals to Estimate Unknown Population Means

Warm Up

Determine the type of sampling method being used in each scenario.

1. A math teacher chooses every third student to present their solution to the class.
2. A cafeteria worker asks the first 100 students that walk into the cafeteria to complete a satisfaction survey.

Learning Goals

- Interpret the margin of error when estimating a population proportion.
- Interpret the margin of error when estimating a population mean.
- Recognize the difference between a sample and a sampling distribution.
- Recognize that data from samples are used to estimate population proportions and population means.
- Use confidence intervals to determine the margin of error of a population proportion estimate.
- Use confidence intervals to determine the margin of error of a population mean estimate.

Key Terms

- margin of error
- population proportion
- sample proportion
- sampling distribution
- confidence interval

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You have learned that the mean of a sample can be used to estimate the population mean. What are methods to measure the possible range of results?

GETTING STARTED

Every Vote Counts

The poll results are categorical data because there are two categories: those who will vote for the mayor and those who won't.

In a poll of 1100 registered voters before an upcoming mayoral election, 594 people, or 54%, said they would vote to re-elect the current mayor, while the remaining voters said they would not vote for the mayor. The *margin of error* for the poll was ±3 percent, which means that the poll predicts that somewhere between 51% (54% − 3%) and 57% (54% + 3%) of people will actually vote to re-elect the mayor. The **margin of error** expresses the maximum expected difference between the true population data and the sample estimate of the data.

1. **Does the poll represent a sample survey, an observational study, or an experiment? Explain your reasoning.**

2. **Based on the poll, can you conclude that the current mayor will be re-elected? Explain your reasoning.**

3. **Is it possible for fewer than 50% of respondents in a new sample to respond that they will vote for the mayor in the election? Is it likely? Explain your reasoning.**

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ACTIVITY

3.1

Sampling Distribution for Categorical Data

Simulation is a technique used to model random events. You first need to describe all the possible outcomes of the event and associate each of these outcomes with a random number. As random numbers are selected, note each outcome. You are attempting to observe the amount of variability that may occur in your sampling and to verify the validity of your real-world results.

1. With your classmates, conduct a simulation to represent polling a new sample of 1100 voters.

a. Divide 1100 by the number of students in your class to determine the size of each student's sample.

b. Generate random numbers equal to the sample size in part (a) to represent responses to the polling question. Generate random numbers between 1 and 100, with the numbers from 1 to 54 representing support for re-electing the mayor and the numbers 55 to 100 representing support for not re-electing the mayor.

Tally the results of your simulation, and then list the total number of tallies for each category.

Think about:

Could the numbers 1 to 46 represent support for not re-electing the mayor and the numbers 47 to 100 represent support for re-electing the mayor?

Number of People Who Respond that They Will Vote to Re-elect the Mayor	Number of People Who Respond that They Will Vote to Not Re-elect the Mayor

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c. **Calculate the percent of people who state that they will vote to re-elect the mayor and the percent of people who state that they will vote to not re-elect the mayor based on your simulation.**

d. **Complete the simulation for the 1100 voters by combining the data from your classmates. List the percent of votes for each category.**

Percent of People Who Respond that They Will Vote to Re-elect the Mayor	Percent of People Who Respond that They Will Vote to Not Re-elect the Mayor

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e. **Are the results of the simulation different from the results of the original poll? Explain your reasoning.**

f. **If you conducted the simulation over and over, would you expect to get the same results or different results each time? Explain your reasoning.**

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The percent of voters who actually vote for the mayor in the election is the **population proportion**. The percent of voters in the sample who respond that they will vote for the mayor is the **sample proportion**.

When you and your classmates generated random numbers to simulate multiple samples of the 1100 voters, you came up with different sample proportions. The set of all of your classmates' sample proportions is part of a *sampling distribution*.

For continuous data, it's called the population mean or sample mean.
For categorical data, it's called the population proportion or sample proportion.

A **sampling distribution** is the set of sample proportions for all possible equal-sized samples. A sampling distribution will be close to a normal distribution, and the center of a sampling distribution is a good estimate of a population proportion—in this case, the percent of people who will actually vote to re-elect the mayor.

But rather than collecting a very large number of samples, a more practical method for estimating a population proportion is to use the sample proportion of a single sample to estimate the standard deviation of the sampling distribution. The standard deviation of a sampling distribution can give you a range in which the population proportion is likely to fall, relative to the sample proportion.

You can learn the details of deriving the formula for the standard deviation of the sampling distribution, $\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$, in a statistics course.

For example, to estimate the standard deviation of the sampling distribution for the sample of 1100 voters, you can use the formula $\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$, where $\hat{p}$ is the sample proportion and n is the sample size.

Worked Example

The sample proportion from the original poll is 54%, or 0.54. This is the percent of the 1100 people in the poll who said they would vote to re-elect the mayor.

The standard deviation of the sampling distribution for this poll is

$$\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} = \sqrt{\frac{0.54(1 - 0.54)}{1100}}$$
$$\approx 0.0150$$

This means that 1 standard deviation below the sample proportion of 54% is 54% − 1.5%, or 52.5%. And 1 standard deviation above the sample proportion of 54% is 54% + 1.5%, or 55.5%.

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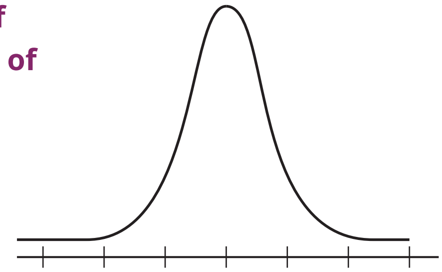
2. **Use the sample proportion and standard deviation of the sampling distribution to label the horizontal axis of the normal curve.**



Percent of People Voting for Mayor's Re-election

3. **Bobbie made an observation about the standard deviation of a sampling distribution.**

> Bobbie
> The standard deviation of a sampling distribution gets smaller and smaller as the size of the sample gets larger and larger.

Is Bobbie's statement correct? Explain why or why not.

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3.2 Confidence Intervals

An estimated range of values that will likely include the population proportion or population mean is called a **confidence interval**. When stating the margin of error, a 95% confidence interval is typically used. However, other confidence intervals may also be used.

Recall the election poll from the previous activity. The standard deviation of the sampling distribution for the election sample is 0.015, or 1.5%. Two standard deviations is 3%, so the margin of error is reported as ±3%.

Confidence intervals for a population proportion are calculated using the sample proportion and the standard deviation of the sampling distribution.

- The lower bound of a 68% confidence interval ranges from 1 standard deviation below the sample proportion to 1 standard deviation above the sample proportion.
- The lower bound of a 95% confidence interval ranges from 2 standard deviations below the sample proportion to 2 standard deviations above the sample proportion.
- The lower bound of a 99.7% confidence interval ranges from 3 standard deviations below the sample proportion to 3 standard deviations above the sample proportion.

1. Determine each confidence interval for the election poll.

a. 68%

b. 95%

c. 99.7%

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2. **Explain the similarities and differences between each confidence interval for the election poll.**

3. **The result of the original poll was 54% with a ±3% margin of error. What confidence interval does ±3% represent? Explain your reasoning.**

4. **Use a 95% confidence interval to determine a margin of error and a range of values for each population proportion.**

   a. **A survey of 1500 teenagers shows that 83% do not like waking up early in the morning.**

   b. **A survey of 200 licensed high school students shows that 16% own their own car.**

   c. **A survey of 500 high school students shows that 90% say math is their favorite class.**

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3.3

Sampling Distribution for Continuous Data

Ask yourself:

Why is the sample mean used instead of sample proportion?

A sample of 50 students at High Marks High School responded to a survey about their amount of sleep during an average night. The sample mean was 7.7 hours and the sample standard deviation was 0.8 hour.

Let's determine an estimate for the population mean sleep time for all High Marks High School students.

1. If you gathered data from many new samples, would you expect the samples to have equal means or different means? Explain your reasoning.

Collecting additional samples of 50 students and plotting each sample mean will result in a sampling distribution. The sampling distribution will be approximately normal, and the mean of the sampling distribution is a good estimate of the population mean.

Just like with the categorical data, a more practical method to estimate the population mean amount of sleep for High Marks High School students is to use the sample mean to calculate an estimate for the standard deviation of the sampling distribution. The formula for the standard deviation of a sampling distribution for continuous data is $\frac{s}{\sqrt{n}}$, where s is the standard deviation of the original sample and n is the sample size.

Remember:

The formula for the standard deviation of a sampling distribution of categorical data is $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.

2. Use the standard deviation from the original sample to determine the standard deviation for the sampling distribution. Explain your work.

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3. **Use the standard deviation of the sampling distribution to determine a 95% confidence interval for the population mean. Explain your work.**

4. **Write the 95% confidence interval in terms of the population mean plus or minus a margin of error.**

5. **Use a 95% confidence interval to determine a range of values for each population mean.**

    a. **A sample of 75 students responded to a survey about the amount of time spent online each day. The sample mean was 3.2 hours, and the standard deviation of the sampling distribution was 0.9 hour.**

    b. **A sample of 1000 teachers responded to a survey about the amount of time they spend preparing for class outside of school hours. The sample mean was 2.5 hours, and the standard deviation of the sampling distribution was 0.5 hour.**

    c. **A sample of 400 adults responded to a survey about the distance from their home to work. The sample mean was 7.8 miles, and the standard deviation of the sampling distribution was 1.6 miles.**

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ACTIVITY

3.4

Online Time Study, Part III

Ask yourself:

How can you apply your new knowledge from this lesson to analyze data in the Online Time Study?

In the previous lesson, you conducted a sample survey. Now, take the next steps to summarize and analyze your data.

To summarize data from a sample survey, observational study, or experiment:

- Calculate the measure of center.
- Calculate the measure of spread.
- Select the most appropriate method to display the data (dot plot, histogram, stem-and-leaf plot, box-and-whisker plot, normal curve).
- Describe the characteristics of the graphical display.

To analyze data from a sample survey, observational study, or experiment use a confidence interval to determine the range of values for the population mean or population proportion.

Recall the study described in previous lessons about the amount of time students in your school spend online each day.

1. Will your study involve estimating a population mean or a population proportion? Explain your reasoning.

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2. Use a 95% confidence interval to determine a range of values for the population mean for each set of your sample survey results.

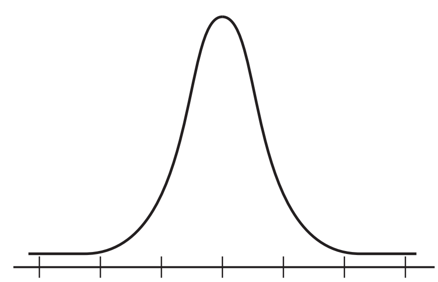
   a. Calculate the sample mean.

   b. Calculate the standard deviation.

   c. Calculate the range of values that corresponds to a 95% confidence interval.

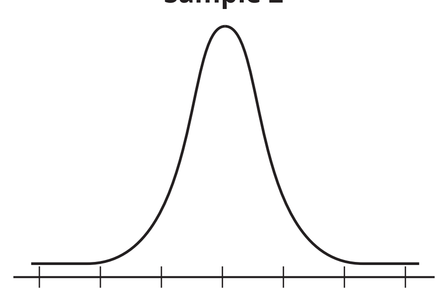
3. Use the sample mean and standard deviation of the sampling distribution to label the horizontal axis of the normal curve for each set of data.

Sample 1



Average Time Spent Online

Sample 2



Average Time Spent Online

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NOTES

TALK the TALK

Continuous or Categorical?

1. **What is the difference between a sample and a sampling distribution?**

2. **What is the difference between a sample proportion and a sample mean?**

3. **A survey of 500 students reports that 82% will attend the winter carnival. Determine a range of values for the population proportion. Use a 95% confidence interval for the population proportion.**

4. **A sample of 300 dog owners responded to a survey about the amount of money they spend on dog chew toys each year. The sample mean was $150 and the sample standard deviation was $12. Determine a range of values for the population mean. Use a 95% confidence interval.**

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Assignment

Write

Write a definition for each term in your own words.

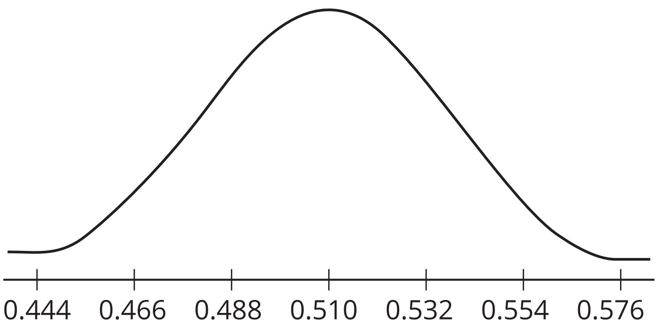
1. population proportion
2. sample proportion
3. sampling distribution
4. confidence interval

Remember

To analyze data from a sample survey, observational study, or experiment, use confidence intervals to determine a range of values for the population mean(s) or proportion(s).

Practice

1. Roosevelt High School is considering a requirement for all 1300 student to wear uniforms to school. Of the 300 parents surveyed by the school, 141 said they were in favor of mandatory school uniforms.
   a. Determine the sample proportion that represents the percent of parents in favor of mandatory school uniforms.
   b. Determine a 95% confidence interval for the population proportion using the sample proportion you determined in part (a). Round your answer to the nearest tenth of a percent.
   c. A group of high school students conducted their own survey of 500 parents. The normal curve displays the sample proportion that represents the percent of parents in favor of mandatory school uniforms and the standard deviation of the sampling distribution. Determine a 95% confidence interval for the population proportion based on the students' sample proportion. Round your answer to the nearest tenth of a percent.



   d. Based on the given information in part (c), how many of the parents who responded to the students' survey are in favor of mandatory school uniforms?
   e. Which survey would you expect to be more accurate, the school's survey or the students' survey? Explain your reasoning.
2. Two hundred teenage boys were surveyed about the number of hours they spend each week playing video games. The sample mean was 11.7 hours and the standard deviation was 3.4 hours.
   a. Determine the standard deviation for the population mean.
   b. Determine a 95% confidence interval for the population mean.
3. Five hundred teenage girls were surveyed about the number of hours they spend each week listening to music. The sample mean was 9.2 hours and the standard deviation was 2.7 hours.
   a. Determine the standard deviation for the population mean.
   b. Determine a 95% confidence interval for the population mean.

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Stretch

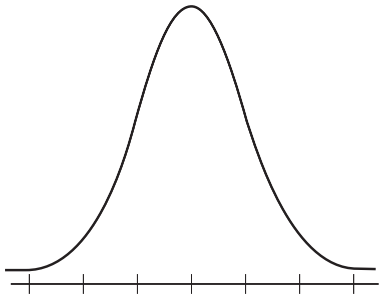
1. The lower and upper bounds for confidence intervals of 68%, 95%, and 99.7% for population proportions are calculated by adding to and subtracting from the sample proportion either 1, 2, or 3 standard deviations. Consider a confidence interval of 90%.
   a. Determine how much area would be in each tail of a standard normal distribution if 90% of the data was in the center.
   b. Determine the lower and upper z-scores of the area under the standard normal curve that would contain 90% of the data.
   c. Revisit the Roosevelt High School mandatory uniform scenario from the Practice section. Use the z-score you determined in part (b) to determine a 90% confidence interval for the population proportion of parents in favor of mandatory school uniforms. Round your answer to the nearest tenth of a percent.
   d. Compare the 90% and 95% confidence intervals. Explain the difference.

Review

1. Twenty students at a small college took the GRE exam. Each student was given an identification number from 01 through 20. Students 01 through 10 are business majors and students 11 through 20 are psychology majors. The table shows the students scores on the verbal reasoning portion of the exam. Scores on the exam range from 130 to 170 in 1 point increments.

Student ID Number	Score	Student ID Number	Score
01	145	11	158
02	130	12	143
03	148	13	140
04	166	14	164
05	133	15	136
06	153	16	147
07	146	17	159
08	139	18	160
09	165	19	170
10	155	20	138

   a. Create a simple random sample of 6 scores from the table. Explain how you created your sample.
   b. Calculate the average of your sample.
   c. The actual average score is approximately 149.75. How does the average of your sample compare?
2. A researcher recorded the cholesterol levels of a sample of women aged 60-70. The average cholesterol level was 201 milligrams per deciliter and the standard deviation was 40 milligrams per deciliter. The resting heart rates follow a normal distribution.



   a. Label the number line so that the curve is a normal curve and follows the properties of a normal distribution. Include 3 standard deviations above and below the mean.
   b. Determine the percent of women who have cholesterol levels between 2 standard deviations below the mean and 1 standard deviation below the mean.
   c. Approximately what percent of women had cholesterol levels above 321 milligrams per deciliter?
3. Convert 210° to radians.
4. Convert $\frac{5\pi}{12}$ to degrees.

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4

How Much Different?

Using Statistical Significance to Make Inferences About Populations

Warm Up

Determine a range of values for the population proportion. Use a 95% confidence interval.

1. A satisfaction survey was given to 220 gym members. Ninety-one percent of the respondents in the sample reported that they workout regularly and intend to renew their contract with the gym for an additional year.

Learning Goals

- Use sample proportions to determine whether differences in population proportions are statistically significant.
- Use sample means to determine whether differences in population means are statistically significant.

Key Term

- statistically significant

You have learned that you can estimate a population proportion or mean by using sample data. How do you determine whether two sample proportions or means are statistically significant?

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GETTING STARTED

Whatta Water

Commercials on a local TV station claim that Whatta Water tastes better than tap water, so a local news anchor sets up an experiment at a local grocery store to test the claim. A representative, unbiased sample of 120 shoppers participate in the tasting survey using unmarked cups. Out of the 120 people, 64 said Whatta Water tastes better than tap water.

1. **If shoppers had to choose one or the other and there is no difference in the taste of the two waters, what proportion of shoppers would you expect to say that Whatta Water tastes better? Explain your reasoning.**

2. **What is the sample proportion of shoppers who stated that Whatta Water tastes better?**

3. **Based on your answers to Questions 1 and 2, what reason(s) can you give to doubt Whatta Water's claim? Explain your reasoning.**

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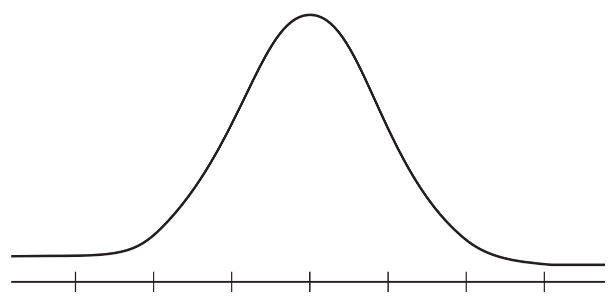
ACTIVITY

4.1 Exploring Categorical Data

The term **statistically significant** is used to indicate that a result is very unlikely to have occurred by chance. Typically, a result that is more than 2 standard deviations from the mean, or outside a 95% confidence interval, is considered statistically significant.

1. Consider the results of the survey in the Getting Started. Use a 95% confidence interval to determine a range of values for the population proportion of people who prefer the taste of Whatta Water. Explain your work.

2. Use the sample proportion and standard deviation of the sampling distribution to label the horizontal axis of the normal curve.



3. Based on the range of values of the 95% confidence interval, what conclusion can you make about Whatta Water's claim that their water tastes better than tap water?

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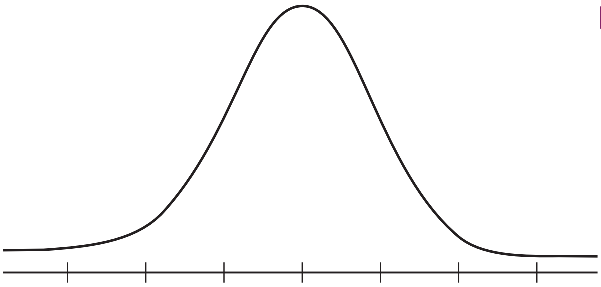
4. The local water company also conducted a survey of 120 people, which showed that people prefer tap water over Whatta Water. Forty-one of the respondents said Whatta Water tastes better.

The assumption is that the results will be 50% if there is no difference between the two kinds of water.

a. Use a 95% confidence interval to determine a range of values for the population proportion of people who prefer Whatta Water. Explain your work.



Percent of People Preferring
Whatta Water to Tap Water

b. Use the sample proportion and standard deviation of the sampling distribution to label the horizontal axis of the normal curve.

c. Based on the range of values of the 95% confidence interval, what conclusion can you draw about the local water company's claim that tap water tastes better than Whatta Water?

Think about:

What numbers are you using to represent those who say Whatta Water tastes better? What numbers are you using to represent those who say tap water tastes better?

5. Use a random number generator to conduct a simulation of the local water company's survey, for a new sample of 120 people. List the results in the table.

Percent of People in Simulation Who Said Whatta Water Tastes Better	Percent of People in Simulation Who Said Tap Water Tastes Better

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6. On the normal curve in Question 4 part (b), locate and mark the sample proportion of your simulation. Describe the location of the sample proportion on the normal curve.

7. Compare the results of your simulation with the water company's study and with Whatta Water's study. Are your results significantly different? Explain your reasoning.

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ACTIVITY

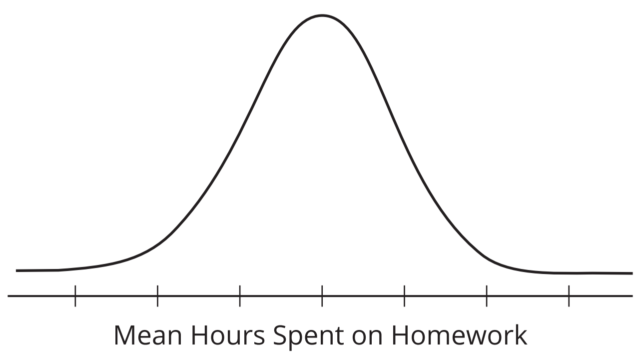
4.2

Exploring Continuous Data

A sample of 40 students at High Marks High School responded to a survey about the average amount of time spent on homework each day. The sample mean is 2.9 hours and the sample standard deviation is 0.8 hour.

1. Use a 95% confidence interval to determine a range of values for the population mean. Explain your work.

2. Label the horizontal axis of the normal curve that represents the sampling distribution.



Ask yourself:

How is this problem similar to the problem in the previous activity? How is it different?

3. A new sample of 40 students was taken and the resulting sample mean is 2.70 hours.

a. On the normal curve in Question 2, locate and mark the sample mean of the new sample. Describe the location of the sample mean on the normal curve.

b. Are the results of the new sample statistically significant? Explain your reasoning.

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4. **What sample mean values are statistically significant? Explain your reasoning.**

5. **Mary shared a comment about the time she spends on homework.**

Mary
I spend an average of 3.5 hours on homework every night. Compared to the sample mean, the average amount of time I spend on homework every night is statistically significant.

Is Mary's reasoning valid? Explain why or why not.

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ACTIVITY
4.3

Comparing Categorical Data

Two hometown newspapers conducted a poll about whether residents are for or against a tax to provide funding for school renovations in the district. Today's News polled 75 residents and 53 stated that they are in favor of the tax increase. Local Time polled 100 residents and 54 stated they are in favor of the tax increase.

1. Calculate the sample proportion for each poll.

2. Use the results from each poll to estimate a range of values for the population proportion using a 95% confidence interval. Explain your work.

3. The Reporter newspaper published a survey of 90 residents, and 38 stated that they are in favor of the tax increase. Use a 95% confidence interval to determine a range of values for the population proportion. Explain your work.

If two confidence intervals overlap, then the difference between the population proportions or population means is not statistically significant. If the confidence intervals do not overlap, then the difference between the population proportions or population means is statistically significant.

4. Compare the population proportion estimates and determine whether their differences are statistically significant. Explain your reasoning.

a. The Reporter and Local Time

b. The Reporter and Today's News

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ACTIVITY

4.4

Comparing Continuous Data

A researcher conducted a randomized experiment to see whether there is a link between a new supplement and blood pressure. She collected data from a representative, unbiased sample of 200 people who had high blood pressure. One hundred of the people were randomly selected to take the supplement and the other 100 people were given a placebo. Recall that a placebo is a treatment that is assumed to have no real effect on the characteristic of interest.

This experiment has two treatments: taking the supplement and taking the placebo.

The participants' blood pressures were recorded at the beginning and at the end of the 12-week experiment, and the difference (end − beginning) was calculated.

1. For the 100-person treatment that took the placebo, what value would you expect for the difference of sample means at the beginning of the experiment and at the end of the experiment? Explain your reasoning.

2. For the 100-person treatment that took the supplement, what value would you expect for the difference of sample means at the beginning of the experiment and at the end of the experiment? Explain your reasoning.

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Suppose that the mean difference in blood pressure of the group who took the supplement is −15 with a standard deviation of 6.2, and the mean difference in blood pressure of the group who took the placebo is 1.7 with a standard deviation of 0.8.

3. Interpret and explain the meaning of a negative mean difference for the treatment that took the supplement and a positive mean difference for the treatment that took the placebo.

4. Use a 95% confidence interval to determine a range of values for the population mean of each treatment. Explain your work.

5. What conclusion can you make about whether the supplement effectively lowers high blood pressure? Explain your reasoning.

Think about:

The results of an experiment may indicate a correlation but not a causation. What is the difference between the two?

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ACTIVITY

4.5

Interpreting Data

The final step of any study is to draw a valid conclusion that answers the question of interest.

1. **A manufacturing company has a policy that states that if significantly more than 2% of computer parts are defective during an 8-hour shift, then the parts from that shift will not be shipped. During an 8-hour shift, 1020 parts were produced and 22 were defective. Should the parts be shipped? Explain your reasoning.**

2. **The mean grade point average (GPA) of a random sample of 50 High Mark High School students who had a part-time job during the previous grading period is 3.15 with a standard deviation of 0.44. The mean GPA of a random sample of 50 High Mark High School students who did not have a part-time job during the previous grading period is 2.77 with a standard deviation of 0.35. Does that data suggest a possible link between High Mark High School students' part-time job status and their GPA?**

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ACTIVITY

4.6

Online Time Study, Part IV

In the previous lesson, you summarized your data and began the process of analyzing it. Now you will further analyze your data by applying the concept of statistical significance.

To analyze data from a sample survey, observational study, or experiment:

- Use a confidence interval to determine the range of values for the population mean or population proportion.
- Use statistical significance to make inferences about a population.

Recall the study you have been planning about the amount of time students in your school spend online each day.

Think about:

How can you use statistical significance to make inferences in the Online Time Study?

1. Use a 95% confidence interval to determine whether the estimate of the population means using each sample is statistically significant. Explain your work.

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TALK the TALK

The End of the Line

Recall the problem from a previous activity about part-time job status and grade point average (GPA).

The population mean interval for the GPA of High Mark High School students who have a part-time job, 3.03 to 3.27, does not overlap with population mean interval for the GPA of High Mark High School students who do not have a part-time job, 2.67 to 2.87.

1. Carmen shared a conclusion about part-time job status and GPA.

Carmen
Because the results of the statistical analysis are statistically significant, I can conclude that holding a part-time job will result in a higher GPA.

Is Carmen's statement correct? Explain why or why not.

The interval for the estimate of the population mean for the GPA of neighboring Great Beginnings High School students who do not have a part-time job is 3.18 to 3.39.

2. Is the GPA of students who do not have a part-time job statistically different at High Mark High and Great Beginnings High School? Explain your reasoning.

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NOTES

3. **The estimate for the population mean for the math GPA of Great Beginnings High School students using a sample of the math club is 3.27 to 3.54. The estimate for the population mean for the math GPA of Great Beginnings High School students using a sample of the government club is 3.11 to 3.40.**

> Max
> The results of the statistical analysis are not statistically significant because the population mean intervals for math GPA overlap.

Is Max's statement correct? Explain why or why not.

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Assignment

Write

Describe how you determine whether two sample proportions or means are statistically significant.

Remember

The term statistically significant is used to indicate that a result is very unlikely to have occurred by chance. Typically, a result that is more than 2 standard deviations from the mean, or outside a 95% confidence interval, is considered statistically significant.

Practice

1. Legislators have been trying to increase public support for the construction of a new bridge in their state's largest city through a broad advertising campaign. Prior to the advertising campaign, 539 out of 1400 people that were polled said they supported the bridge project. Following the advertising campaign, 561 out of 1100 people that were polled said they supported the project.
   a. Determine the sample proportion of people who support the new bridge for each poll.
   b. Determine whether the results of the 2 polls are statistically significant. Use a 95% confidence interval when making your calculations.
   c. Based on your findings, what can you conclude about the impact of the ad campaign?
2. A random sample of 150 qualifying speeds is collected from data on stock car races. The mean qualifying speed is 191.8 miles per hour with a standard deviation of 2.1 miles per hour.
   a. Determine a range of values for the population mean using a 95% confidence interval.
   b. Burn Rubber Tires introduces a new tire they claim is revolutionary. They guarantee these tires will increase the speeds of stock cars. A random sample of 150 qualifying speeds of cars using these new tires is collected. The average qualifying speed of cars with these tires is 192.08 miles per hour. Are the results of the sample using the new tires statistically significant? What can you conclude about the effectiveness of the new tires? Explain your reasoning.
3. On average, 48% of all babies born in the United States are girls. In the past year, 473 of the 860 babies born in one particular county were girls. Determine whether the birth rate for girls in this county is statistically significant. Use a 95% confidence interval when making your calculations.

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Stretch

1. A recent study of the blood pressure of employees in a particular company found the rate of employees who have high blood pressure had a 95% confidence interval of 0.4598 to 0.4816.
   a. What was the sample proportion?
   b. What was the standard deviation from the sample?
   c. If 480 employees in the sample had high blood pressure, how many employees were in the sample?
   d. The population proportion of employees in the company with high blood pressure is 0.46. Are the results of the sample of employees statistically significant? Explain your reasoning.

Review

1. A small town of 15,680 adults is thinking of building a community recreation center with a pool. Of the 1200 adults surveyed by the town, 445 said they were in favor of a community recreation center.
   a. Determine the sample proportion that represents the percent of adults in the town that favor building a community recreation center. Round your answer to the nearest tenth of a percent.
   b. Determine a 95% confidence interval for the population proportion using the sample proportion. Round your answer to the nearest tenth of a percent.
2. The speed of cars on a stretch of highway is normally distributed. The average speed of a car is 65 miles per hour with a standard deviation of 6 miles per hour.
   a. What percent of cars are driving less than 70 miles per hour?
   b. What percent of cars are driving between 57 and 62 miles per hour?
   c. Determine the speed of a car in the 65th percentile.
3. Write an equation of a sine curve with amplitude 2, period $\frac{\pi}{4}$, phase shift 3, and vertical shift 6.

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5

DIY

Designing a Study and Analyzing the Results

Warm Up

Consider each question and determine whether a sample survey, observational study, or experiment is most appropriate.

1. Are pet owners healthier than people who do not own a pet?
2. Does a particular herb increase memory?
3. Which population uses more electronic reading devices, adults or teenagers?

Learning Goals

- Analyze the validity of conclusions based on statistical analysis of data.
- Design a sample survey, observational study, or experiment to answer a question.
- Conduct a sample survey, observational study, or experiment to collect data.
- Summarize the data of your sample survey, observational study, or experiment.
- Analyze the data of your sample survey, observational study, or experiment.
- Summarize the results and justify conclusions of your sample survey, observational study, or experiment.

You have learned to determine statistical significance between two population proportions or means. How do you use statistical analysis to draw valid conclusions that answers a question of interest?

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GETTING STARTED

Check Yourself

Throughout this topic, you have learned how to structure a study and analyze the results.

1. Review the pieces that you've completed so far in your online time study.

I. Design a sample survey, observational study, or experiment.

- Determine a characteristic of interest to learn about from a sample survey, observational study, or experiment.
- Identify the population.
- Write a question(s) that can be answered by collecting quantitative data.
- Collect a sample using a method that avoids bias.
- Eliminate elements of the design that may introduce confounding.

II. Conduct the sample survey, observational study, or experiment.

- Use the sampling method to collect data for your sample survey, observational study, or experiment.

III. Summarize the data of the sample survey, observational study, or experiment.

- Calculate the measure of center.
- Calculate the measure of spread.
- Select the most appropriate method to display the data (dot plot, histogram, stem-and-leaf plot, box-and-whisker plot, normal curve).
- Describe the characteristics of the graphical display.

IV. Analyze the data of the sample survey, observational study, or experiment.

- Use a confidence interval to determine a range of values for the population mean or population proportion.
- Use statistical significance to make inferences about a population.

V. Draw a valid conclusion that answers the question of interest of the sample survey, observational study, or experiment.

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ACTIVITY
5.1

Online Time Study, Part V

You have designed, conducted, summarized, and analyzed data from your Online Time Study.

1. Write a conclusion that answers the question of interest in the Online Time Study sample survey. Use the data and data analysis to justify your conclusion.

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ACTIVITY 5.2 Designing a Study and Analyzing the Results

U.S. high school juniors who took the ACT last year were asked to check a box in the answer booklet stating whether they had completed an ACT prep course prior to taking the exam. To determine whether a prep course improved ACT scores, researchers collected a random sample of ACT test scores from 1500 U.S. high school juniors who completed a prep course prior to taking the ACT exam. They also collected a random sample of ACT test scores from 1500 U.S. high school juniors who did not complete a prep course prior to taking the ACT exam. Those who completed the prep course had a mean score of 22.1 with a standard deviation of 3.9. Those who did not complete the prep course had a mean score of 21.3 with a standard deviation of 4.3.

1. **Describe the design of the study.**

2. **Summarize the data collected.**

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3. **Determine whether the difference between the 2 population means statistically significant? Explain your reasoning.**

4. **Draw a conclusion about the effectiveness of the ACT prep course. Discuss the effect of any potential bias on the results.**

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NOTES

TALK the TALK

Let's Wrap This Up

1. **Is it easier to design and conduct a sample survey, observational study, or experiment? Why?**

2. **When designing a sample survey, observational study, or experiment, what is the difference between selecting a characteristic of interest and selecting a question that can be answered by collecting data?**

3. **What should be considered when identifying the population for a sample survey, observational study, or experiment?**

4. **What are important elements associated with designing a sample survey, observational study, or experiment?**

5. **What are important elements associated with summarizing the data of a sample survey, observational study, or experiment?**

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Assignment

Write

Explain in your own words the difference between a sample survey, an observational study, and an experiment.

Remember

To study a characteristic of interest, you must design a sample survey, observational study, or experiment; conduct the sample survey, observational study, or experiment; summarize the data; analyze the results; and draw a valid conclusion.

Practice

A company has come up with a new organic fertilizer. In order to test whether the new fertilizer works, they treated some staked tomato plants with the organic fertilizer and did not treat other staked tomato plants. The company collected a random sample of the yield of 85 tomato plants treated by the fertilizer and a random sample of 85 tomato plants not treated by the fertilizer. The plants treated with the fertilizer had a mean yield of 8.25 pounds of tomatoes with a standard deviation of 0.37 pound. The plants not treated with the fertilizer had a mean yield of 7.95 pounds of tomatoes with a standard deviation of 0.42 pound.

1. Identify the samples and the population.
2. Identify the characteristic of interest.
3. Is the company conducting a sample survey, an observational study, or an experiment? Explain your reasoning.
4. Determine the standard deviation for both population means.
5. Determine a 95% confidence interval for both population means.
6. Is the difference between the 2 population means statistically significant? Explain your reasoning.
7. What can you conclude about the effectiveness of the organic fertilizer? Discuss the effect of any potential bias on the results.

Stretch

1. Design an observational study for the height of male students in your grade and the grade below at your school.
   - Identify the samples and the population.
   - Identify the characteristic of interest.
   - Determine the standard deviation for both population means.
   - Determine a 95% confidence interval for both population means.
   - Determine whether the difference between the 2 population means is statistically significant.
   - Draw a conclusion about the heights of male students in the two grades. Discuss the effect of any potential bias on the results.

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Review

1. A community college tries a new program designed to increase student satisfaction with the college. Prior to the program, 1623 out of 2151 students polled were satisfied. After the program, 1718 out of 2201 students polled were satisfied.
   a. Determine the sample proportion of students who were satisfied for each poll.
   b. Determine whether the results of the 2 polls are statistically significant. Use a 95% confidence interval when making your calculations.
   c. Based on your findings, what can you conclude about the impact of the program?
2. The principal of a school is going to order pizzas for a staff luncheon. The delivery times for Sebastion's Pizzaria have a mean of 35 minutes and a standard deviation of 4.5 minutes. The delivery times for Lorenzo's Pizzaria have a mean of 33 minutes and a standard deviation of 2.5 minutes. Delivery times are normally distributed.
   a. The principal would like the pizza to be delivered in less than 30 minutes. From which pizzaria would the principal have a higher probability of getting the pizza delivered in the time she wants? Explain your reasoning.
   b. Due to a meeting running late, the principal would like the pizza to be delivered in between 35 and 40 minutes. From which pizzaria would the principal have a higher probability of getting the pizza delivered in the time she wants? Explain your reasoning.
3. Divide the rational expressions.
   $\frac{8x^3 - 16x^2}{x - 5} \div \frac{40x^3 + 16x^2}{5x + 2}$

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Making Inferences and Justifying Conclusions Summary

KEY TERMS

- characteristic of interest
- sample survey
- random sample
- biased sample
- observational study
- experiment
- treatment
- experimental unit
- confounding
- convenience sample
- subjective sample
- volunteer sample
- simple random sample
- stratified random sample
- cluster sample
- cluster
- systematic sample
- parameter
- statistic
- margin of error
- population proportion
- sample proportion
- sampling distribution
- confidence interval
- statistically significant

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LESSON 1 Data, Data, Everywhere

You can use data to help answer questions about the world. The specific question that you are trying to answer or the specific information that you are trying to gather is called a **characteristic of interest**.

For example, you can use data to help determine which drug is most effective, teenagers' favorite television program, or how often doctors wash their hands.

One way of collecting data is to use a sample survey. A **sample survey** poses one or more questions of interest to obtain sample data from a population.

When sample data are collected in order to describe a characteristic of interest, it is important that such a sample be as representative of the population as possible. One way to collect a representative sample is to use a random sample. A **random sample** is a sample that is selected from the population in such a way that every member of the population has the same chance of being selected. A **biased sample** is a sample that is collected in a way that makes it unrepresentative of the population.

In an **observational study**, data are gathered about a characteristic of the population by simply observing and describing events in their natural settings. Recording the number of children who use the swings at a local park would be an example of a simple observational study.

An **experiment** gathers data on the effect of one or more **treatments**, or experimental conditions, on the characteristic of interest. Members of a sample, also known as **experimental units**, are randomly assigned to a treatment group.

For example, consider a situation in which researchers conduct an experiment to test the effectiveness of a new asthma drug by collecting data from a sample of 200 asthma patients. One hundred of the patients received a placebo treatment along with an inhaler. The other one hundred patients received the new drug along with an inhaler. Monthly blood and breathing tests were performed on all 200 patients to determine whether the new drug was effective.

The population of the study is asthma patients with the sample being the 200 people participating in the experiment. The characteristic of interest is their response to the asthma medication.

Confounding occurs when there are other possible reasons, called confounds, for the results to have occurred that were not identified prior to the study.

For example, in the study about the effectiveness of the new asthma medication, the experiment could have been confounded if the placebo group had not been given an inhaler. This would lead to a visible difference in the groups and would make the patients question the treatment.

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LESSON 2 Ample Sample Examples

A **convenience sample** is a sample whose data is based on what is convenient for the person choosing the sample. A **subjective sample** is a sample drawn by making a judgment about which data items to select. Another type of sample is a volunteer sample. A **volunteer sample** is a sample whose data consists of those who volunteer to be part of a sample. These types of sampling methods could lead to bias.

For example, consider a situation in which a cereal company conducts taste tests for a new cereal on a random sample of its employees. This is an example of a convenience sample. There is bias in this study because the taste test is only conducted on the company's employees. It is possible that the employees will prefer the cereal of the company that employs them for other reasons than taste.

A **simple random sample** is a sample composed of data elements that were equally likely to have been chosen from the population. Using a random digit table is one option for selecting a simple random sample. To use the table, begin at any digit and follow the numbers in a systematic way, such as moving across a row until it ends and then moving to the beginning of the next row. If the same two-digit number comes up more than once, skip it each time it is repeated and go to the next number. You can also use technology to generate a random list of numbers.

A **stratified random sample** is a random sample obtained by dividing a population into different groups, or strata, according to a characteristic and randomly selecting data from each group.

For example, the data set shows the number of late student arrivals at five elementary schools each week for five weeks.

Number of Late Arrivals				
Week 1	**Week 2**	**Week 3**	**Week 4**	**Week 5**
49	37	45	44	43
47	41	45	46	48
39	43	38	44	42
43	47	39	39	42
52	55	50	54	55

You can create a stratified random sample with 5 data values to describe the number of late arrivals by randomly choosing one school from each of the 5 weeks and recording the number of late arrivals: {39, 37, 50, 46, 42}.

A **cluster sample** is a random sample that is obtained by creating clusters. Then, one cluster is randomly selected for the sample. Each **cluster** contains the characteristics of a population.

A **systematic sample** is a random sample obtained by selecting every *n*th data value in a population.

The mean of a sample, $\bar{x}$, can be used to estimate the population mean, μ. The population mean is an example of a **parameter**, because it is a value that refers to a population. The sample mean is an example of a **statistic**, because it is a value that refers to a sample.

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LESSON 3

A Vote of Confidence

The **margin of error** expresses the maximum expected difference between the true population data and the sample estimate of the data.

For example, consider the following situation.

In a poll of 1100 registered voters before an upcoming mayoral election, 594 people, or 54%, said they would vote to re-elect the current mayor, while the remaining voters said they would not vote for the mayor. The margin of error for the poll was ± 3 percent, which means that the poll predicts that somewhere between 51% (54% − 3%) and 57% (54% + 3%) of people will actually vote to re-elect the mayor.

Simulation is a technique used to model random events. First, describe all the possible outcomes of the event and associate each of these outcomes with a random number. As random numbers are selected, note each outcome. Simulation is used to observe the amount of variability that may occur in your sampling and to verify the validity of your real-world results.

For example, to conduct a simulation to represent polling a new sample of 1100 voters, you could create 11 samples of 100 people. In each sample, generate random numbers between 1 and 100, with numbers from 1 to 54 representing support for re-electing the mayor and the numbers 55 to 100 representing support for not re-electing the mayor. Then, tally the results and compare to the original sample.

The percent of voters who actually vote for the mayor in the election is the **population proportion**. The percent of voters in the sample who respond that they will vote for the mayor is the **sample proportion**. For continuous data, it is called the population or sample mean. For categorical data, it is called the population or sample proportion.

The set of all of the sample proportions created when conducting a simulation is part of a sampling distribution. A **sampling distribution** is the set of sample proportions for all possible equal-sized samples. A sampling distribution will be close to a normal distribution, and the center of a sampling distribution is a good estimate of a population proportion—in this case, the percent of people who will actually vote to re-elect the mayor. But rather than collecting a very large number of samples, a more practical method for estimating a population proportion is to use the sample proportion of a single sample to estimate the standard deviation of the sampling distribution. The standard deviation of a sampling distribution can give you a range in which the population proportion is likely to fall, relative to the sample proportion.

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For example, to estimate the standard deviation of the sampling distribution for the sample of 1100 voters, you can use the formula $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ where $\hat{p}$ is the sample proportion and n is the sample size.

The sample proportion from the original poll is 54%, or 0.54. This is the percent of the 1100 people in the poll who said they would vote to re-elect the mayor. The standard deviation of the sampling distribution for this poll is $\sqrt{\frac{\hat{p}\,(1-\hat{p})}{n}} = \sqrt{\frac{0.54(1-0.54)}{1100}} \approx 0.0150$.

This means that 1 standard deviation below the sample proportion of 54% is 54% − 1.5%, or 52.5%, and 1 standard deviation above the sample proportion of 54% is 54% + 1.5%, or 55.5%.

An estimated range of values that will likely include the population proportion or population mean is called a **confidence interval**. When stating the margin of error, a 95% confidence interval is typically used. However, other confidence intervals may also be used.

For example, recall the situation of the election poll. The standard deviation of the sampling distribution for the election sample is 0.015, or 1.5%. Two standard deviations is 3%, so the margin of error is reported as ±3%. Confidence intervals for a population proportion are calculated using the proportion of a sample and the standard deviation of the sampling distribution.

The formula for the standard deviation of a sampling distribution for continuous data is $\frac{s}{\sqrt{n}}$, where s is the standard deviation of the original sample and n is the sample size.

For example, consider the following situation.

A sample of 250 women responded to a survey about the amount of money they spend on cosmetics each month. The sample mean was \$45.50 and the sample standard deviation was \$10.75.

$$\frac{s}{\sqrt{n}} = \frac{10.75}{\sqrt{250}} \approx 0.68$$

A 95% confidence interval for the population mean is \$45.50 − 2(\$0.68), or \$44.14, to \$45.50 + 2(\$0.68), or \$46.86.

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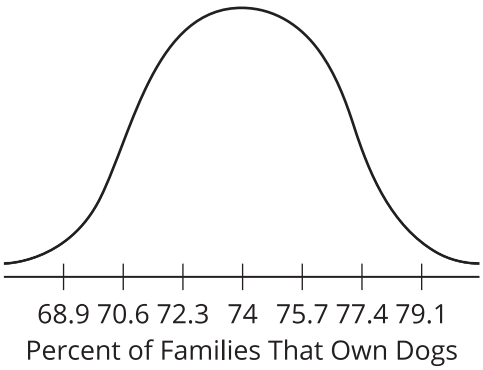
LESSON 4

How Much Different?

The term **statistically significant** is used to indicate that a result is very unlikely to have occurred by chance. Typically, a result that is more than 2 standard deviations from the mean, or outside a 95% confidence interval, is considered statistically significant.

Use the sample proportion and standard deviation of the sampling distribution to label the horizontal axis of the normal curve. Then, determine what sample proportions would be statistically significant.



For example, consider the situation in which a sample proportion of families that own dogs is 74%, and the standard deviation is 0.017.

Sample proportion values less than 70.6% and greater than 77.4% are statistically significant because those values are outside of the 95% confidence interval.

If two confidence intervals overlap, then the difference between the population proportions or population means is not statistically significant. If the intervals do not overlap, then the difference between the population proportions or population means is statistically significant.

For example, analyze the following situation to determine whether the results are statistically significant.

Two hometown newspapers conducted a poll about whether residents are for or against a tax to provide funding for school renovations in the district. Today's News polled 75 residents and 53 stated that they are in favor of the tax increase. Local Time polled 100 residents and 54 stated they are in favor of the tax increase.

Today's News: $\hat{p} = \frac{53}{75} \approx 0.71 \qquad \sigma = \sqrt{\frac{0.71(1 - 0.71)}{75}} \approx 0.052$

Confidence interval $= 0.71 \pm 2(0.052) = 0.814$ or 0.606

Local Time: $\hat{p} = \frac{54}{100} \approx 0.54 \qquad \sigma = \sqrt{\frac{0.54(1 - 0.54)}{100}} \approx 0.050$

Confidence interval $= 0.54 \pm 2(0.050) \approx 0.64$ or 0.44

Since these two confidence intervals overlap, the results of the poll are not statistically significant.

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LESSON 5

DIY

To design a sample survey, observational study, or experiment, consider these steps:

- Identify the characteristic of interest.
- Identify the population.
- Identify methods to collect the sample so that the sample is not biased.
- Ensure that participants are randomly assigned to a treatment.
- Eliminate elements of the design that may introduce confounding.

To summarize data from a sample survey, observational study, or experiment:

- Calculate measures of center.
- Calculate measures of spread.
- Select the most appropriate method(s) to display the data (dot plot, histogram, stem-and-leaf plot, box-and-whisker plot, normal curve).
- Describe the characteristics of the graphical display.

To analyze data from a sample survey, observational study, or experiment:

- Use confidence intervals to determine a range of values for the population mean(s) or proportion(s).

You can determine which type of sample technique would be most appropriate to answer a question for a sample survey, observational study, or experiment.

For example, suppose you want to estimate the number of senior citizens in a town that are on public assistance. You can assign all the senior citizens in the town an ID number and use a computer to randomly generate a sample of senior citizens. This technique provides a random sample of the population of the senior citizens in the town, and random sampling is typically representative of a population.

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Glossary

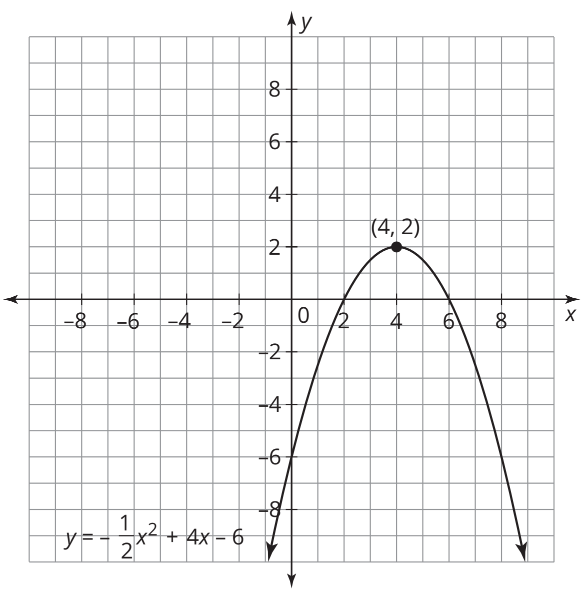
A

absolute maximum

A function has an absolute maximum if there is a point that has a *y*-coordinate that is greater than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (4, 2) is the absolute maximum of the graph of the function $f(x) = -\frac{1}{2}x^2 + 4x - 6$.

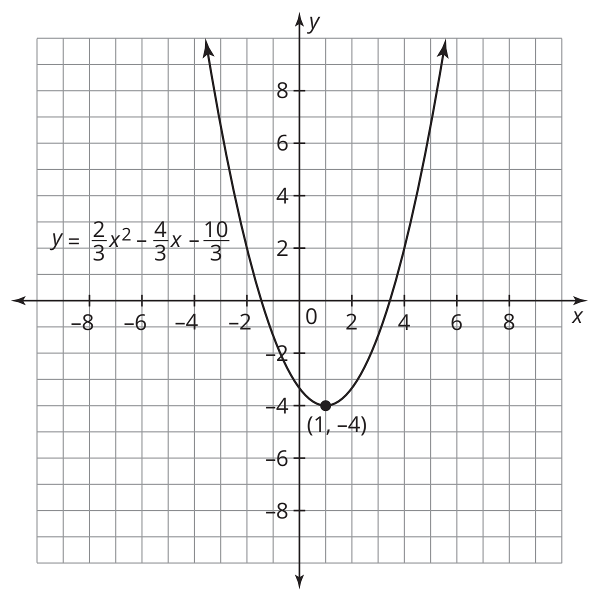


absolute minimum

A function has an absolute minimum if there is a point that has a *y*-coordinate that is less than the *y*-coordinates of every other point on the graph.

Example

The ordered pair (1, −4) is the absolute minimum of the graph of the function $y = \frac{2}{3}x^2 - \frac{4}{3}x - \frac{10}{3}$.



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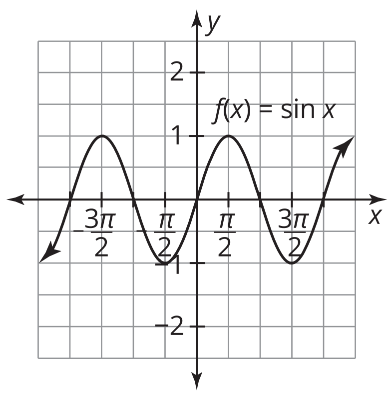
amplitude

The amplitude of a periodic function is half of the distance between the maximum and minimum values of the function.

Example

The function $y = \sin x$ has a maximum of 1 and a minimum of -1. The distance between the maximum and minimum is 2. So, the amplitude of $y = \sin x$ is 1.



average rate of change

The average rate of change of a function is the ratio of the independent variable to the dependent variable over a specific interval. The formula for average rate of change is $\frac{f(b) - f(a)}{b - a}$ for an interval (a, b). The expression $a - b$ represents the change in the input of the function f. The expression $f(b) - f(a)$ represents the change in the function f as the input changes from a to b.

Example

Consider the function $f(x) = x^2$.

The average rate of change of the interval (1, 3) is $\frac{3^2 - 1^2}{3 - 1} = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4$.

B

biased sample

A biased sample is a sample that does not accurately represent all of a population.

Example

A survey is conducted asking students their favorite class. Only students in the math club are surveyed. The sample of students is a biased sample.

Binomial Theorem

The Binomial Theorem states that it is possible to extend any power of $(a + b)$ into a sum of the form shown.

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Example

Use the Binomial Theorem to find the third term of $(x + y)^{20}$.

$$(x + y)^{20} = \binom{20}{2} x^{20-2} y^2 = \frac{20!}{18!2!} x^{18} y^2$$

$$= \frac{20 \cdot 19}{2 \cdot 1} x^{18} y^2 = 190\, x^{18} y^2$$

C

Change of Base Formula

The Change of Base Formula allows you to calculate an exact value for a logarithm by rewriting it in terms of a different base. It is especially helpful when using a calculator.

The Change of Base Formula states:

$\log_b(c) = \frac{\log_a(c)}{\log_a(b)}$, where $a, b, c > 0$ and $a, b \neq 1$.

Example

$$\log_4(50) = \frac{\log 50}{\log 4} \approx 2.821928095$$

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characteristic of interest

A characteristic of interest is the specific question that you are trying to answer or specific information that a study is trying to gather.

Example

In a sample survey to determine teenagers' online habits, a characteristic of interest is the amount of time that a teenager spends online per day.

closed under an operation

A set is closed under an operation if the operation is performed on any of the numbers in the set and the result is a number that is also in the same set.

Example

The set of whole numbers is closed under addition. The sum of any two whole numbers is always another whole number.

cluster

A cluster is a sample of the population that contains the characteristics of the population.

Example

A city manager randomly selects one block in the city and surveys all of the residents of that block. Each block is considered a cluster.

cluster sample

A cluster sample is a sample obtained by creating clusters, with each cluster containing the characteristics of the population, and randomly selecting a cluster.

Example

If students in a high school are divided into clusters of 20 students based on their student I.D. number and then one cluster is randomly selected, this is a cluster sample.

coefficient of determination

The coefficient of determination (R^2) measures the "strength" of the relationship between the original data and its regression equation. The value of the coefficient of determination ranges from 0 to 1 with a value of 1 indicating a perfect fit between the regression equation and the original data.

common logarithm

A common logarithm is a logarithm with a base of 10. Common logarithms are usually written without a base.

Example

$\log(10x)$ or $\log x$ are examples of a common logarithm.

complex numbers

The set of complex numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers. The set of complex numbers consists of the set of imaginary numbers and the set of real numbers.

Example

The numbers $1 + 2i$, 7, and $-3i$ are complex numbers.

composition of functions

Composition of functions is the process of substituting one function for the variable in another function.

Example

If $f(x) = 3x - 5$ and $g(x) = x^2$, then the composition of the functions $f(g(x))$ can be written as $f(g(x)) = 3(x^2) - 5 = 3x^2 - 5$.

The composition of functions $g(f(x))$ can be written as $g(f(x)) = (3x + 5)^2$.

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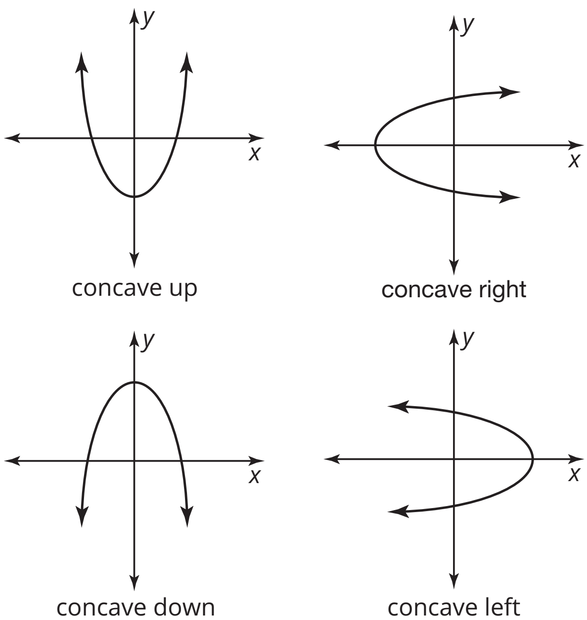
concavity of a parabola

The concavity of a parabola describes the orientation of the curvature of the parabola.

Example



confidence interval

A confidence interval is an estimated range of values, based on the results of a sample survey, that will likely include the population proportion. Typically, a confidence interval of 95%, or 2 standard deviations from the mean, is used. The formula for calculating the confidence interval for proportions is $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, where $\hat{p}$ is the sample population and n is the sample size. The formula $\frac{s}{\sqrt{n}}$, where s is the standard deviation of the sample and n is the sample size, is used for continuous data.

Example

A survey of 2000 teenagers reports that 42% have a part-time job.

$$\sqrt{\frac{0.42(1-0.42)}{2000}}$$

$$\sqrt{\frac{0.42(0.58)}{2000}}$$

$$\approx 0.011$$

The interval from 40.9% to 43.1% represents a 95% confidence interval for the population proportion.

confounding

Confounding is the process of overlooking factors and situations that distort the final results when seeking to gather information or data.

Example

Suppose that a study is conducted to determine if there is a link between a certain type of insulin that some diabetic patients use and cancer. Confounding can occur due to the fact that there are other potential causes of cancer that could be involved in the sample.

continuous data

Continuous data are data that have an infinite number of possible values.

Example

The heights of students is an example of continuous data.

convenience sample

A convenience sample is a sample whose data are based on what is convenient for the person choosing the sample.

Example

If you choose the students sitting closest to you in math class as your sample, you have a convenience sample.

cosine function

The cosine function is a periodic function. It takes angle measures (θ values) as inputs and then outputs real number values which correspond to the coordinates of points on the unit circle.

Example

The function $h(\theta) = 4\cos(\theta + \pi)$ is a cosine function.

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cube root function

The cube root function is the inverse of the power function $f(x) = x^3$.

Example

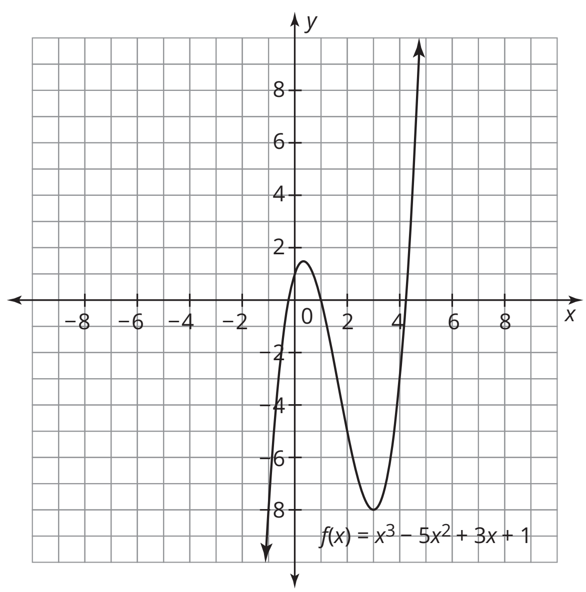
The cube root function is $g(x) = \sqrt[3]{x}$.

cubic function

A cubic function is a function that can be written in the standard form $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$.

Example

The function $f(x) = x^3 - 5x^2 + 3x + 1$ is a cubic function.



D

damping function

A damping function is a function that is multiplied to a periodic function to decrease its amplitude over time. It can be from a multitude of function families, including linear, quadratic, or exponential.

Example

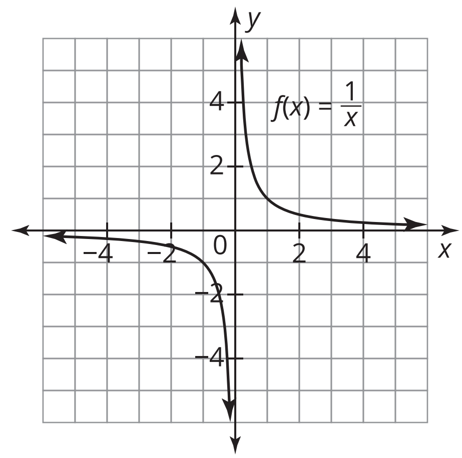
In the function $f(x) = 2^x \cdot \sin x + 1$, the exponential function 2^x is the damping function.

discontinuous function

A discontinuous function is a function that is not a continuous curve—it has points that are isolated from each other.

Example

The function $f(x) = \frac{1}{x}$ is a discontinuous function.



discrete data

Discrete data are data that have a finite number of possible values.

Example

If you roll a number cube 10 times and record the results, the results are discrete data.

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E

Empirical Rule for Normal Distributions

The Empirical Rule for Normal Distributions states that:

- Approximately 68% of the area under the normal curve is within one standard deviation of the mean.
- Approximately 95% of the area under the normal curve is within two standard deviations of the mean.
- Approximately 99.7% of the area under the normal curve is within three standard deviations of the mean.

Example

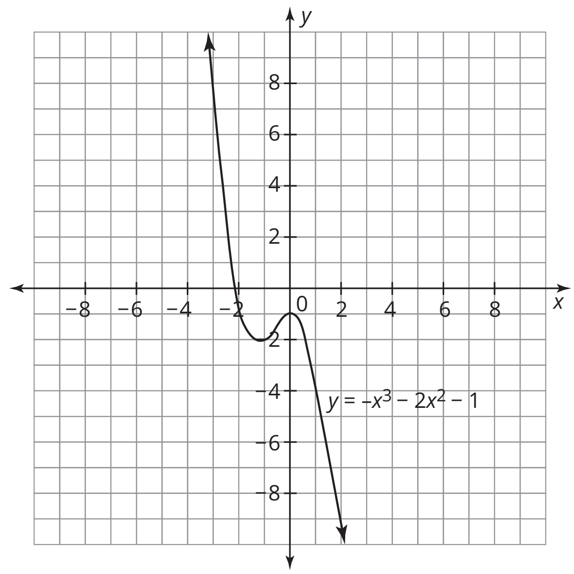
For a data set that is normally distributed with a mean of 10 and a standard deviation of 1, the following are true:

- Approximately 68% of the data values are between 9 and 11.
- Approximately 95% of the data values are between 8 and 12.
- Approximately 99.7% of the data values are between 7 and 13.

end behavior

The end behavior of the graph of a function is the behavior of the graph as x approaches infinity and as x approaches negative infinity.

Example



The end behavior of the graph shown can be described as follows:

As x approaches infinity, y approaches negative infinity.

As x approaches negative infinity, y approaches infinity.

Euclid's Formula

Euclid's Formula is a formula used to generate Pythagorean triples given any two positive integers. Given positive integers r and s, where $r > s$, Euclid's Formula is $(r^2 + s^2)^2 = (r^2 - s^2)^2 + (2rs)^2$.

Example

Let $r = 3$ and $s = 1$.

$(3^2 + 1^2)^2 = (3^2 - 1^2)^2 + (2 \cdot 3 \cdot 1)^2$

$10^2 = 8^2 + 6^2$

So, one Pythagorean triple is 6, 8, 10.

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even function

An even function f is a function for which $f(-x) = f(x)$ for all values of x in the domain.

Example

The function $f(x) = x^2$ is an even function because $(-x)^2 = x^2$.

experiment

An experiment gathers data on the effect of one or more treatments, or experimental conditions, on the characteristic of interest.

Example

The following is an example of an experiment.

A sample of 200 asthma patients participated in the clinical trial for a new asthma drug. One hundred of the patients received a placebo treatment along with an inhaler, while the remaining 100 patients received the new drug along with an inhaler. Monthly blood and breathing tests were performed on all 200 patients to determine if the new drug was effective.

experimental unit

An experimental unit is a member of a sample in an experiment.

Example

Suppose that an experiment is conducted to test the effects of a new drug on a sample of patients. Each patient is an experimental unit in the experiment.

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extraneous solution

Extraneous solutions are solutions that result from the process of solving an equation; but are not valid solutions to the equation.

Example

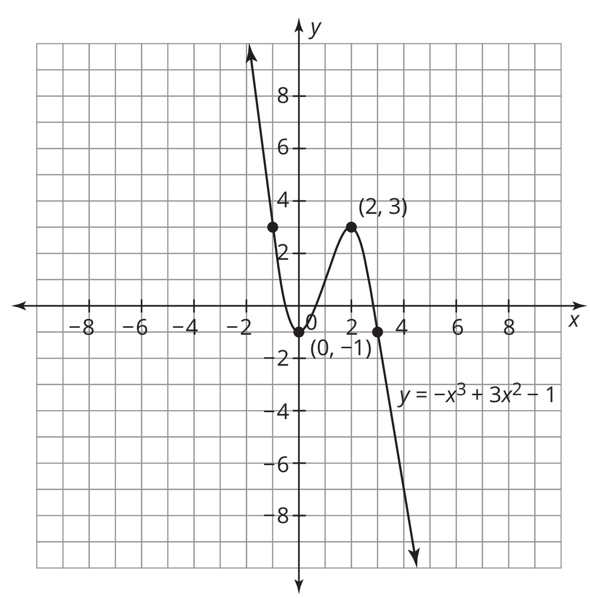
$$\log_2 (x) + \log_2 (x + 7) = 3$$
$$\log_2 (x^2 + 7x) = 3$$
$$x^2 + 7x = 2^3$$
$$x^2 + 7x = 8$$
$$x^2 + 7x - 8 = 0$$
$$(x + 8)(x - 1) = 0$$
$$x + 8 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = -8 \qquad\qquad x = 1$$

The solution $x = -8$ is an extraneous solution because the argument of a logarithm must be greater than zero.

extrema

Extrema are the set of all relative maximums, relative minimums, absolute maximums, and absolute minimums for a graph.

Example



The graph shown has 2 extrema, a relative maximum at (2, 3) and a relative minimum at (0, −1).

F

Factor Theorem

The Factor Theorem states that a polynomial is divisible by $(x - r)$ if the value of the polynomial at r is zero.

Example

The polynomial $x^3 - 2x^2 + 2x - 1$ is divisible by $x - 1$ because $(1)^3 - 2(1)^2 + 2(1) - 1 = 0$.

factored form of a quadratic function

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.

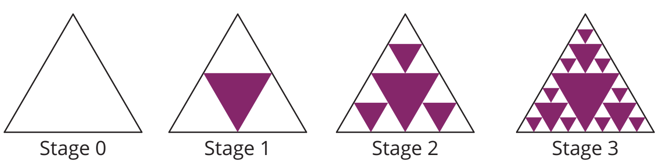
Example

The function $h(x) = x^2 - 8x + 12$ written in factored form is $h(x) = (x - 6)(x - 2)$.

fractal

A fractal is a complex geometric shape that is constructed by a mathematical pattern. Fractals are infinite and self-similar.

Example



frequency

The frequency of a periodic function is the reciprocal of the period and specifies the number of repetitions of the graph of a periodic function per unit. It is calculated by the formula $\frac{|B|}{2\pi}$.

Example

The function $f(x) = 3 \cos (2x)$ has a B-value of 2, so the frequency is $\frac{|2|}{2\pi}$ or $\frac{1}{\pi}$ units.

function

A function is a relation such that for each element of the domain there exists exactly one element in the range.

Example

The equation $y = 2x$ is a function. Every x-value has exactly one corresponding y-value.

function notation

Function notation is a way of representing functions algebraically. The function $f(x)$ is read as "f of x" and indicates that x is the input and $f(x)$ is the output.

Example

The function $f(x) = 0.75x$ is written using function notation.

Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that any polynomial equation of degree n must have exactly n complex roots or solutions; also, every polynomial function of degree n must have exactly n complex zeros. However, any root or zero may be a multiple root or zero.

Example

The polynomial equation $x^5 + x^2 - 6 = 0$ has 5 complex roots because the polynomial $x^5 + x^2 - 6$ has a degree of 5.

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G

geometric series

A geometric series is the sum of the terms of a geometric sequence.

Example

The geometric series corresponding to the geometric sequence 2, 4, 8, 16 is 2 + 4 + 8 + 16, or 30.

H

half-life

A half-life is the amount of time it takes a substance to decay to half of its original amount.

Example

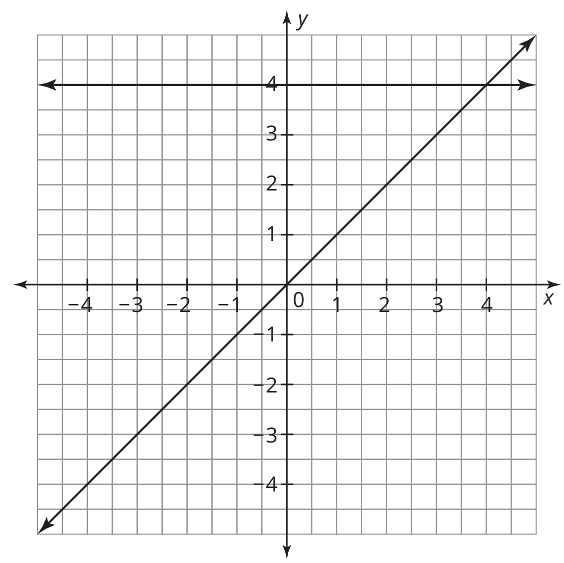
The radioactive isotope strontium-90 has a half-life of about 30 years. A 1000-gram sample of strontium-90 will decay to 500 grams in 30 years.

Horizontal Line Test

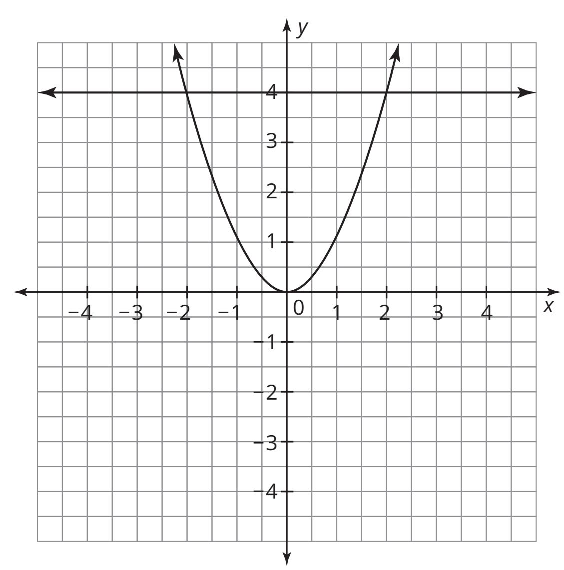
The Horizontal Line Test is a test to determine if a function is one to one. To use the test, imagine drawing every possible horizontal line on the coordinate plane. If no horizontal line intersects the graph of a function at more than one point, then the function is one to one.

Example

The function $y = x$ passes the Horizontal Line Test because no horizontal line can be drawn that intersects the graph at more than one point. So, the function is one to one.



The function $y = x^2$ does not pass the Horizontal Line Test because a horizontal line can be drawn that intersects the graph at more than one point. So, the function is not one to one.



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I

imaginary numbers

The set of imaginary numbers is the set of all numbers written in the form $a + bi$, where a and b are real numbers and b is not equal to 0.

Example

The numbers $2 - 3i$ and $5i$ are imaginary numbers. The number 6 is not an imaginary number.

imaginary part of a complex number

In a complex number of the form $a + bi$, the term bi is called the imaginary part of a complex number.

Example

The imaginary part of the complex number $3 + 2i$ is $2i$.

imaginary roots (imaginary zeros)

Equations and functions that have imaginary solutions requiring i have imaginary roots or imaginary zeros.

Example

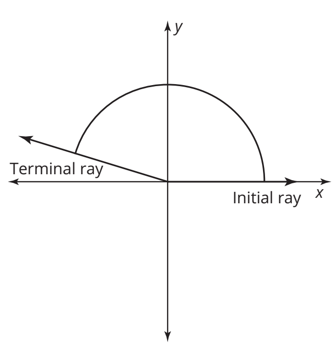
The quadratic equation $x^2 - 2x + 2 = 0$ has two imaginary roots: $1 + i$ and $1 - i$.

initial ray of an angle

The initial ray of an angle in standard position is the ray with its endpoint at the origin and extending along the positive x-axis.

Example

The initial ray of the angle is labeled in the diagram.



inverse cosine ($\cos^{-1}$)

The $\cos^{-1}$ function is the inverse of the cosine function. The inverse cosine function is written as arccos or $\cos^{-1}$.

Example

$\cos(60°) = \frac{1}{2}$ so $\cos^{-1}\left(\frac{1}{2}\right) = 60°$

inverse of a function

The inverse of a one-to-one function is a function that results from exchanging the independent and dependent variables. A function $f(x)$ with coordinates $(x, f(x))$ will have an inverse with coordinates $(f(x), x)$.

Example

The inverse of the function $y = 2x$ can be found by exchanging the variables x and y.

The inverse of $y = 2x$ is $x = 2y$.

inverse sine ($\sin^{-1}$)

The $\sin^{-1}$ function is the inverse of the sine function. The inverse sine function is written as arcsin or $\sin^{-1}$.

Example

$\sin(30°) = \frac{1}{2}$ so $\sin^{-1}\left(\frac{1}{2}\right) = 30°$

inverse tangent ($\tan^{-1}$)

The $\tan^{-1}$ function is the inverse of the tangent function. The inverse tangent function is written as arctan or $\tan^{-1}$.

Example

$\tan(45°) = 1$ so $\tan^{-1}(1) = 45°$

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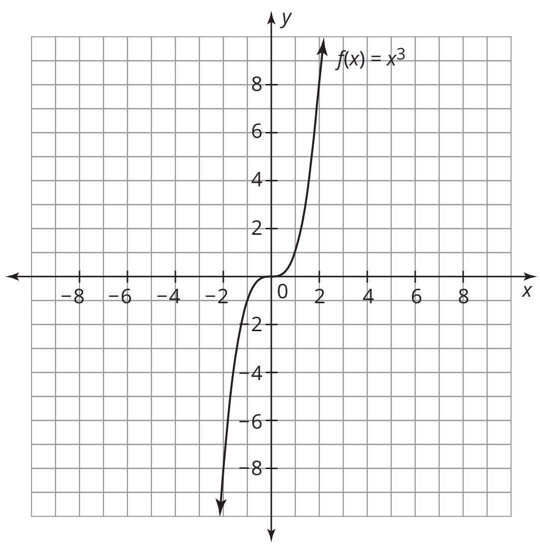
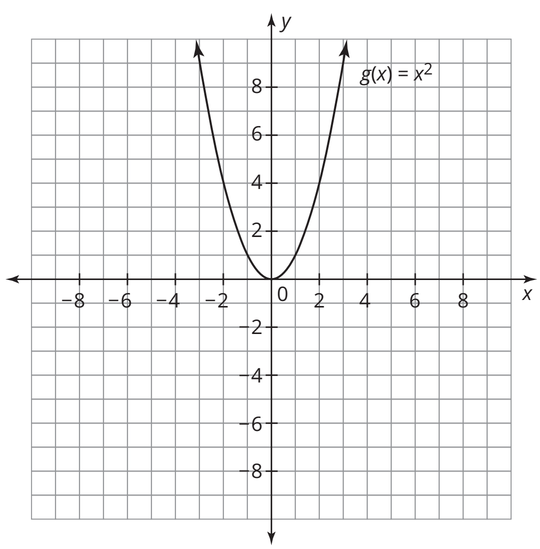
invertible function

An invertible function is a function whose inverse exists. It is one-to-one and passes the Horizontal Line Test, so its inverse will also be a function.

Example





The graph of $f(x) = x^3$ is an invertible function because it is one-to-one and passes the Horizontal Line Test. Therefore its inverse will also be a function.

The graph of $g(x) = x^2$ is not an invertible function because it does not pass the Horizontal Line Test. Its inverse does not exist.

iterative process

An iterative process is one in which the output from one iteration is used as the input for the next iteration.

Example

A recursive sequence uses an iterative process to generate its terms.

$a_n = 3a_{n-1} + 1$

$a_1 = 2$

Begin with the first term, which is 2, and substitute it into the sequence to get the next term.

$a_2 = 3a_1 + 1$

$= 3(2) + 1$

$= 7$

Then substitute a_2 into the sequence to produce a_3, and so on.

L

logarithm

The logarithm of a positive number is the exponent to which the base must be raised to result in that number.

Example

Because $10^2 = 100$, the logarithm of 100 to the base 10 is 2.

$\log 100 = 2$

Because $2^3 = 8$, the logarithm of 8 to the base 2 is 3.

$\log_2 (8) = 3$

logarithm with same base and argument

The logarithm of a number, with the base equal to the same number, is always equal to 1.

$$\log_b (b) = 1$$

Example

$\log_4 (4) = 1$

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logarithmic equation

A logarithmic equation is an equation that contains a logarithm.

Example

The equation $\log_2(x) = 4$ is a logarithmic equation.

logarithmic function

A logarithmic function is a function involving a logarithm.

Example

The function $f(x) = 3 \log x$ is a logarithmic function.

M

margin of error

The margin of error expresses the maximum expected difference between the true population data and the sample estimate of the data.

Example

In a poll of 1100 registered voters, 54% said they would vote to re-elect the current mayor. The margin of error for the poll is ± 3%, which means that somewhere between 51% (54% − 3%) and 57% (54% + 3%) will actually vote to re-elect the current mayor.

mean (μ)

The mean of a data set is the sum of all of the values of the data set divided by the number of values in the data set. The mean is also called the average.

Example

The mean of the numbers 7, 9, 13, 4, and 7 is $\frac{7 + 9 + 13 + 4 + 7}{5}$, or 8.

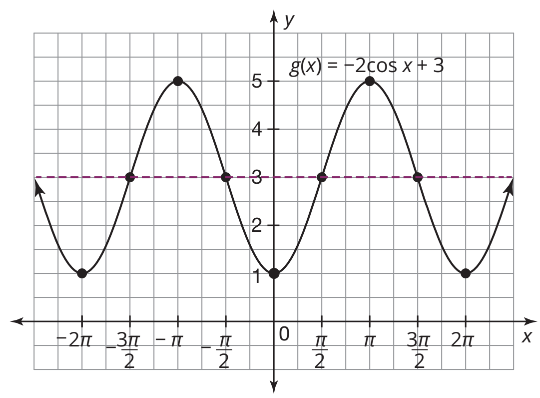
The mean of a set of normally distributed data is aligned with the peak of the normal curve.

midline

The midline of a periodic function is a reference line whose equation is the average of the minimum and maximum values of the function.

Example

In the graph of $g(x) = -2 \cos x + 3$ the midline occurs at $y = 3$ because the maximum value is 5 and the minimum value is 1.



multiplicity

Multiplicity is how many times a particular number is a zero for a given function.

Example

The equation $x^2 + 2x + 1 = 0$ has a double root at $x = -1$. The root -1 has a multiplicity of 2.

$x^2 + 2x + 1 = 0$

$(x + 1)(x + 1) = 0$

$x + 1 = 0$ or $x + 1 = 0$

$x = -1$ or $x = -1$

N

natural base *e*

The natural base e is an irrational number equal to approximately 2.71828.

Example

$e^2 \approx 2.7183^2 \approx 7.3892$

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natural logarithm

A natural logarithm is a logarithm with a base of e. Natural logarithms are usually written as ln.

Example

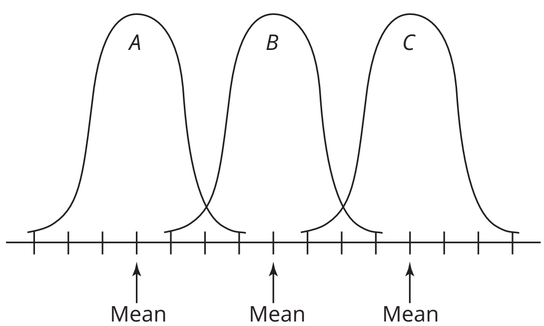
$\log_e(x)$ or $\ln x$ is a natural logarithm.

normal curve

A normal curve is a curve that is bell-shaped and symmetric about the mean.

Example

A normal curve is shown.



normal distribution

A normal distribution, or normal probability distribution, describes a continuous data set that can be modeled using a normal curve.

Example

Adult IQ scores, gas mileage of certain cars, and SAT scores are all continuous data that follow a normal distribution.

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O

observational study

An observational study gathers data about a characteristic of the population without trying to influence the data.

Example

The following is an example of an observational study. New research funded by a pediatric agency found that nearly 70% of in-house day care centers show as much as 2.5 hours of television to the children in the center per day. The study examined 132 day care programs in 2 Midwestern states.

odd function

An odd function f is a function for which $f(-x) = -f(x)$ for all values of x in the domain.

Example

The function $f(x) = x^3$ is an odd function because $(-x)^3 = -x^3$.

P

parameter

When data are gathered from a population, the characteristic used to describe the population is called a parameter.

Example

If you wanted to find out the average height of the students at your school, and you measured every student at the school, the characteristic "average height" would be a parameter.

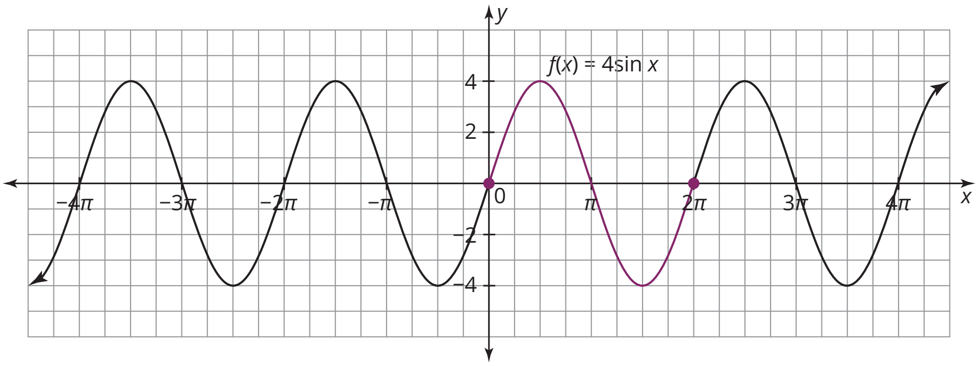
percentile

A percentile is a data value for which a certain percent of the data is below the data value in a normal distribution.

period

A period of a periodic function is the length of the smallest interval over which the function repeats.

Example

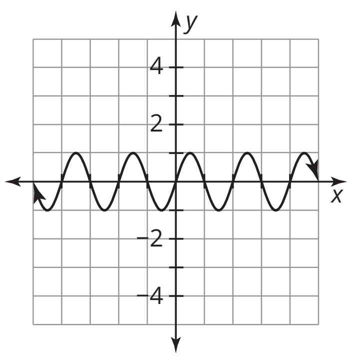


periodic function

A periodic function is a function whose graph consists of repeated instances of a portion of the graph.

Example

The function $f(x) = \sin x$ is a periodic function. The portion of the graph between $x = 0$ and $x = 2\pi$ repeats.



periodicity identity

A periodicity identity is a trigonometric identity based on the period of the trigonometric functions.

Example

The six periodicity identities are:

$\sin(x + 2\pi) = \sin x$; $\cos(x + 2\pi) = \cos x$

$\sec(x + 2\pi) = \sec x$; $\csc(x + 2\pi) = \csc x$

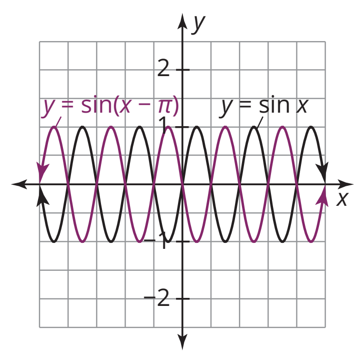
$\tan(x + \pi) = \tan x$; $\cot(x + \pi) = \cot x$

phase shift

A phase shift of a periodic function is a horizontal translation.

Example

The function $y = \sin(x - \pi)$ has a phase shift of π units from the basic function $y = \sin x$.



polynomial function

A polynomial function is a function that can be written in the form

$p(x) = \square x^n + \square x^{n-1} + \cdots + \square x^2 + \square x + \square$,

where the coefficients, represented by each $\square$, are complex numbers and the exponents are nonnegative integers.

Example

The function $f(x) = 5x^3 + 3x^2 + x + 1$ is a polynomial function.

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polynomial long division

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree.

Example

$$
\begin{array}{r}
4x^2 - 6x + 3 \\
2x + 3 \overline{\smash{)}\, 8x^3 + 0x^2 - 12x - 7} \\
\underline{-(8x^3 + 12x^2)} \qquad\quad \\
-12x^2 - 12x \quad\;\; \\
\underline{-(-12x^2 - 18x)} \quad\;\; \\
6x - 7 \\
\underline{-(6x + 9)} \\
\text{Remainder } -16
\end{array}
$$

population

The population is the entire set of items from which data can be selected. When you decide what you want to study, the population is the set of all elements in which you are interested. The elements of that population can be people or objects.

Example

If you wanted to find out the average height of the students at your school, the number of students at the school would be the population.

population proportion

A population proportion is the percentage of an entire population that yields a favorable outcome in an experiment.

Example

In an election, the population proportion represents the percentage of people in the entire town who vote to re-elect the mayor.

power function

A power function is a function of the form $P(x) = ax^n$ where n is a non-negative integer.

Example

The functions $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$ are power functions.

Power Rule of Logarithms

The Power Rule of Logarithms states that the logarithm of a power is equal to the product of the exponent and the logarithm of the base of the power.

$$\log_b (x)^n = n \cdot \log_b (x)$$

Example

$\ln (x)^2 = 2 \ln x$

Product Rule of Logarithms

The Product Rule of Logarithms states that the logarithm of a product is equal to the sum of the logarithms of the factors.

$$\log_b (xy) = \log_b (x) + \log_b (y)$$

Example

$\log (5x) = \log 5 + \log x$

pure imaginary number

A pure imaginary number is a number of the form bi, where b is not equal to 0.

Example

The imaginary numbers $-4i$ and $15i$ are pure imaginary numbers.

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Pythagorean identity

A Pythagorean identity is a trigonometric identity based on the Pythagorean Theorem.

Example

The three Pythagorean identities are:

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Q

Quadratic Formula

The Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c$, where a, b, and c represent real numbers and $a \neq 0$.

quartic function

A quartic function is a polynomial function with a degree of four.

Examples

The function $f(x) = 3x^4 - 2x + 5$ is a quartic function.

quintic function

A quintic function is a polynomial function with a degree of five.

Examples

The function $f(x) = 5x^5 + 3x^4 + x^3$ is a quintic function.

Quotient Rule of Logarithms

The Quotient Rule of Logarithms states that the logarithm of a quotient is equal to the difference of the logarithms of the dividend and the divisor.

$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

Examples

$\log\left(\frac{x}{2}\right) = \log x - \log 2$

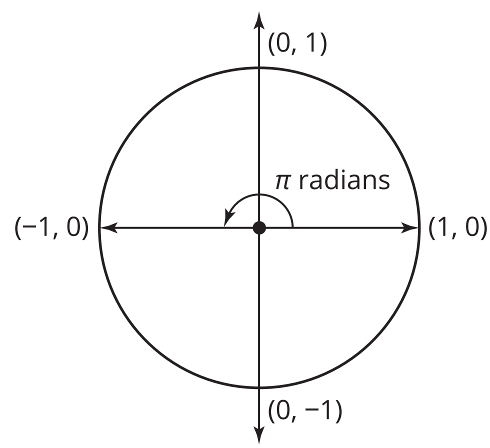
R

radians

A radian is a unit of measurement for an angle in standard position. The ratio of the intercepted arc length of a central angle to the length of the radius is the measure of the central angle in radians.

Example

The angle shown has a radian measure of π radians.



radical function

A radical function is a function that contains one or more radical expressions.

Example

The function $f(x) = \sqrt{3x + 5}$ is a radical function.

random sample

A random sample is a method of collecting data in which every member of a population has an equal chance of being selected.

Example

Choosing 100 fans at random to participate in a survey from a crowd of 5000 people is an example of a random sample.

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rational equation

A rational equation is an equation that contains one or more rational expressions.

Example

The equation $\frac{1}{x-1} + \frac{1}{x+1} = 4$ is a rational equation.

rational function

A rational function is any function that can be written as the ratio of two polynomial functions. A rational function can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions, and $Q(x) \neq 0$.

Example

The function $f(x) = \frac{1}{x-1} + \frac{1}{x+1}$ is a rational function.

real part of a complex number

In a complex number of the form $a + bi$, the term a is called the real part of a complex number.

Example

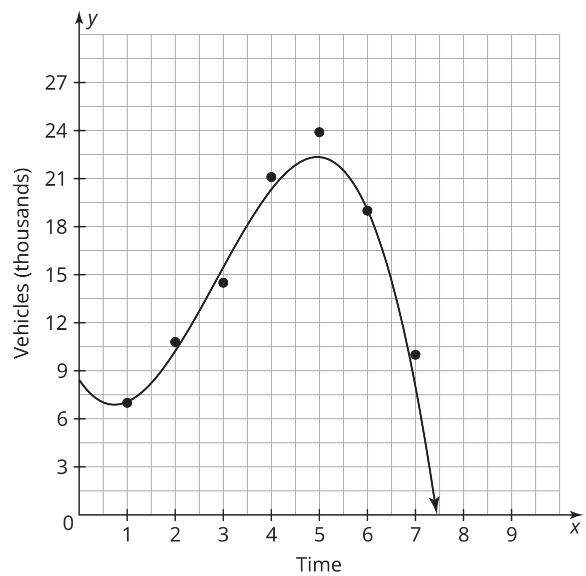
The real part of the complex number $3 + 2i$ is 3.

regression equation

A regression equation is a function that models the relationship between two variables in a scatter plot.

Example

The regression equation $y = -0.41x^3 + 3.50x^2 - 4.47x + 8.44$ models the relationship between time and the number of vehicles.



relation

A relation is the mapping between a set of input values called the domain and a set of output values called the range.

Example

The set of points {(0, 1), (1, 8), (2, 5), (3, 7)} is a relation.

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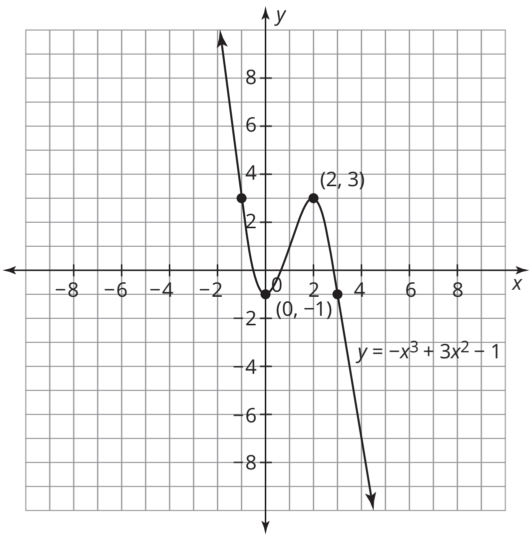
relative maximum

A relative maximum is the highest point in a particular section of a graph.

Example

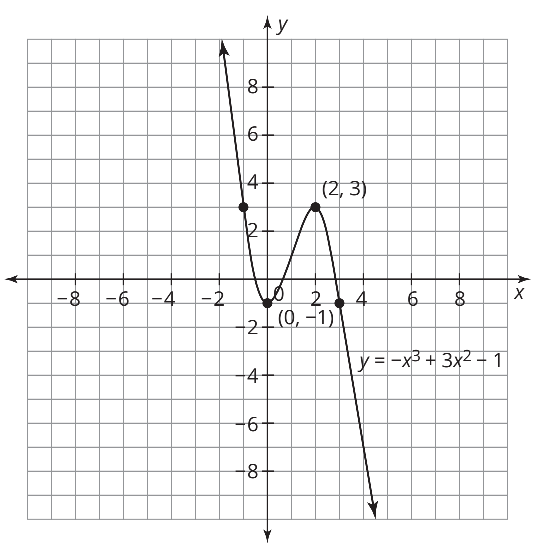


The graph shown has a relative maximum at (2, 3).

relative minimum

A relative minimum is the lowest point in a particular section of a graph.

Example



The graph shown has a relative minimum at (0, −1).

Remainder Theorem

The Remainder Theorem states that the remainder when dividing a polynomial by $(x - r)$ is the value of the polynomial at r.

Example

The value of the polynomial $x^2 + 5x + 2$ at 1 is $(1)^2 + 5(1) + 2 = 8$. So, the remainder when $x^2 + 5x + 2$ is divided by $x - 1$ is 8.

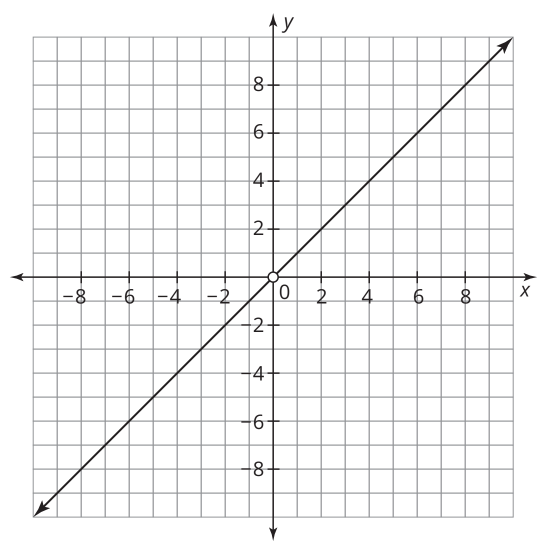
$$\begin{array}{r} x + 6 \\ x - 1\overline{)x^2 + 5x + 2} \\ \underline{x^2 - x} \\ 6x + 2 \\ \underline{6x - 6} \\ 8 \end{array}$$

removable discontinuity

A removable discontinuity is a single point at which the graph of a function is not defined.

Example

The graph of the function $f(x) = \frac{x^2}{x}$ has a removable discontinuity at $x = 0$.



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S

sample

Where data are collected from a selection of the population, the data are called a sample.

Example

If you wanted to find out the average height of the students in your school, you could choose just a certain number of students and measure their heights. The heights of the students in this group would be the sample.

sample proportion

A sample proportion is the percentage of a sample that yields a favorable outcome in an experiment. This is often used to make predictions about a population.

Example

In an election, a sample of townspeople is surveyed. The sample proportion represents the percentage of the survey results that indicate that they will vote to re-elect the mayor.

sample survey

A sample survey poses a question of interest to a random sample of the targeted population.

Example

The following is an example of a sample survey.

A recent survey of nearly 1200 young people from across the U.S. shows that 40% of 16- to 20-year-olds who have a driver's license admit to texting on a regular basis while they are driving.

sampling distribution

A sampling distribution consists of every possible sample of equal size from a given population. A sampling distribution provides an estimate for population parameters. The mean or proportion of a sampling distribution is estimated by the mean or proportion of a sample. For categorical data, the standard deviation of a sampling distribution is estimated by calculating $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ where $\hat{p}$ (p-hat) is the sample proportion and n is the sample size. For continuous data, the standard deviation of a sampling distribution is estimated by calculating $\frac{s}{\sqrt{n}}$ where s is the standard deviation of the original sample and n is the sample size.

Example

A sleep survey of 50 teens resulted in a sample mean of 7.7 hours and sample standard deviation of 0.8 hours.

The estimated mean of the sampling distribution is 7.7 hours. The estimated standard deviation of the sampling distribution is approximately 0.11 hours.

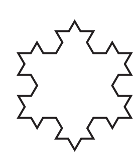
$$\frac{s}{\sqrt{n}} = \frac{0.8}{\sqrt{50}} \approx 0.11$$

self-similar

A self-similar object is exactly or approximately similar to a part of itself.

Example

A Koch snowflake is considered to be self-similar.



simple random sample

A simple random sample is a sample in which every member of the population has the same chance of being selected.

Example

Using a random number generator to select a sample is an example of simple random sampling.

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sine function

The sine function is a periodic function. It takes angle measures (θ values) as inputs and then outputs real number values which correspond to the coordinates of points on the unit circle.

Example

The function $h(\theta) = -\sin(2\theta) + 1$ is a sine function.

square root function

The square root function is the inverse of the power function $f(x) = x^2$ when the domain is restricted to $x \geq 0$.

Example

The square root function is $g(x) = \sqrt{x}$.

standard deviation (σ)

Standard deviation is a measure of the variation of the values in a data set from the mean of the data. A lower standard deviation represents data that are more tightly clustered near the mean. A higher standard deviation represents data that are more spread out from the mean. Use the formula below to calculate standard deviation.

$$\text{standard deviation} = \sqrt{\frac{\sum_{i=1}^{n}(x_1 - \overline{x})^2}{n}}$$

where $\overline{x}$ is the mean and n is the number of data values in the data set $\{x_1, x_2, \ldots, x_n\}$.

Example

In the data set of test scores 60, 70, 80, 90, 100, the mean $\overline{x}$ is 80 and the number of data elements n is 5.

$$\sigma = \sqrt{\frac{(60-80)^2 + (70-80)^2 + (80-80)^2 + (90-80)^2 + (100-80)^2}{5}}$$

$$= \sqrt{\frac{1000}{5}}$$

$$= \sqrt{200}$$

$$\approx 14.14.$$

standard form (general form) of a quadratic function

A quadratic function written in the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is in standard form, or general form.

Example

The function $f(x) = -5x^2 - 10x + 1$ is written in standard form.

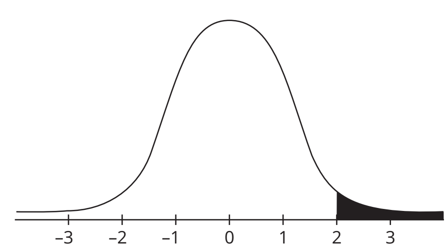
standard normal distribution

The standard normal distribution is a normal probability distribution with the following properties:

- The mean is equal to 0.
- The standard deviation is 1.
- The curve is bell-shaped and symmetric about the mean.

Example

A standard normal distribution curve is shown.

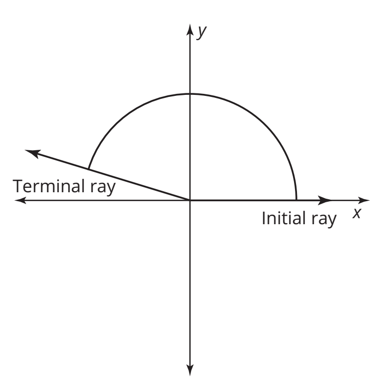


standard position of an angle

The standard position of an angle occurs when the vertex of the angle is at the origin and one ray of the angle is on the x-axis.

Example

The angle shown is in standard position.



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statistic

When data are gathered from a sample, the characteristic used to describe the sample is called a statistic.

Example

If you wanted to find out the average height of the students in your school, and you chose just a certain number of students randomly and measured their heights, the characteristic "average height" would be called a statistic.

statistically significant

A survey that has a result that is statistically significant indicates that the result did not likely occur by chance, but is likely linked to a specific cause. Typically, a result that is more than 2 standard deviations away from the mean is considered statistically significant.

Example

A survey of 2000 teenagers reports that 42% have a part-time job. The interval from 40.9% to 43.1% represents a 95% confidence interval for the population proportion. A survey that yields a report of 50% of teenagers with a part-time job would be considered statistically significant.

stratified random sample

A stratified random sample is a sample obtained by dividing the population into different groups, or strata, according to a characteristic, and randomly selecting data from each group.

Example

If students in a high school are divided by class, and random samples are then taken from each class, the result is a stratified random sample.

subjective sample

A subjective sample is a sample that is chosen based on some criteria, rather than at random.

Example

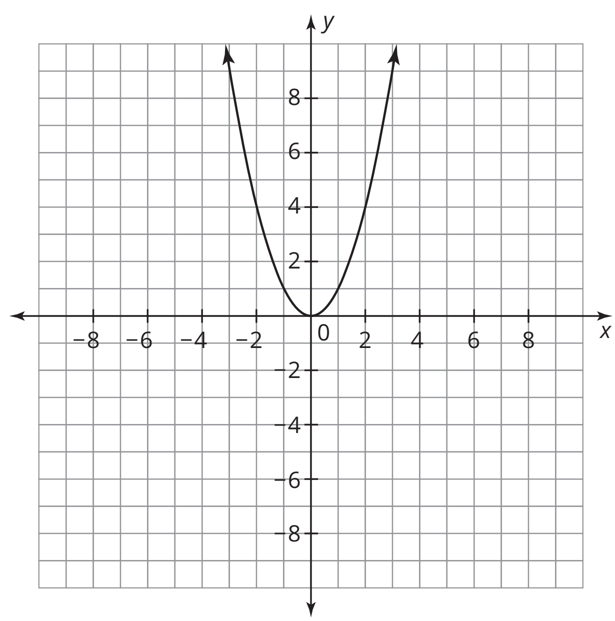
From a set of students, "choosing five students you know" is a subjective sample. In contrast, "choosing five students at random" is a random sample.

symmetric about a line

If a graph is symmetric about a line, the line divides the graph into two identical parts.

Example

The graph of $f(x) = x^2$ is symmetric about the line $x = 0$.



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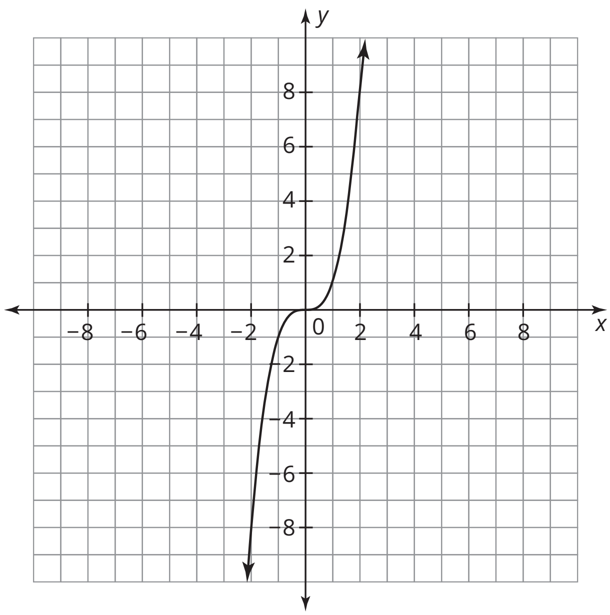
symmetric about a point

A function is symmetric about a point if each point on the graph has a point the same distance from the central point, but in the opposite direction.

Example

The graph of $f(x) = x^3$ is symmetric about the point (0, 0).



synthetic division

Synthetic division is a method for dividing a polynomial by a linear factor of the form $(x - r)$.

Example

The quotient of $2x^2 - 3x - 9$ and $x - 3$ can be calculated using synthetic division.

3	2	−3	−9
		6	9
	2	3	0

The quotient of $2x^2 - 3x - 9$ and $x - 3$ is $2x + 3$.

systematic sample

A systematic sample is a sample obtained by selecting every *n*th data in the population.

Example

If you choose every 12th student that walks into school, your sample is a systematic sample.

T

tangent function

The tangent function is a periodic function. It takes angle measures (θ values) as inputs and then outputs real number values which correspond to the coordinates of points on the unit circle.

Example

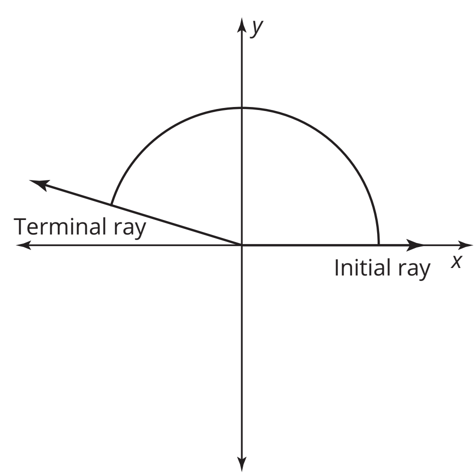
The function $f(\theta) = \tan\left(\frac{\theta}{2}\right)$ is a tangent function.

terminal ray of an angle

The terminal ray of an angle in standard position is the ray with its endpoint at the origin that is not the initial ray.

Example

The terminal ray of the angle is labeled in the diagram.



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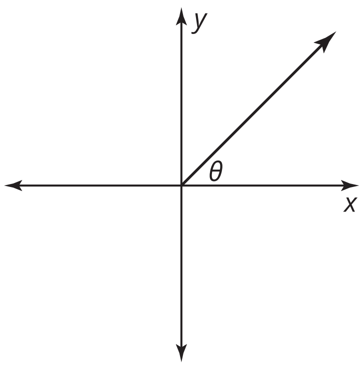
the number *i*

The number i is a number such that $i^2 = -1$.

theta (θ)

Theta is a symbol typically used to represent the measure of an angle in standard position.

Example



treatment

A treatment is a condition in an experiment.

Example

Suppose that an experiment is conducted to test the effects of a new drug on a sample of patients. The distribution of the drug to the patients is the treatment in the experiment.

trigonometric equation

A trigonometric equation is an equation that includes one or more trigonometric functions.

Example

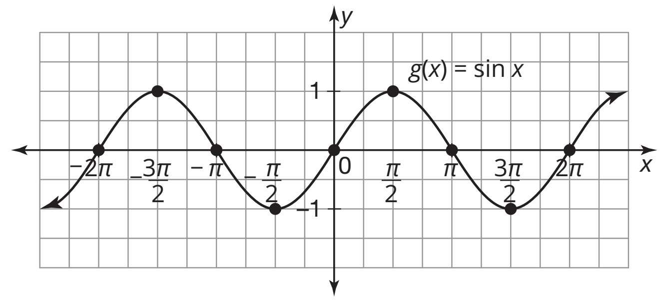
The equation $\cos x = \frac{\sqrt{2}}{2}$ is a trigonometric equation.

trigonometric function

A trigonometric function is a periodic function that takes angle measures (θ values) as inputs and then outputs real number values which correspond to the coordinates of points on the unit circle.

Example

The function $g(x) = \sin x$ is a trigonometric function. The graph of the sine function $g(\theta) = \sin \theta$ is obtained by evaluating the θ values of the unit circle and graphing the coordinates.



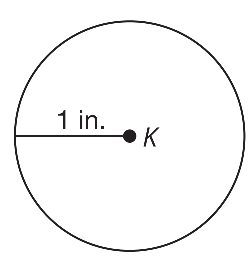
θ	$g(\theta) = \sin \theta$	$(\theta, g(\theta))$
0	$\sin(0) = 0$	$(0, 0)$
$\frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\left(\frac{\pi}{2}, 1\right)$
π	$\sin(\pi) = 0$	$(\pi, 0)$
$\frac{3\pi}{2}$	$\sin\left(\frac{3\pi}{2}\right) = -1$	$\left(\frac{3\pi}{2}, -1\right)$
2π	$\sin(2\pi) = 0$	$(2\pi, 0)$

U

unit circle

A unit circle is a circle whose radius is one unit of distance.

Example



Circle K is a unit circle.

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V

vertex form of a quadratic function

A quadratic function written in vertex form is in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$.

Example

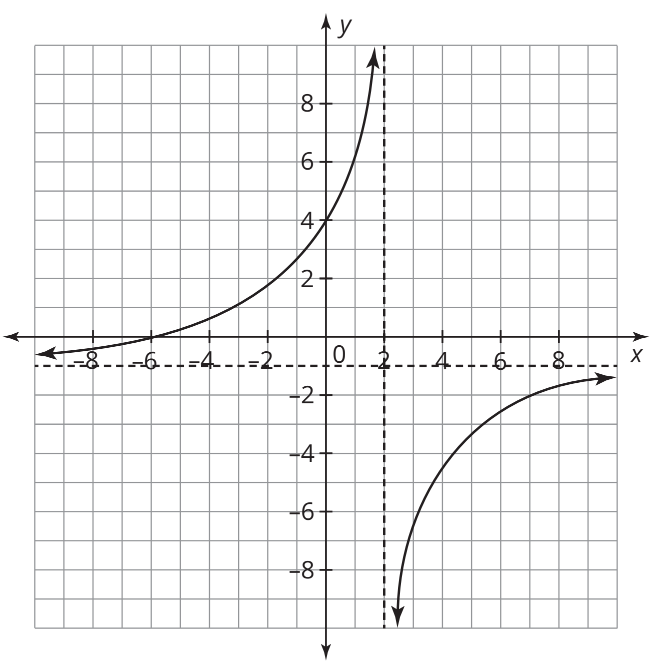
The quadratic equation $y = 2(x - 5)^2 + 10$ is written in vertex form. The vertex of the graph is the point (5, 10).

vertical asymptote

A vertical asymptote is a vertical line that a function gets closer and closer to, but never intersects. The asymptote does not represent points on the graph of the function. It represents the output value that the graph approaches.

Example

The graph has two asymptotes: a vertical asymptote $x = 2$ and a horizontal asymptote $y = -1$.



volunteer sample

A volunteer sample is a sample whose data consists of those who volunteer to be part of the sample.

Example

If you ask students in your school to complete and submit an optional survey so that you can collect data, your sample is a volunteer sample.

Z

z-score

A z-score is a number that describes how many standard deviations from the mean a particular value is. The following formula can be used to calculate a z-score for a particular value, where z represents the z-score, x represents the particular data value, μ represents the mean, and σ represents the standard deviation.

$$z = \frac{x - \mu}{\sigma}$$

Example

Suppose that a set of data follows a normal distribution with a mean of 22 and a standard deviation of 2.4.

The z-score for a data value of 25 is $z = \frac{25 - 22}{2.4} = 1.25$.

Zero Product Property

The Zero Product Property states: "If the product of two or more factors is equal to zero, then at least one factor must be equal to zero." This is also called the Converse of Multiplication Property of Zero.

Example

According to the Zero Product Property, if $(x - 2)(x + 3) = 0$ then $x - 2 = 0$ or $x + 3 = 0$.

Zero Property of Logarithms

The Zero Property of Logarithms states that the logarithm of 1, with any base, is always equal to 0.

$$\log_b (1) = 0$$

Example

$\log_3 (1) = 0$

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