CHAPTER 12

Sequences and Series

GOLDEN RECTANGLES

The Fibonacci sequence has connections to geometry, art, and architecture. Explore them by using golden rectangles.

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ARE YOU READY?

Vocabulary

Match each term on the left with a definition on the right.

1. exponential function
2. function
3. linear equation
4. quadratic function

A. a pairing in which there is exactly one output value for each input value

B. an equation whose graph is a straight line

C. a function defined by a quotient of two polynomials

D. a function of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$

E. a function of the form $f(x) = ab^x$, where $a \neq 0$ and $b \neq 1$

Simplify Radical Expressions

Simplify each expression.

5. $\sqrt{25} \cdot \sqrt{36}$
6. $\sqrt{121} - \sqrt{81}$
7. $\sqrt{\frac{1}{49}}$
8. $\frac{\sqrt{16}}{\sqrt{64}}$

Evaluate Powers

Evaluate.

9. $(-3)^3$
10. $(-5)^4$
11. $1 - (-2^3)^3$
12. $\frac{2^2 \cdot 2^7}{(2^2)^5}$

Solve for a Variable

Solve each equation for x.

13. $y = 12x - 5$
14. $y = -\frac{x}{3} + 1$
15. $y = -9 + x^2$
16. $y = -4(x^2 - 9)$

Evaluate Expressions

Evaluate each expression for $x = 2$, $y = 12$, and $z = 24$.

17. $\frac{y(y+1)}{3x}$
18. $z + (y - 1)x$
19. $y\left(\frac{x+z}{2}\right)$
20. $z\left(\frac{1-y}{1-x}\right)$

Counterexamples

Find a counterexample to show that each statement is false.

21. $n^2 = n$, where n is a real number
22. $n^3 \geq n^2 \geq n$, where n is a real number
23. $\frac{1}{n} > \frac{1}{n^2}$, where n is a real number
24. $\frac{2}{n} \neq \frac{n}{2}$, where n is a real number

CHAPTER 12

Study Guide: Preview

Where You've Been

Previously, you

- studied sets of numbers, including natural numbers and perfect squares.
- used patterns of differences or ratios to classify data.
- graphed and evaluated linear and exponential functions.

In This Chapter

You will study

- patterns of numbers, called *sequences*, and their sums, called *series*.
- patterns to determine whether sequences are arithmetic or geometric.
- how to write and evaluate sequences and series.

Where You're Going

You can use the skills learned in this chapter

- in future math classes, especially Precalculus and Calculus.
- in Physics classes to model patterns, such as the heights of bouncing objects.
- outside of school to calculate the growth of financial investments.

Key Vocabulary/Vocabulario

converge	convergir
diverge	divergir
explicit formula	fórmula explícita
finite sequence	sucesión finita
infinite sequence	sucesión infinita
iteration	iteración
limit	límite
recursive formula	fórmula recurrente
sequence	sucesión
series	serie
term of a sequence	término de una sucesión

Vocabulary Connections

To become familiar with some of the vocabulary terms in the chapter, consider the following. You may refer to the chapter, the glossary, or a dictionary if you like.

1. What does the word **sequence** mean in everyday usage? What might a number sequence refer to?
2. The word *finite* means "having definite or definable limits." Give examples of sentences that use the word *finite*. Explain what a **finite sequence** might refer to.
3. Using the previous definition of *finite*, give examples of sentences that use the word *infinite*. Explain what an **infinite sequence** might refer to.
4. What does a television series refer to? What might a mathematical **series** mean?
5. State what a term of a polynomial refers to. Then write a possible description for a **term of a sequence**.

Reading and Writing Math

Writing Strategy: Write a Convincing Argument

Being able to write a convincing argument about a math concept shows that you understand the material well. You can use a four-step method to write an effective argument as shown in the response to the exercise below.

From Lesson 11-2

35. Write About It Describe the difference between theoretical probability and experimental probability. Give an example in which they may differ.

Step 1 **Identify the goal.**

The goal is to describe the difference between theoretical and experimental probability.

Step 2 **Provide a statement of response to the goal.**

Theoretical probability is based purely on mathematics, but experimental probability is based on the results of an experiment.

Step 3 **Provide the evidence to support your statement.**

In a coin toss, the theoretical probability of tossing heads is $\frac{\text{number of favorable outcomes}}{\text{number of outcomes in the sample space}} = \frac{1}{2}$.

The experimental probability of tossing heads is $\frac{\text{number of times the event occurs}}{\text{number of trials}}$.

In one trial, the result is either heads or tails, so the experimental probability will be 1 or 0. The theoretical probability is still $\frac{1}{2}$.

Step 4 **Summarize your argument.**

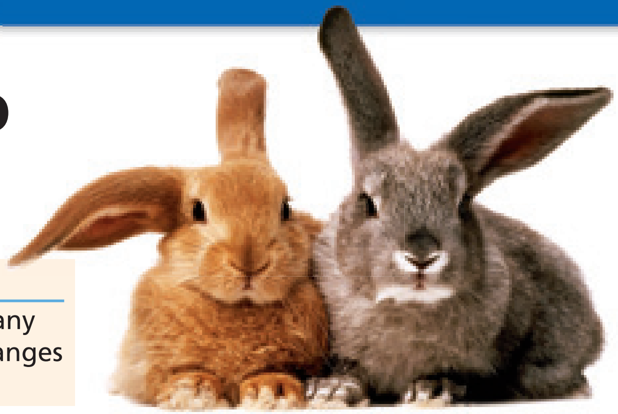
Because theoretical probability is based solely on the mathematical outcomes, it never changes. Experimental probability is based on actual results, so it may change with each trial of an experiment.

Try This

Use the four-step method described above to answer each question.

1. A number cube is rolled 20 times and lands on the number 3 twice. What is the fewest number of rolls needed for the experimental probability of rolling a 3 to equal the theoretical probability of rolling a 3? Explain how you got your answer.
2. Aidan has narrowed his college choices down to 9 schools. He plans to visit 3 or 4 schools before the end of his junior year. How many more ways can he visit a group of 4 schools than a group of 3 schools? Explain.

12-1 Introduction to Sequences



Objectives

Find the nth term of a sequence.

Write rules for sequences.

Vocabulary

sequence
term of a sequence
infinite sequence
finite sequence
recursive formula
explicit formula
iteration

Who uses this?

Sequences can be used to model many natural phenomena, such as the changes in a rabbit population over time.

In 1202, Italian mathematician Leonardo Fibonacci described how fast rabbits breed under ideal circumstances. Fibonacci noted the number of pairs of rabbits each month and formed a famous pattern called the *Fibonacci sequence*.

A **sequence** is an ordered set of numbers. Each number in the sequence is a **term of the sequence**. A sequence may be an **infinite sequence** that continues without end, such as the natural numbers, or a **finite sequence** that has a limited number of terms, such as $\{1, 2, 3, 4\}$.

Reading Math

a_n is read "a sub n."

You can think of a sequence as a function with sequential natural numbers as the domain and the terms of the sequence as the range. Values in the domain are called *term numbers* and are represented by n. Instead of function notation, such as $a(n)$, sequence values are written by using subscripts. The first term is a_1, the second term is a_2, and the nth term is a_n. Because a sequence is a function, each number n has only one term value associated with it, a_n.

Term number	n	1	2	3	4	5	Domain
Term value	a_n	1	1	2	3	5	Range

In the Fibonacci sequence, the first two terms are 1 and each term after that is the sum of the two terms before it. This can be expressed by using the rule $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$, where $n \geq 3$. This is a *recursive formula*. A **recursive formula** is a rule in which one or more previous terms are used to generate the next term.

EXAMPLE 1 Finding Terms of a Sequence by Using a Recursive Formula

Find the first 5 terms of the sequence with $a_1 = 5$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

The first term is given, $a_1 = 5$.

Substitute a_1 into the rule to find a_2.

Continue using each term to find the next term.

n	$2a_{n-1} + 1$	a_n
1	*Given*	5
2	$2(5) + 1$	11
3	$2(11) + 1$	23
4	$2(23) + 1$	47
5	$2(47) + 1$	95

The first 5 terms are 5, 11, 23, 47, and 95.

CHECK IT OUT! **Find the first 5 terms of each sequence.**

1a. $a_1 = -5$, $a_n = a_{n-1} - 8$ **1b.** $a_1 = 2$, $a_n = -3a_{n-1}$

In some sequences, you can find the value of a term when you do not know its preceding term. An **explicit formula** defines the nth term of a sequence as a function of n.

EXAMPLE 2 Finding Terms of a Sequence by Using an Explicit Formula

Find the first 5 terms of the sequence $a_n = 2^n - 3$.

Make a table. Evaluate the sequence for $n = 1$ through $n = 5$.

n	$2^n - 3$	a_n
1	$2^1 - 3$	−1
2	$2^2 - 3$	1
3	$2^3 - 3$	5
4	$2^4 - 3$	13
5	$2^5 - 3$	29

The first 5 terms are −1, 1, 5, 13, and 29.

Check Use a graphing calculator.
Enter $y = 2^x - 3$ and make a table.

CHECK IT OUT! **Find the first 5 terms of each sequence.**

2a. $a_n = n^2 - 2n$ **2b.** $a_n = 3n - 5$

You can use your knowledge of functions to write rules for sequences.

EXAMPLE 3 Writing Rules for Sequences

Write a possible explicit rule for the nth term of each sequence.

A **3, 6, 12, 24, 48, …**

Examine the differences and ratios.

Ratios 2 2 2 2

Terms	3	6	12	24	48

1st differences 3 6 12 24

2nd differences 3 6 12

The ratio is constant. The sequence is exponential with a base of 2.

Look for a pattern with powers of 2.

$a_1 = 3 = 3(2)^0$, $a_2 = 6 = 3(2)^1$, $a_3 = 12 = 3(2)^2$, …

A pattern is $3(2)^{n-1}$. One explicit rule is $a_n = 3(2)^{n-1}$.

B **2.5, 4, 5.5, 7, 8.5, …**

Examine the differences.

Terms	2.5	4	5.5	7	8.5

1st differences 1.5 1.5 1.5 1.5

The first differences are constant, so the sequence is linear.

The first term is 2.5, and each term is 1.5 more than the previous.

A pattern is $2.5 + 1.5(n - 1)$, or $1.5n + 1$. One explicit rule is $a_n = 1.5n + 1$.

Remember!

Linear patterns have constant first differences. Quadratic patterns have constant second differences. Exponential patterns have constant ratios. (Lesson 9-6)

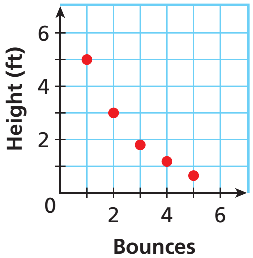
CHECK IT OUT! **Write a possible explicit rule for the nth term of each sequence.**

3a. 7, 5, 3, 1, −1, … **3b.** $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

EXAMPLE 4 *Physics Application*

A ball is dropped and bounces to a height of 5 feet. The ball rebounds to 60% of its previous height after each bounce. Graph the sequence and describe its pattern. How high does the ball bounce on its 9th bounce?

Because the ball first bounces to a height of 5 feet and then bounces to 60% of its previous height on each bounce, the recursive rule is $a_1 = 5$ and $a_n = 0.6a_{n-1}$. Use this rule to find some other terms of the sequence and graph them.



$a_2 = 0.6(5) = 3$

$a_3 = 0.6(3) = 1.8$

$a_4 = 0.6(1.8) = 1.08$

The graph appears to be exponential. Use the pattern to write an explicit rule.

$a_n = 5(0.6)^{n-1}$, where n is the bounce number

Use this rule to find the bounce height for the 9th bounce.

$a_9 = 5(0.6)^{9-1} \approx 0.084$ foot, or approximately 1 inch.

The ball is about 0.084 feet high on the 9th bounce.

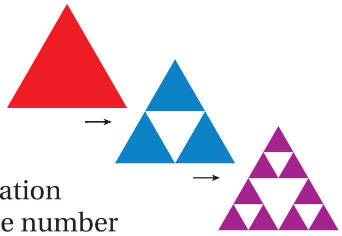
CHECK IT OUT!

4. An ultra-low-flush toilet uses 1.6 gallons every time it is flushed. Graph the sequence of total water used after n flushes, and describe its pattern. How many gallons have been used after 10 flushes?

Recall that a fractal is an image made by repeating a pattern (Lesson 5-5). Each step in this repeated process is an **iteration**, the repetitive application of the same rule.

EXAMPLE 5 Iteration of Fractals

The Sierpinski triangle is a fractal made by taking an equilateral triangle, removing an equilateral triangle from the center, and repeating for each new triangle. Find the number of triangles in the next 2 iterations.



By removing the center of each triangle, each iteration turns every triangle into 3 smaller triangles. So the number of triangles triples with each iteration.

The number of triangles can be represented by the sequence $a_n = 3^{n-1}$.

The 4th and 5th terms are $a_4 = 3^{4-1} = 27$ and $a_5 = 3^{5-1} = 81$.

The next two iterations result in 27 and 81 triangles.

CHECK IT OUT!

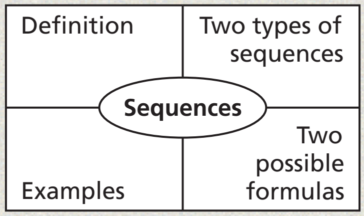
5. The Cantor set is a fractal formed by repeatedly removing the middle third of a line segment as shown. Find the number of segments in the next 2 iterations.

THINK AND DISCUSS

1. Explain the difference between a recursive rule and an explicit rule.
2. Identify three possible next terms for the sequence 1, 2, 4, ...
3. Describe how a sequence is a function. Do all sequences have the same domain? Explain.

Know it! Note

4. **GET ORGANIZED** Copy and complete the graphic organizer. Summarize what you have learned about sequences.



12-1 Exercises

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GUIDED PRACTICE

1. **Vocabulary** A formula that uses one or more previous terms to find the next term is a(n) __?__ formula. (*explicit* or *recursive*)

SEE EXAMPLE 1 p. 862

Find the first 5 terms of each sequence.

2. $a_1 = 1, a_n = 4a_{n-1} - 1$
3. $a_1 = 3, a_n = a_{n-1} + 11$
4. $a_1 = 500, a_n = \frac{a_{n-1}}{5}$

SEE EXAMPLE 2 p. 863

5. $a_n = 12(n - 2)$
6. $a_n = \left(-\frac{1}{2}\right)^{n-1}$
7. $a_n = -3n^2$
8. $a_n = n(n - 1)$
9. $a_n = 4^{n-1}$
10. $a_n = (n + 1)^2$

SEE EXAMPLE 3 p. 863

Write a possible explicit rule for the *n*th term of each sequence.

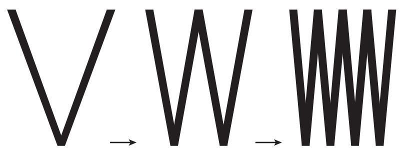
11. 6, 9, 12, 15, 18, ...
12. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
13. 25, 15, 5, −5, −15, ...

SEE EXAMPLE 4 p. 864

14. **Income** Billy earns $25,000 the first year and gets a 5% raise each year. Graph the sequence, and describe its pattern. How much will he earn per year after 5 years? 10 years?

SEE EXAMPLE 5 p. 864

15. **Patterns** Find the number of segments in the next 2 terms of the pattern shown.



PRACTICE AND PROBLEM SOLVING

Find the first 5 terms of each sequence.

16. $a_1 = 7, a_n = a_{n-1} - 3$
17. $a_n = \frac{1}{n^2}$
18. $a_1 = 4, a_n = 1.5a_{n-1} - 2$
19. $a_n = (2)^{n-1} + 8$
20. $a_n = 2n^2 - 12$
21. $a_1 = -2, a_n = -3a_{n-1} - 1$

Independent Practice	
For Exercises	See Example
16–18	1
19–21	2
22–24	3
25	4
26	5

Extra Practice

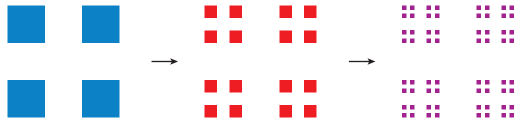
Skills Practice p. S26
Application Practice p. S43

Write a possible explicit rule for the *n*th term of each sequence.

22. 2, 8, 18, 32, 50, . . .　**23.** 9, 5, 1, −3, −7, . . .　**24.** 5, 0.5, 0.05, 0.005, . . .

25. Architecture Chairs for an orchestra are positioned in a curved form with the conductor at the center. The front row has 16 chairs, and each successive row has 4 more chairs. Graph the sequence and describe its pattern. How many chairs are in the 6th row?

26. Fractals Find the number of squares in the next 2 iterations of Cantor dust as shown.



Find the first 5 terms of each sequence.

27. $a_1 = 12,\ a_n = \frac{1}{2}a_{n-1} + 2$　**28.** $a_1 = 1,\ a_n = \frac{2}{a_{n-1}}$　**29.** $a_1 = -10,\ a_n = -a_{n-1} + 10$

30. $a_n = 2n^2 - 12$　**31.** $a_n = 8 - \frac{1}{10}n$　**32.** $a_n = 5(-1)^{n+1}(3)^{n-1}$

33. ERROR ANALYSIS Two attempts to find the first 5 terms of the sequence $a_1 = 3$ and $a_n = 2n + 1$ are shown. Which is incorrect? Explain the error.

A 3, 5, 7, 9, 11

B 3, 7, 15, 31, 63

LINK

Math History

The Fibonacci sequence can also be used to discover the golden ratio. The ratios of successive terms become closer and closer to the golden ratio, $\frac{1+\sqrt{5}}{2}$.

Write a possible explicit rule for each sequence, and find the 10th term.

34. $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$　**35.** $\frac{15}{9}, \frac{14}{9}, \frac{13}{9}, \frac{12}{9}, \frac{11}{9}, \ldots$　**36.** −5.0, −2.5, 0, 2.5, 5.0, . . .

37. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$　**38.** 0.04, 0.4, 4, 40, 400, . . .　**39.** 24, 21, 16, 9, 0, . . .

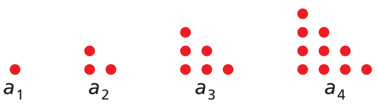
40. Fibonacci Recall from the lesson that the Fibonacci sequence models the number of pairs of rabbits after a certain number of months. The sequence begins 1, 1, . . ., and each term after that is the sum of the two terms before it.

a. Find the first 12 terms of the Fibonacci sequence.

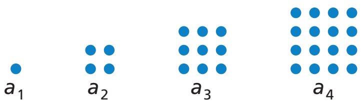
b. How many pairs of rabbits are produced under ideal circumstances at the end of one year?

Find the number of dots in the next 2 figures for each dot pattern.

41.



42.



43. Chess Ronnie is scheduling a chess tournament in which each player plays every other player once. He created a table and found that each new player added more than one game.

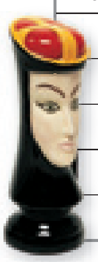
a. Graph the sequence and describe its pattern. What are the next 2 terms in the sequence?

b. Use a regression to find an explicit rule for the sequence.

c. What if...? How would the schedule change if each player played every other player *twice*? Make a table, and describe how the sequence is transformed.

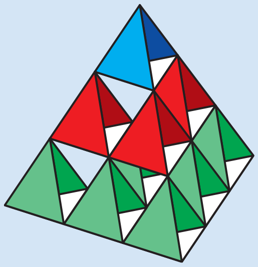
Single-Play Chess Tournament

Number of Players	Number of Games
1	0
2	1
3	3
4	6
5	10



44. This problem will prepare you for the Multi-Step Test Prep on page 888.

A kite is made with 1 tetrahedron in the first (top) layer, 3 tetrahedrons in the second layer, 6 tetrahedrons in the third layer, and so on. Each tetrahedron is made by joining six sticks of equal length.



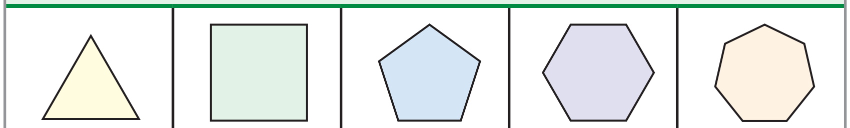
a. The rule $a_n = a_{n-1} + 6n$ gives the number of sticks needed to make the nth layer of the kite, where $a_1 = 6$. Find the first five terms of the sequence.

b. Use regression to find an explicit rule for the sequence.

c. How many sticks are needed to build the 10th layer of the kite?

45. **Geometry** The sum of the interior angle measures for the first 5 regular polygons is shown.

Sum of Interior Angle Measures



180°	360°	540°	720°	900°

a. Write an explicit rule for the nth term of the sequence of the sum of the angle measures. What is the sum of the measures of the interior angles of a 12-sided regular polygon?

b. Recall that all angles are congruent in a regular polygon. Make a table for the measure of an interior angle for each regular polygon. Graph the data, and describe the pattern.

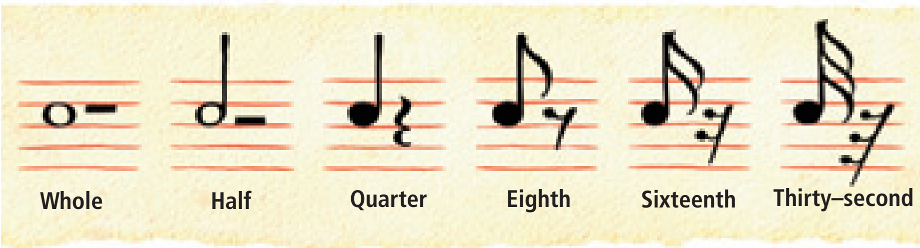
c. Write an explicit rule for the nth term of the sequence described in part **b.**

d. Find the measure of an interior angle of a 10-sided regular polygon.

46. **Estimation** Estimate the 20th term of the sequence 7.94, 8.935, 9.93, 10.925, 11.92, … Explain how you reached your estimate.

47. **Music** Music involves arranging different pitches through time. The musical notation below indicates the duration of various notes (and rests).

Symbols for Musical Notes and Rests



a. Write a numerical sequence that shows the progression of notes (and rests). Write a recursive formula and an explicit formula to generate this sequence.

b. In 4/4 time, a whole note represents 4 beats, a half note represents 2 beats, a quarter note represents 1 beat, and so on. Write a sequence for the number of beats that each note in the progression represents. Then write a recursive formula and an explicit formula to generate this sequence. How is this sequence related to the sequence in part **a**?

48. **Critical Thinking** Can the recursive rule and the explicit rule for a formula ever be the same?

49. **Write About It** Explain how an infinite sequence is different from a finite sequence.

TEST PREP

50. Which is the next term in the sequence $-9, -6, -3, 0, \ldots$?

(A) -3 (B) 0 (C) 3 (D) 6

51. Which rule describes the given sequence 4, 12, 36, 108, …?

(F) $a_n = 4 + 3n$
(G) $a_n = 3 + 4n$
(H) $a_1 = 4, a_n = 3a_{n-1}, n \geq 2$
(J) $a_1 = 3, a_n = 4a_{n-1}, n \geq 2$

52. Which sequence is expressed by the rule $a_n = \frac{2n}{n+1}$?

(A) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, \ldots$
(B) $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \ldots$
(C) $0, 1, 2, \frac{3}{2}, \frac{8}{5}, \ldots$
(D) $2, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{12}{7}, \ldots$

53. Which sequence is expressed by the rule $a_1 = 6$ and $a_n = 12 - 2a_{n-1}, n \geq 2$?

(F) $6, 4, 2, 0, -2, -4, \ldots$
(G) $0, 12, -12, 36, -60, \ldots$
(H) $6, 0, 12, -12, 36, \ldots$
(J) $6, 0, -6, -12, -18, \ldots$

54. **Gridded Response** Find the next term in the sequence $-32, 16, -8, 4, -2, \ldots$.

CHALLENGE AND EXTEND

Write an explicit rule for each sequence and find the 10th term.

55. $-\frac{2}{3}, \frac{5}{3}, 8, \frac{61}{3}, \frac{122}{3}, \ldots$

56. $-2, 6, -12, 20, -30, \ldots$

57. $0.9, 0.8, 0.6, 0.3, -0.1, \ldots$

58. **Geometry** Draw 5 circles. Place 1 point on the first circle, 2 points on the second, 3 points on the third, and so forth. Then connect every point with every other point in each circle, and count the maximum number of nonoverlapping regions that are formed inside each circle.

a. Write the resulting sequence.

b. Although the sequence appears to double, the sixth circle has less than 32 possible regions. Try to find them all by carefully drawing this figure. How many did you get?

SPIRAL REVIEW

Simplify. Assume that all expressions are defined. *(Lesson 8-2)*

59. $\frac{x^2 - 9}{x^2 + 5x + 6}$

60. $\frac{4x^2 - 5x}{8x^2 + 18x - 35}$

61. $\frac{4x - 12}{x^2 - 25} \div \frac{8x - 24}{2x - 10}$

62. $\frac{x^2 - 5x - 6}{x^2 - 3x - 18} \cdot \frac{x^2 + x - 6}{x^2 - x - 2}$

Add or subtract. *(Lesson 8-4)*

63. $\frac{2x - 3}{x + 1} + \frac{4x - 9}{x - 1}$

64. $\frac{9x}{8x - 4} - \frac{10x + 3}{12x - 6}$

65. $\frac{x^2}{2x + 7} - \frac{x}{x + 2}$

66. **Literature** Christopher is reading a book containing 854 pages at a rate of 1.5 pages per minute. Create a table, equation, and graph to represent the number of pages remaining to be read P in relation to time t. *(Lesson 9-1)*

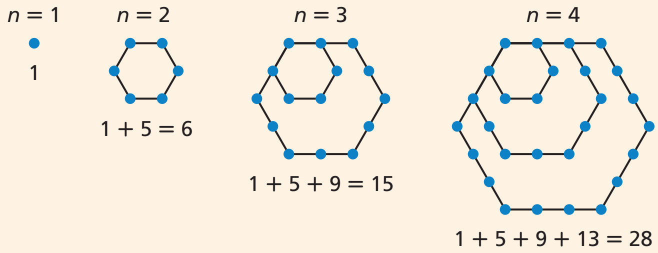
Connecting Algebra to **Geometry**

See Skills Bank page S66

Geometric Patterns and Tessellations

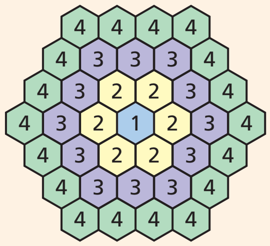
Sequences of figures can often be described with number patterns.

Polygonal numbers can be represented by dots arranged in the form of polygons. The first four hexagonal numbers are illustrated. The sums show a pattern. The sequence for the total number of dots is 1, 6, 15, 28, ...



Example

The figure shows how regular hexagons can be used to tessellate, or cover, the plane. Write a sequence for the number of hexagons added at each stage. Describe the pattern in the sequence, and find the next term.



Show the number of hexagons added at each stage.

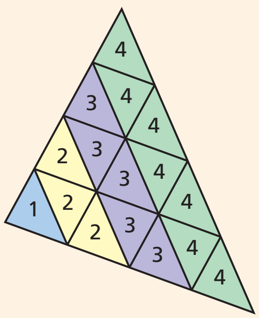
Level	1	2	3	4
Number of Hexagons	1	6	12	18

From the second term on, the number added appears to increase by 6 each time. The next level probably would have 24 hexagons. You can check your conjecture by constructing the next stage of the tessellation and counting the hexagons.

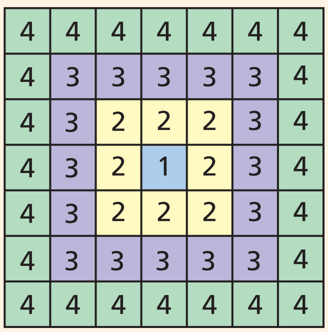
Try This

Write a sequence for the number of polygons added at each stage of the figure. Describe the pattern in the sequence, and find the next term.

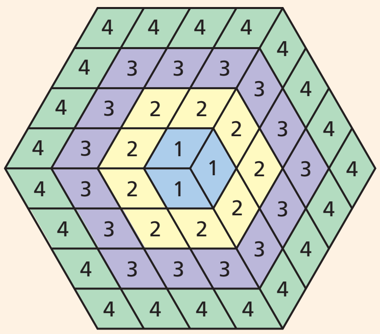
1.



2.



3.



12-2 Series and Summation Notation

Objective
Evaluate the sum of a series expressed in sigma notation.

Vocabulary
series
partial sum
summation notation

Why learn this?
You can use sums of sequences to find the size of a house of cards. (See Example 4.)

In Lesson 12-1, you learned how to find the nth term of a sequence. Often we are also interested in the sum of a certain number of terms of a sequence. A **series** is the indicated sum of the terms of a sequence. Some examples are shown in the table.

Sequence	1, 2, 3, 4	2, 4, 6, 8, ...	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$
Series	1 + 2 + 3 + 4	2 + 4 + 6 + 8 + ⋯	$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

Because many sequences are infinite and do not have defined sums, we often find partial sums. A **partial sum**, indicated by S_n, is the sum of a specified number of terms of a sequence.

For the even numbers:

$S_1 = 2$ *Sum of first term*
$S_2 = 2 + 4 = 6$ *Sum of first 2 terms*
$S_3 = 2 + 4 + 6 = 12$ *Sum of first 3 terms*
$S_4 = 2 + 4 + 6 + 8 = 20$ *Sum of first 4 terms*

A series can also be represented by using **summation notation**, which uses the Greek letter $\sum$ (capital *sigma*) to denote the sum of a sequence defined by a rule, as shown.

$$\sum_{k=1}^{5} 2k$$

5 ← Last value of k
$2k$ ← Explicit formula for sequence
$k=1$ ← First value of k

EXAMPLE 1 Using Summation Notation

Write each series in summation notation.

A $3 + 6 + 9 + 12 + 15$

Find a rule for kth term of the sequence.

$a_k = 3k$ *Explicit formula*

Write the notation for the first 5 terms.

$\sum_{k=1}^{5} 3k$ *Summation notation*

B $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$

Find a rule for the kth term.

$a_k = (-1)^{k+1}\left(\frac{1}{2}\right)^k$ *Explicit formula*

Write the notation for the first 6 terms.

$\sum_{k=1}^{6} (-1)^{k+1}\left(\frac{1}{2}\right)^k$ *Summation notation*

Caution!
For sequences with alternating signs:
Use $(-1)^{k+1}$ if $a_1 = +$.
Use $(-1)^{k}$ if $a_1 = -$.

CHECK IT OUT! Write each series in summation notation.

1a. $\frac{2}{4} + \frac{2}{9} + \frac{2}{16} + \frac{2}{25} + \frac{2}{36}$ **1b.** $-2 + 4 - 6 + 8 - 10 + 12$

EXAMPLE 2 Evaluating a Series

Expand each series and evaluate.

A $\sum_{k=3}^{6} \frac{1}{2^k}$

$$\sum_{k=3}^{6} \frac{1}{2^k} = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}$$ *Expand the series by replacing k.*

$$= \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}$$ *Evaluate powers.*

$$= \frac{8}{64} + \frac{4}{64} + \frac{2}{64} + \frac{1}{64} = \frac{15}{64}$$ *Simplify.*

B $\sum_{k=1}^{4} (10 - k^2)$

$$\sum_{k=1}^{4} (10 - k^2) = (10 - 1^2) + (10 - 2^2) + (10 - 3^2) + (10 - 4^2)$$ *Expand.*

$$= 9 + 6 + 1 + (-6)$$ *Simplify.*

$$= 10$$

Caution!

Watch out! A series can have a first value other than $k = 1$, such as in Example 2A, where k begins at 3.

CHECK IT OUT! Expand each series and evaluate.

2a. $\sum_{k=1}^{4} (2k - 1)$ **2b.** $\sum_{k=1}^{5} -5(2)^{k-1}$

Finding the sum of a series with many terms can be tedious. You can derive formulas for the sums of some common series.

In a *constant series,* such as $3 + 3 + 3 + 3 + 3$, each term has the same value.

$$\sum_{k=1}^{5} 3 = \underbrace{3 + 3 + 3 + 3 + 3}_{\text{5 terms}} = 5 \cdot 3 = 15$$

The formula for the sum of a constant series is $\sum_{k=1}^{n} c = nc$, as shown.

$$\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ terms}} = nc$$

A *linear series* is a counting series, such as the sum of the first 10 natural numbers. Examine when the terms are rearranged.

$$\sum_{1}^{10} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$= (1 + 10) + (2 + 9) + (3 + 8) + (4 + 7) + (5 + 6)$$
$$= \underbrace{11 + 11 + 11 + 11 + 11}_{\text{5 terms}} = 5(11) = 55$$

Notice that 5 is half of the number of terms and 11 represents the sum of the first and the last term, $1 + 10$. This suggests that the sum of a linear series is $\sum_{k=1}^{n} k = \frac{n}{2}(1 + n)$, which can be written as $\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$.

Similar methods will help you find the sum of a *quadratic series.*

Know it! Note

Summation Formulas

CONSTANT SERIES	LINEAR SERIES	QUADRATIC SERIES
$\sum_{k=1}^{n} c = nc$	$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}$	$\sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6}$

EXAMPLE 3 Using Summation Formulas

Evaluate each series.

A $\sum_{k=5}^{10} 8$ *Constant series*

Method 1 Use the summation formula.

There are 6 terms.

$$\sum_{k=5}^{10} 8 = nc = 6(8) = 48$$

Method 2 Expand and evaluate.

$$\sum_{k=5}^{10} 8 = \underbrace{8 + 8 + 8 + 8 + 8 + 8}_{\text{6 terms}} = 48$$

B $\sum_{k=1}^{5} k$ *Linear series*

Method 1 Use the summation formula.

$$\sum_{k=1}^{5} k = \frac{n(n+1)}{2} = \frac{5(6)}{2} = 15$$

Method 2 Expand and evaluate.

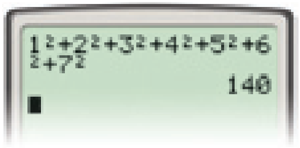
$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

C $\sum_{k=1}^{7} k^2$ *Quadratic series*

Method 1 Use the summation formula.

$$\sum_{k=1}^{7} k^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{7(7+1)(2 \cdot 7 + 1)}{6}$$
$$= \frac{56(15)}{6}$$
$$= 140$$

Method 2 Use a graphing calculator.



> **Caution!**
>
> When counting the number of terms, you must include both the first and the last. For example, $\sum_{k=5}^{10} 8$ has six terms, not five.
> $k = 5, 6, 7, 8, 9, 10$

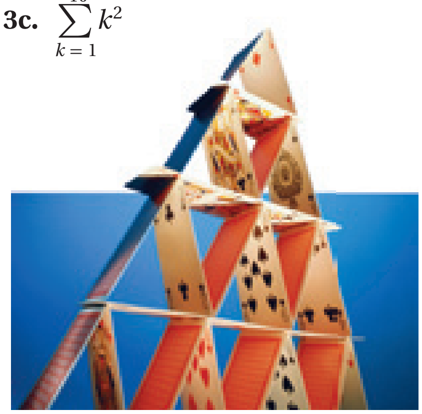
CHECK IT OUT! Evaluate each series.

3a. $\sum_{k=1}^{60} 4$ **3b.** $\sum_{k=1}^{15} k$ **3c.** $\sum_{k=1}^{10} k^2$

EXAMPLE 4 Problem-Solving Application

PROBLEM SOLVING

Ricky is building a card house similar to the one shown. He wants the house to have as many stories as possible with a deck of 52 playing cards. How many stories will Ricky's house have?



1 Understand the Problem

The **answer** will be the number of stories, or rows, in the card house.

List the important information:

- He has 52 playing cards.
- The house should have as many stories as possible.

2 Make a Plan

Make a diagram of the house to better understand the problem. Find a pattern for the number of cards in each story. Write and evaluate the series.

3 Solve

Make a table and a diagram.

Row	1	2	3	4
Diagram				
Cards	2	5	8	11

The number of cards increases by 3 in each row. Write a series to represent the total number of cards in n rows.

$\sum_{k=1}^{n}(3k-1)$, where k is the row number and n is the total number of rows

Evaluate the series for several n-values.

$$\sum_{k=1}^{4}(3k-1) = [3(1)-1]+[3(2)-1]+[3(3)-1]+[3(4)-1]$$
$$= 26$$

$$\sum_{k=1}^{5}(3k-1) = [3(1)-1]+[3(2)-1]+[3(3)-1]+[3(4)-1]+[3(5)-1]$$
$$= 40$$

$$\sum_{k=1}^{6}(3k-1) = [3(1)-1]+[3(2)-1]+[3(3)-1]+[3(4)-1]+$$
$$[3(5)-1]+[3(6)-1]$$
$$= 57$$

Because Ricky has only 52 cards, the house can have at most 5 stories.

4 Look Back

Use the table to continue the pattern. The 5th row would have 14 cards. $S_5 = 2 + 5 + 8 + 11 + 14 = 40$. The next row would have more than 12 cards, so the total would be more than 52.

CHECK IT OUT!

4. A flexible garden hose is coiled for storage. Each subsequent loop is 6 inches longer than the preceding loop, and the innermost loop is 34 inches long. If there are 6 loops, how long is the hose?

THINK AND DISCUSS

1. Explain the difference between a sequence and a series.

2. Explain what each of the variables represents in the notation $\sum_{k=m}^{n} k$.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Write the general notation and an example for each term.

	Sequence	Series
Notation		
Example		

12-2 Exercises

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Homework Help Online
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Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** Give an example of *summation notation.*

SEE EXAMPLE 1 p. 870

Write each series in summation notation.

2. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$
3. $-3 + 6 - 9 + 12 - 15$
4. $1 + 10 + 100 + 1000 + 10{,}000$
5. $100 + 95 + 90 + 85 + 80$

SEE EXAMPLE 2 p. 871

Expand each series and evaluate.

6. $\sum_{k=1}^{5} k^3$
7. $\sum_{k=1}^{4} (-1)^{k+1} \frac{12}{k^2}$
8. $\sum_{k=5}^{10} -5k$

SEE EXAMPLE 3 p. 872

Evaluate each series.

9. $\sum_{k=1}^{21} k$
10. $\sum_{k=1}^{20} k^2$
11. $\sum_{k=15}^{35} 6$

SEE EXAMPLE 4 p. 872

12. **Finance** Melinda makes monthly car payments of \$285 each month. How much will she have paid after 2 years? 5 years?

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
13–16	1
17–19	2
20–22	3
23	4

Extra Practice

Skills Practice p. S26

Application Practice p. S43

Write each series in summation notation.

13. $1.1 + 2.2 + 3.3 + 4.4 + 5.5$
14. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$
15. $11 - 12 + 13 - 14 + 15 - 16$
16. $1 + 2 + 4 + 8 + 16 + 32$

Expand each series and evaluate.

17. $\sum_{k=1}^{5} [8(k+1)]$
18. $\sum_{k=2}^{7} (-2)^k$
19. $\sum_{k=1}^{4} \frac{k-1}{k+1}$

Evaluate each series.

20. $\sum_{k=1}^{99} k$
21. $\sum_{k=11}^{88} 2.5$
22. $\sum_{k=1}^{25} k^2$

23. **Retail** A display of soup cans is arranged with 1 can on top and each row having an additional can. How many cans are in a display of 20 rows?

Write each series in summation notation.

24. $-1 + 4 - 9 + 16 - 25 + 36$
25. $25 + 24 + 23 + \cdots + 1$
26. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243}$
27. $-800 - 80 - 8 - 0.8 - 0.08$
28. $10.8 + 10.5 + 10.2 + 9.9$
29. $9 - 16 + 25 - 36 + 49 - 64$
30. $-3.9 + 4.4 - 4.9 + 5.4 - 5.9$
31. $0 + 3.4 + 6.8 + 10.2 + 13.6$
32. $3 + \frac{3}{2} + 1 + \frac{3}{4} + \frac{3}{5}$
33. $1000 + 100 + 10 + 1 + \frac{1}{10}$

34. **Travel** The distance from St. Louis, Missouri, to Los Angeles, California, is 1596 miles. Michael plans to travel half the distance on the first day and half the remaining distance each day after that. Write a series in summation notation for the total distance he will travel in 5 days. How far will Michael travel in the 5 days?

35. **Safety** An employer uses a telephone tree to notify employees in the case of an emergency closing. When the office manager makes the decision to close, she calls 3 people. Each of these people calls 3 other people, and so on.

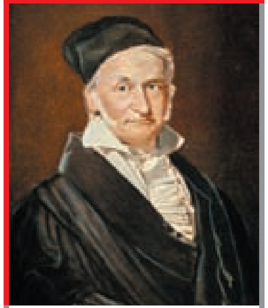
a. Make a tree diagram with 3 levels to represent the problem situation.

b. Write and evaluate a series to find the total number of people notified after 5 levels of calls.

c. **What if...?** Suppose that the phone tree involves calling 5 people at each level. How many more people would be notified after 5 levels of calls?

LINK

Math History



The Granger Collection, New York

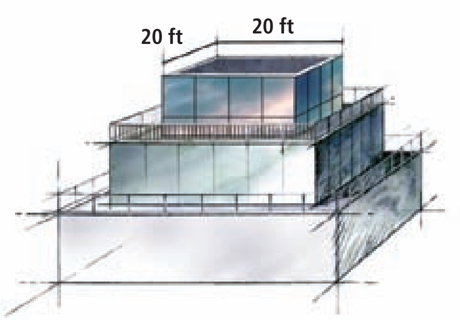
At the age of 10, German mathematician Carl Friedrich Gauss discovered a quick method for adding the first 100 natural numbers. His method gave us the summation formula for a linear series.

Expand each series and evaluate.

36. $\sum_{k=1}^{6}(k^2+1)$ **37.** $\sum_{k=1}^{6}(-1)^k 5k$ **38.** $\sum_{k=3}^{6}\frac{1}{2k}$

39. $\sum_{k=1}^{6}(3k-2)$ **40.** $\sum_{k=6}^{11}12(k-2)$ **41.** $\sum_{k=1}^{5}\frac{k^2}{5k}$

42. **Architecture** A hotel is being built in the shape of a pyramid, as shown in the diagram. Each square floor is 10 feet longer and 10 feet wider than the floor above it.



a. Write a series that represents the total area of n floors of the hotel.

b. How many stories must the hotel be to have at least 50,000 square feet of floor area?

Estimation **Use mental math to estimate each sum. Then compare your answer to the sum obtained by using a calculator.**

43. $10 + 11 + 12 + \cdots + 29 + 30$

44. $1 + 3 + 5 + \cdots + 97 + 99$

45. $-2 + (-4) + (-6) + \cdots + (-98) + (-100)$

46. **Physics** The distance that an object falls in equal time intervals is represented in the table. Rules created by Leonardo da Vinci and Galileo are shown. (In this general case, specific units do not apply to time or distance.)

Distance Fallen in Each Time Interval

Time Interval	Leonardo's Rule	Galileo's Rule
1	1	1
2	2	3
3	3	5
4	4	7
5	5	9

a. Write the series for each rule for 5 intervals, and find the respective sums. What does the sum of the series for 5 intervals represent?

b. Write each series in summation notation. Then evaluate each series for $n = 10$.

c. By the current rule, the distance fallen in each interval is 1, 4, 9, 16, 25.... How do Leonardo's rule and Galileo's rule compare to the current rule?

47. **Critical Thinking** Some mathematical properties may be applied to series.

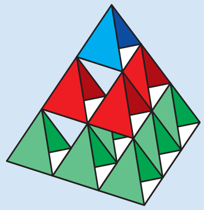
a. Evaluate $\sum_{k=1}^{10}3n$ and $3\sum_{k=1}^{10}n$. Make a conjecture based on your answer.

b. Evaluate $\sum_{k=1}^{10}n + \sum_{k=1}^{10}2$ and $\sum_{k=1}^{10}(n+2)$. Make a conjecture based on your answer.

MULTI-STEP TEST PREP

48. This problem will prepare you for the Multi-Step Test Prep on page 888.

The series $\sum_{k=1}^{n}\left(3k^2+3k\right)$ gives the total number of sticks needed to make a tetrahedral kite with n layers.

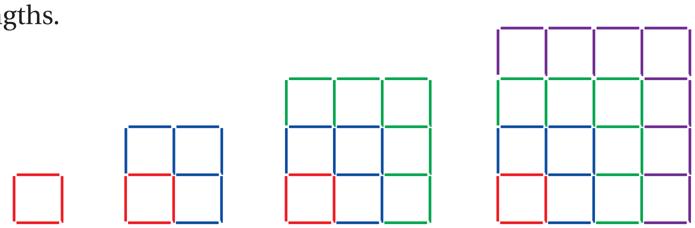


a. Expand and evaluate the series to find out how many sticks are needed to make a kite with 5 layers.

b. Use the properties
$\sum_{k=1}^{n}\left(a_k+b_k\right)=\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}b_k$ and $\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$
to rewrite the series as a multiple of a quadratic series plus a multiple of a linear series.

c. Use summation formulas to determine how many sticks are needed to make a kite with 17 layers.

49. **Multi-Step** Examine the pattern made by toothpick squares with increasing side lengths.



a. Write a sequence for the number of toothpicks added to form each new square.

b. Write and evaluate a series in summation notation to represent the total number of toothpicks in a square with a side length of 6 toothpicks.

50. **Critical Thinking** Are the sums of $1+3+5+7+9$ and $9+7+5+3+1$ the same? Do these series have the same summation notation? Explain.

51. **Write About It** Explain why S_n represents a partial sum and not a complete sum of the terms of a sequence.

TEST PREP

52. Which notation accurately reflects the series $\sum_{k=1}^{7}(-1)^k(3k)$?

Ⓐ $3+6+9+12+15+18+21$
Ⓑ $3-6+9-12+15-18+21$
Ⓒ $-3+6-9+12-15+18-21$
Ⓓ $-3-6-9-12-15-18-21$

53. Which notation accurately reflects the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}$?

Ⓕ $\sum_{k=1}^{4}\frac{k}{2}$
Ⓖ $\sum_{k=1}^{4}\frac{1}{2^k}$
Ⓗ $\sum_{k=1}^{4}\frac{1}{2k}$
Ⓙ $\sum_{k=1}^{4}\frac{1}{k+2}$

54. What is the value of $\sum_{k=1}^{6}k^2$?

Ⓐ 36
Ⓑ 55
Ⓒ 91
Ⓓ 273

55. Find the sum of the series $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\frac{1}{24}$?

Ⓕ $\frac{1}{45}$
Ⓖ $\frac{4}{45}$
Ⓗ $\frac{7}{12}$
Ⓙ $\frac{5}{8}$

56. **Short Response** The number of cans in each row of a pyramidal stack is 1, 4, 9, 16, ... Would a sequence or series be used to find the number of cans in the 20th row? Explain.

CHALLENGE AND EXTEND

Write each series in summation notation. Then find the sum.

57. $1 + 2 + 3 + \cdots + 1000$

58. $1^2 + 2^2 + 3^2 + \cdots + 25^2$

Prove each summation property for the sequences a_k and b_k.

59. $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

60. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

61. **Critical Thinking** What might the sum of the sequence $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ be if it continues forever? Explain.

SPIRAL REVIEW

Find the intercepts of each line, and graph the lines. *(Lesson 2-3)*

62. $3x - 4y = 12$

63. $-6x + 3y = -18$

64. $10x + 15y = -5$

65. **Architecture** The height in feet of an elevator above ground is modeled by $h(t) = 8|t - 6| + 10$, where t is the time in seconds. What is the minimum height of the elevator? *(Lesson 2-9)*

Find the first 5 terms of each sequence. *(Lesson 12-1)*

66. $a_n = \left(\frac{1}{2}n + 2\right)^2$

67. $a_1 = 2,\ a_n = \left(a_{n-1}\right)^2 - 1$

68. $a_n = \frac{4^n}{2}$

Career Path

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KEYWORD: MB7 Career

Steven Howe
Nursing student

Q: What high school math classes did you take?

A: Algebra 1, Geometry, Business Math, and Algebra 2

Q: Why did you decide to pursue a career in nursing?

A: I've always liked science, especially biology and anatomy. I also wanted a career where I can help people, so a job in medicine seemed perfect.

Q: How is math used in nursing?

A: Nurses have to make a lot of calculations to ensure that patients receive the correct dosages of medicine. They also use complicated instruments to measure and monitor different body functions.

Q: What are your future plans?

A: I'll finish my associate's degree in nursing at the end of the semester. I hope to get a job at one of the local hospitals.

12-2 Technology LAB

Evaluate Sequences and Series

Graphing calculators have built-in features that help you generate the terms of a sequence and find the sums of series.

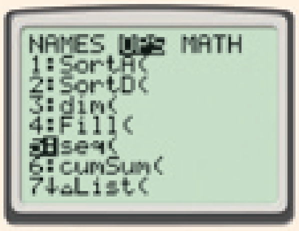
Use with Lesson 12-2

Activity

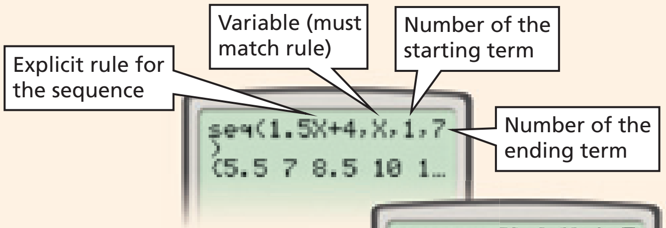
Use a graphing calculator to find the first 7 terms of the sequence $a_n = 1.5n + 4$. Then find the sum of those terms.

1 Find the first 7 terms of the sequence.

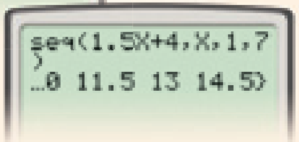
Enter the **LIST** operations menu by pressing 2nd STAT (LIST) and scrolling right to the **OPS** menu. Then select the sequence command **5:seq(**.



The sequence command takes the following four expressions separated by commas.



Enter the rule, using x as the variable. Enter 1 for the starting term and 7 for the ending term. Close the parentheses, and then press ENTER.



The terms of the sequence will be displayed in brackets. Use the arrow keys to scroll to see the rest of the terms.

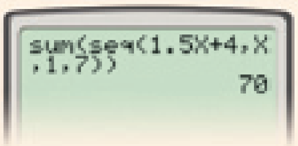
The first 7 terms are 5.5, 7, 8.5, 10, 11.5, 13, and 14.5.

2 Find the sum of the terms.

Enter the **LIST** math menu by pressing 2nd STAT (LIST), and scrolling right to the **MATH** menu. Then select the sum command **5:sum(**.

Follow the steps for entering a sequence as shown in Step 1.

The sum of the first 7 terms is 70.



Try This

Find the first 8 terms of each sequence. Then find the sum of those 8 terms.

1. $a_n = 2n^2 - 5$ **2.** $a_n = \frac{1}{4}(2)^{n-1}$ **3.** $a_n = 0.3n + 1.6$

4. $a_n = 20n$ **5.** $a_n = n^3 - 2n$ **6.** $a_n = 0.1(5)^n$

7. Critical Thinking Find the next 5 terms of the sequence 200, 182, 164, 146, 128, ... Then find the sum of those 5 terms.

12-3 Arithmetic Sequences and Series

Objectives

Find the indicated terms of an arithmetic sequence.

Find the sums of arithmetic series.

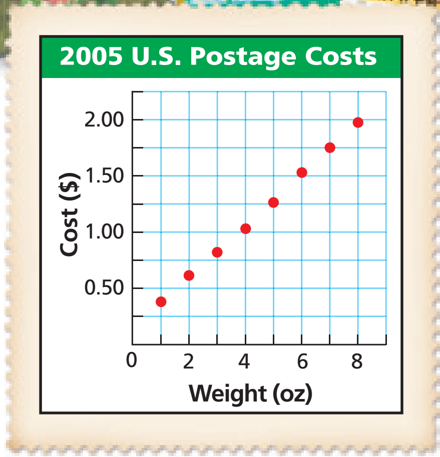
Vocabulary

arithmetic sequence
arithmetic series

Who uses this?

You can use arithmetic sequences to predict the cost of mailing letters.

The cost of mailing a letter in 2005 gives the sequence 0.37,0.60,0.83,1.06, …. This sequence is called an **arithmetic sequence** because its successive terms differ by the same number d $(d \neq 0)$, called the *common difference*. For the mail costs, d is **0.23**, as shown.



Term	a_1	a_2	a_3	a_4
Value	0.37	0.60	0.83	1.06

Differences 0.23 0.23 0.23

Recall that linear functions have a constant first difference. Notice also that when you graph the ordered pairs (n, a_n) of an arithmetic sequence, the points lie on a straight line. Thus, you can think of an arithmetic sequence as a linear function with sequential natural numbers as the domain.

EXAMPLE 1 Identifying Arithmetic Sequences

Determine whether each sequence could be arithmetic. If so, find the common first difference and the next term.

A $-3, 2, 7, 12, 17, \ldots$

$-3,\quad 2,\quad 7,\quad 12,\quad 17$

Differences 5 5 5 5

The sequence could be arithmetic with a common difference of 5. The next term is $17 + 5 = 22$.

B $-4, -12, -24, -40, -60, \ldots$

$-4,\quad -12,\quad -24,\quad -40,\quad -60$

Differences -8 -12 -16 -20

The sequence is not arithmetic because the first differences are not common.

CHECK IT OUT! **Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.**

1a. $1.9, 1.2, 0.5, -0.2, -0.9, \ldots$ **1b.** $\frac{11}{2}, \frac{11}{3}, \frac{11}{4}, \frac{11}{5}, \frac{11}{6}, \ldots$

Each term in an arithmetic sequence is the sum of the previous term and the common difference. This gives the recursive rule $a_n = a_{n-1} + d$. You also can develop an explicit rule for an arithmetic sequence.

Notice the pattern in the table. Each term is the sum of the first term and a multiple of the common difference.

This pattern can be generalized into a rule for all arithmetic sequences.

Postage Costs per Ounce

n	a_n
1	$a_1 = 0.37 + 0(0.23)$
2	$a_2 = 0.37 + 1(0.23)$
3	$a_3 = 0.37 + 2(0.23)$
4	$a_4 = 0.37 + 3(0.23)$
n	$a_n = 0.37 + (n - 1)(0.23)$

Know it! Note

General Rule for Arithmetic Sequences

The nth term a_n of an arithmetic sequence is given by

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term and d is the common difference.

EXAMPLE 2 Finding the *n*th Term Given an Arithmetic Sequence

Find the 10th term of the arithmetic sequence 32, 25, 18, 11, 4, … .

Step 1 Find the common difference: $d = 25 - 32 = -7$.

Step 2 Evaluate by using the formula.

$a_n = a_1 + (n - 1)d$	*General rule*
$a_{10} = 32 + (10 - 1)(-7)$	*Substitute 32 for a_1, 10 for n, and −7 for d.*
$= -31$	*Simplify.*

The 10th term is −31.

Check Continue the sequence.

n	1	2	3	4	5	6	7	8	9	10
a_n	32	25	18	11	4	−3	−10	−17	−24	−31 ✔

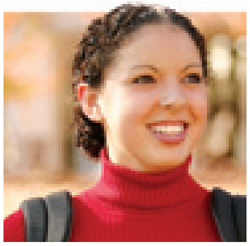
CHECK IT OUT! **Find the 11th term of each arithmetic sequence.**

2a. −3, −5, −7, −9, …

2b. 9.2, 9.15, 9.1, 9.05, …

Student to Student

Finding the nth Term

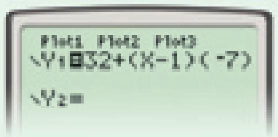


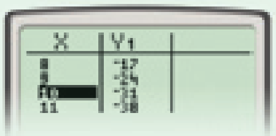
Diana Watson
Bowie High School

I like to check the value of a term by using a graphing calculator.

I enter the function for the nth, or general, term. For Example 2A, enter $y = 32 + (x - 1)(-7)$.

I then use the table feature. Start at 1 (for $n = 1$), and use a step of 1. Then find the desired term (y-value) as shown for $n = 10$.





EXAMPLE 3 Finding Missing Terms

Find the missing terms in the arithmetic sequence 11, ▢, ▢, ▢, −17.

Step 1 Find the common difference.

$$a_n = a_1 + (n-1)d \qquad \textit{General rule}$$

$$-17 = 11 + (5-1)d \qquad \textit{Substitute } -17 \textit{ for } a_n, 11 \textit{ for } a_1, \textit{ and 5 for } n.$$

$$-7 = d \qquad \textit{Solve for } d.$$

Step 2 Find the missing terms using $d = -7$ and $a_1 = 11$.

$a_2 = 11 + (2-1)(-7)$	$a_3 = 11 + (3-1)(-7)$	$a_4 = 11 + (4-1)(-7)$
$= 4$	$= -3$	$= -10$

The missing terms are 4, −3, and −10.

CHECK IT OUT! **3.** Find the missing terms in the arithmetic sequence 2, ▢, ▢, ▢, 0.

Because arithmetic sequences have a common difference, you can use any two terms to find the difference.

EXAMPLE 4 Finding the *n*th Term Given Two Terms

Find the 6th term of the arithmetic sequence with $a_9 = 120$ and $a_{14} = 195$.

Step 1 Find the common difference.

$$a_n = a_1 + (n-1)d$$

$$a_{14} = a_9 + (14-9)d \qquad \textit{Let } a_n = a_{14} \textit{ and } a_1 = a_9. \textit{ Replace 1 with 9.}$$

$$a_{14} = a_9 + 5d \qquad \textit{Simplify.}$$

$$195 = 120 + 5d \qquad \textit{Substitute 195 for } a_{14} \textit{ and 120 for } a_9.$$

$$75 = 5d$$

$$15 = d$$

Step 2 Find a_1.

$$a_n = a_1 + (n-1)d \qquad \textit{General rule}$$

$$120 = a_1 + (9-1)(15) \qquad \textit{Substitute 120 for } a_9, \textit{ 9 for } n, \textit{ and 15 for } d.$$

$$120 = a_1 + 120 \qquad \textit{Simplify.}$$

$$0 = a_1$$

Step 3 Write a rule for the sequence, and evaluate to find a_6.

$$a_n = a_1 + (n-1)d \qquad \textit{General rule}$$

$$a_n = 0 + (n-1)(15) \qquad \textit{Substitute 0 for } a_1 \textit{ and 15 for } d.$$

$$a_6 = 0 + (6-1)15 \qquad \textit{Evaluate for } n = 6.$$

$$= 75$$

The 6th term is 75.

CHECK IT OUT! **Find the 11th term of each arithmetic sequence.**

4a. $a_2 = -133$ and $a_3 = -121$ **4b.** $a_3 = 20.5$ and $a_8 = 13$

In Lesson 12-2 you wrote and evaluated series. An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence. You can derive a general formula for the sum of an arithmetic series by writing the series in forward and reverse order and adding the results.

$$
\begin{aligned}
S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \dots + a_n \\
S_n &= a_n + (a_n - d) + (a_n - 2d) + \dots + a_1 \\
2S_n &= \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \dots + (a_1 + a_n)}_{(a_1 + a_n) \text{ is added } n \text{ times}}
\end{aligned}
$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n(a_1 + a_n)}{2}, \text{ or } S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Know it! Note

Sum of the First n Terms of an Arithmetic Series

WORDS	NUMBERS	ALGEBRA
The sum of the first n terms of an arithmetic series is the product of the number of terms and the average of the first and last terms.	The sum of $2 + 4 + 6 + 8 + 10$ is $5\left(\frac{2 + 10}{2}\right) = 5(6) = 30$.	$S_n = n\left(\frac{a_1 + a_n}{2}\right)$, where n is the number of terms, a_1 is the first term, and a_n is the nth term.

EXAMPLE 5

Finding the Sum of an Arithmetic Series

Find the indicated sum for each arithmetic series.

A S_{15} for $25 + 12 + (-1) + (-14) + \cdots$

Find the common difference.

$d = 12 - 25 = -13$

Find the 15th term.

$a_{15} = 25 + (15 - 1)(-13)$

$= -157$

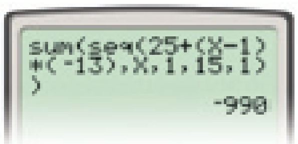
Find S_{15}.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ *Sum formula*

$S_{15} = 15\left(\frac{25 + (-157)}{2}\right)$ *Substitute.*

$= 15(-66) = -990$

Check Use a graphing calculator.



✔

B $\sum_{k=1}^{12} (3 + 4k)$

Find the 1st and 12th terms.

$a_1 = 3 + 4(1) = 7$

$a_{12} = 3 + 4(12) = 51$

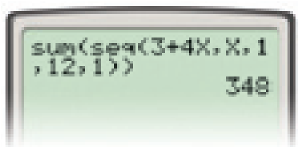
Find S_{12}.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$

$S_{12} = 12\left(\frac{7 + 51}{2}\right)$

$= 348$

Check Use a graphing calculator.



✔

Remember!

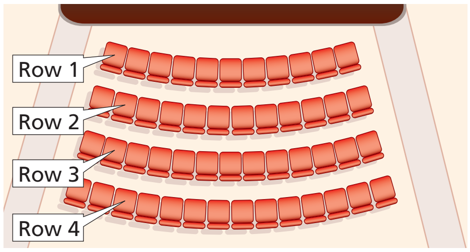
These sums are actually *partial sums.* You cannot find the complete sum of an infinite arithmetic series because the term values increase or decrease indefinitely.

CHECK IT OUT! **Find the indicated sum for each arithmetic series.**

5a. S_{16} for $12 + 7 + 2 + (-3) + \cdots$ **5b.** $\sum_{k=1}^{15} (50 - 20k)$

EXAMPLE 6 Theater Application

The number of seats in the first 14 rows of the center orchestra aisle of the Marquis Theater on Broadway in New York City form an arithmetic sequence as shown.



A **How many seats are in the 14th row?**

Write a general rule using $a_1 = 11$ and $d = 1$.

$$a_n = a_1 + (n - 1)d \qquad \textit{Explicit rule for nth term}$$

$$a_{14} = 11 + (14 - 1)1 \qquad \textit{Substitute.}$$

$$= 11 + 13$$

$$= 24 \qquad \textit{Simplify.}$$

There are 24 seats in the 14th row.

B **How many seats in total are in the first 14 rows?**

Find S_{14} using the formula for finding the sum of the first n terms.

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) \qquad \textit{Formula for first n terms}$$

$$S_{14} = 14\left(\frac{11 + 24}{2}\right) \qquad \textit{Substitute.}$$

$$= 14\left(\frac{35}{2}\right)$$

$$= 245 \qquad \textit{Simplify.}$$

There are 245 seats in rows 1 through 14.

CHECK IT OUT!

6. **What if...?** Suppose that each row after the first had 2 additional seats.

a. How many seats would be in the 14th row?

b. How many total seats would there be in the first 14 rows?

THINK AND DISCUSS

1. Compare an arithmetic sequence with a linear function.

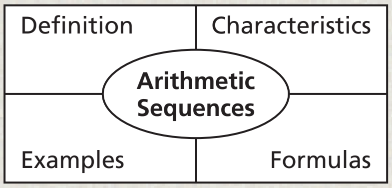
2. Describe the effect that a negative common difference has on an arithmetic sequence.

3. Explain how to find the 6th term in a sequence when you know the 3rd and 4th terms.

4. Explain how to find the common difference when you know the 7th and 12th terms of an arithmetic sequence.

Know it! Note

5. **GET ORGANIZED** Copy and complete the graphic organizer. Write in each rectangle to summarize your understanding of arithmetic sequences.



12-3 Exercises

GUIDED PRACTICE

1. **Vocabulary** The expression 10 + 20 + 30 + 40 + 50 is an __?__. (*arithmetic sequence* or *arithmetic series*)

SEE EXAMPLE 1 p. 879

Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.

2. 46, 39, 32, 25, 18, …
3. 28, 21, 15, 10, 6, …
4. $\frac{12}{3}, \frac{10}{3}, \frac{8}{3}, \frac{6}{3}, \frac{4}{3}, \dots$

SEE EXAMPLE 2 p. 880

Find the 8th term of each arithmetic sequence.

5. 3, 8, 13, 18, …
6. $10, 9\frac{3}{4}, 9\frac{1}{2}, 9\frac{1}{4}, \dots$
7. −3.2, −3.4, −3.6, −3.8, …

SEE EXAMPLE 3 p. 881

Find the missing terms in each arithmetic sequence.

8. 13, $\square$, $\square$, 25
9. 9, $\square$, $\square$, $\square$, 37
10. 1.4, $\square$, $\square$, $\square$, −1

SEE EXAMPLE 4 p. 881

Find the 9th term of each arithmetic sequence.

11. $a_4 = 27$ and $a_5 = 19$
12. $a_3 = 12.2$ and $a_4 = 12.6$
13. $a_3 = -5$ and $a_6 = -11$
14. $a_{10} = 100$ and $a_{20} = 50$
15. $a_7 = -42$ and $a_{11} = -28$
16. $a_4 = \frac{3}{4}$ and $a_8 = \frac{1}{2}$

SEE EXAMPLE 5 p. 882

Find the indicated sum for each arithmetic series.

17. S_{15} for $5 + 9 + 13 + 17 + \cdots$
18. $\sum_{k=1}^{12}(-2 + 6k)$
19. S_{18} for $3.2 + 2.9 + 2.6 + 2.3 + \cdots$

SEE EXAMPLE 6 p. 883

20. **Salary** Juan has taken a job with an initial salary of \$26,000 and annual raises of \$1250.
    a. What will his salary be in his 6th year?
    b. How much money in total will Juan have earned after six years?

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
21–23	1
24–26	2
27–29	3
30–32	4
33–35	5
36	6

Extra Practice
Skills Practice p. S26
Application Practice p. S43

Determine whether each sequence could be arithmetic. If so, find the common difference and the next term.

21. 288, 144, 72, 36, 18, …
22. −2, −12, −22, −32, −42, …
23. 0.99, 0.9, 0.81, 0.72, …

Find the 11th term of each arithmetic sequence.

24. 12, 11.9, 11.8, 11.7, …
25. $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \dots$
26. −3.0, −2.5, −2.0, −1.5, …

Find the missing terms in each arithmetic sequence.

27. 77, $\square$, $\square$, $\square$, 33
28. −29, $\square$, $\square$, −2
29. 2.3, $\square$, $\square$, $\square$, 1.5

Find the 12th term of each arithmetic sequence.

30. $a_4 = 18.4$ and $a_5 = 16.2$
31. $a_4 = -2$ and $a_8 = 46$
32. $a_{22} = -49$ and $a_{25} = -58$

Find the indicated sum for each arithmetic series.

33. S_{15} for $-18 + (-16) + (-14) + \cdots$
34. $\sum_{k=1}^{20}(88 - 3k)$
35. $\sum_{k=1}^{14}\left(14 - \frac{1}{2}k\right)$

36. Consumer Economics Clarissa is buying a prom dress on layaway. She agrees to make a $15 payment and increase the payment by $5 each week.

a. What will her payment be in the 9th week?

b. How much money in total will Clarissa have paid after 9 weeks?

LINK

Clocks

One of London's most recognizable landmarks, Big Ben sits atop the Palace of Westminister, where the British Parliament meets. The name Big Ben actually refers to the 13.8-ton bell that chimes the hours.

37. Clocks A clock chimes every hour. The clock chimes once at 1 o'clock, twice at 2 o'clock, and so on.

a. How many times will the clock chime from 1 P.M. through midnight? in exactly one 24-hour period?

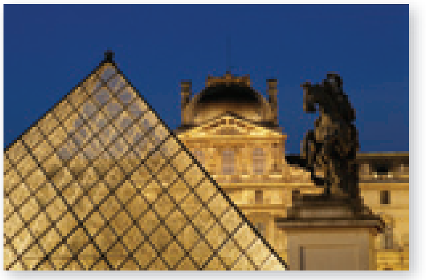
b. **What if...?** Another clock also chimes once on every half hour. How does this affect the sequence and the total number of chimes per day?

Find the indicated sum for each arithmetic series.

38. $\sum_{k=1}^{16}(555 - 11k)$ **39.** $\sum_{k=1}^{15}(4 - 0.5k)$ **40.** $\sum_{k=1}^{18}\left(-33 + \frac{5}{2}k\right)$

41. S_{16} for $7.5 + 7 + 6.5 + 6.0 + \cdots$ **42.** S_{18} for $2 + 9 + 16 + 23 + \cdots$

43. Architecture The Louvre pyramid in Paris, France, is built of glass panes. There are 4 panes in the top row, and each additional row has 4 more panes than the previous row.



a. Write a series in summation notation to describe the total number of glass panes in n rows of the pyramid.

b. If the pyramid were made of 18 complete rows, how many panes would it have?

c. The actual pyramid has 11 panes less than a complete 18-row pyramid because of the space for the entrance. Find the total number of panes in the Louvre pyramid.

44. Physics Water towers are tall to provide enough water pressure to supply all of the houses and businesses in the area of the tower. Each foot of height provides 0.43 psi (pounds per square inch) of pressure.

a. Write a sequence for the pressure in psi for each foot of height.

b. What is the minimum height that supplies 50 psi, a typical minimum supply pressure?

c. What is the minimum height that supplies 100 psi, which is a typical maximum pressure?

d. Graph the sequence, and discuss the relationship between the height found for a pressure of 50 psi and the height found at 100 psi.

45. Exercise Sheila begins an exercise routine for 20 minutes each day. Each week she plans to add 5 minutes per day to the length of her routine.

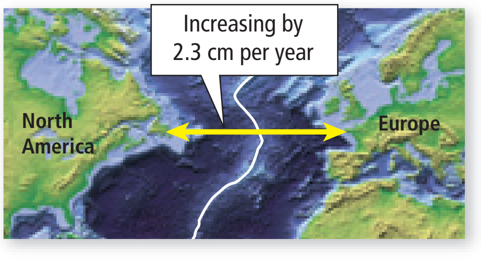
a. For how many minutes will she exercise each day of the 6th week?

b. What happens to the length of Sheila's exercise routine if she continues this increasing pattern for 2 years?

46. Geology Every year the continent of North America moves farther away from Europe.

a. How much farther from Europe will North America be in 50 years?

b. How many years until an extra mile is added? (*Hint:* 1 mi ≈ 1609 m)



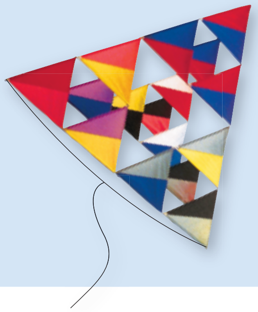
MULTI-STEP TEST PREP

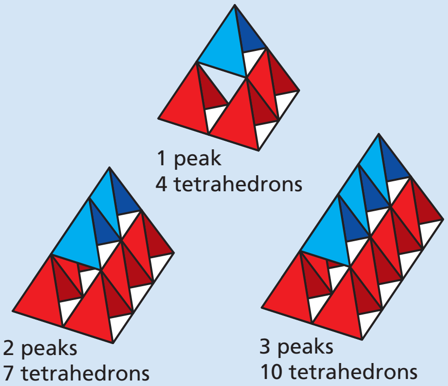
47. This problem will prepare you for the Multi-Step Test Prep on page 888.

You can make a simple tetrahedral kite with one peak by using 4 tetrahedrons. You can also make long kites with multiple peaks by successively adding 3 tetrahedrons as shown.

a. How many tetrahedrons are needed to make a kite with 20 peaks?

b. A kite maker wants to build one example of every kite with 1 to 20 peaks. How many tetrahedrons will be needed?



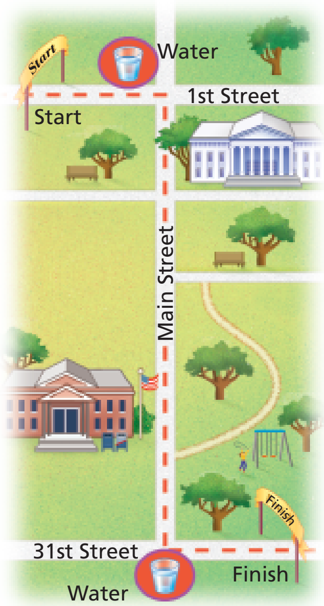


48. **Finance** The starting salary for a summer camp counselor is \$395 per week. In each of the subsequent weeks, the salary increases by \$45 to encourage experienced counselors to work for the entire summer. If the salary is \$710 in the last week of the camp, for how many weeks does the camp run?

49. **Sports** A town is planning a 5K race. The race route will begin at 1st street, travel 30 blocks down Main Street, and finish on 31st Street. The race planners want to have water stations at each turn. In addition, they want to place 5 more water stations evenly distributed between 1st Street and 31st Street on Main Street.

a. At what street intersections should the water stations be placed?

b. If each block is 0.1 mile, what is the maximum distance a runner will be from a water station while on Main Street?



50. **Critical Thinking** What is the least number of terms you need to write the general rule for an arithmetic sequence? How many points do you need to write an equation of a line? Are these answers related? Explain.

51. **Write About It** An arithmetic sequence has a positive common difference. What happens to the nth term as n becomes greater and greater? What happens if the sequence has a negative common difference?

TEST PREP

52. Which sequence could be an arithmetic sequence?

Ⓐ $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Ⓑ 2.2, 4.4, 6.6, 8.8, …

Ⓒ 2, 4, 8, 16, …

Ⓓ 2, 4, 7, 11, …

53. A catering company charges a setup fee of \$45 plus \$12 per person. Which of the following sequences accurately reflects this situation?

Ⓕ $a_n = 45 + 12(n - 1)$

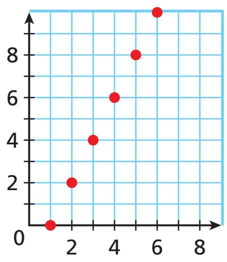
Ⓖ 45, 57, 69, 81, 93, …

Ⓗ $a_n = 57 + 12n$

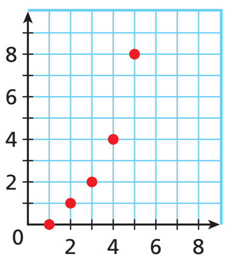
Ⓙ 57, 69, 81, 93, 105, …

54. Which graph might represent the terms of an arithmetic sequence?

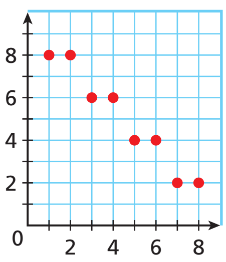
(A)



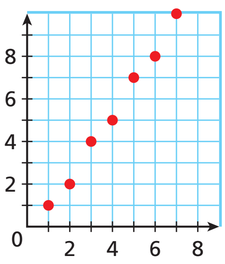
(B)



(C)



(D)



55. Given the arithmetic sequence 4, ■, ■, ■, 40, what are the three missing terms?

(F) 11, 22, 33 (H) 14, 24, 34

(G) 13, 22, 31 (J) 16, 24, 36

56. Which represents the sum of the arithmetic series 19 + 16 + 13 + 10 + 7 + 4?

(A) $\sum_{k=1}^{6} 19 - 3k$ (C) $\sum_{k=1}^{6} (22 - 3k)$

(B) $\sum_{k=1}^{6} 19 - 4k$ (D) $\sum_{k=1}^{6} [22 - 3(k - 1)]$

57. **Gridded Response** What is the 13th term of the arithmetic sequence 54, 50, 46, 42, ...

CHALLENGE AND EXTEND

58. Consider the two terms of an arithmetic series a_n and a_m.

   a. Show that the common difference is $d = \frac{a_n - a_m}{n - m}$.

   b. Use the new formula to find the common difference for the arithmetic sequence with $a_{12} = 88$ and $a_{36} = 304$.

59. Find a formula for the sum of an arithmetic sequence that does NOT include the last term. When might this formula be useful?

60. The sum of three consecutive terms of an arithmetic sequence is 60. If the product of these terms is 7500, what are the terms?

61. **Critical Thinking** What does $a_{2n} = 2a_n$ mean and for what arithmetic sequences is it true?

SPIRAL REVIEW

Tell whether the function shows growth or decay. Then graph. *(Lesson 7-1)*

62. $f(x) = 1.25(0.75)^x$ 63. $f(x) = 1.43(5.32)^x$ 64. $f(x) = 0.92(0.64)^x$

65. **Sound** The loudness of sound is given by $L = 10\log\left(\frac{I}{I_0}\right)$, where L is the loudness of sound in decibels, I is the intensity of sound, and I_0 is the intensity of the softest audible sound. A sound meter at an auto race had a relative intensity of $10^{9.2}\, I_0$. Find the loudness of the sound in decibels. *(Lesson 7-3)*

Write each series in summation notation. *(Lesson 12-2)*

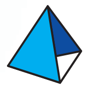
66. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ 67. $\frac{4}{5} + \frac{8}{5} + \frac{12}{5} + \frac{16}{5} + 4$

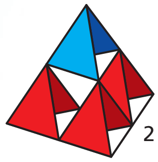
68. $-1 + 2 + 7 + 14 + 23$ 69. $-\frac{1}{3} - \frac{2}{3} - 1 - \frac{4}{3} - \frac{5}{3}$

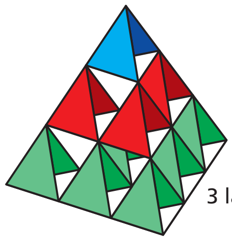
MULTI-STEP TEST PREP

Exploring Arithmetic Sequences and Series

Go Fly a Kite! Alexander Graham Bell, the inventor of the telephone, is also known for his work with tetrahedral kites. In 1902, Bell used the kites to prove that it is possible to build an arbitrarily large structure that will fly. The kites are made up of tetrahedrons (four-sided triangular figures) with two sides covered with fabric. As shown in the figure, the size of a tetrahedral kite is determined by how many layers it has.


1 layer


2 layers


3 layers

1. The first layer of a tetrahedral kite has 1 tetrahedron, the second layer has 3 tetrahedrons, and the third layer has 6 tetrahedrons. Write a sequence that shows how many tetrahedrons are in each of the first 10 layers.
2. Write a recursive formula for the sequence.
3. Write an explicit rule for the nth term of the sequence.
4. How many tetrahedrons are there in the 25th layer?
5. Write a series in summation notation that gives the total number of tetrahedrons in a kite with 25 layers.
6. Evaluate the series in problem 5 to find the total number of tetrahedrons in a kite with 25 layers. (*Hint:* Use the properties $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.)
7. You see someone flying a large tetrahedral kite at a kite festival. You look up and estimate that the bottom layer of the kite contains between 100 and 110 tetrahedrons. How many layers does the kite have? How many tetrahedrons did it take to build the kite?

READY TO GO ON?

CHAPTER 12

SECTION 12A

Quiz for Lessons 12-1 Through 12-3

12-1 Introduction to Sequences

Find the first 5 terms of each sequence.

1. $a_n = \frac{2}{3}n$

2. $a_n = 4^{n-1}$

3. $a_1 = -1$ and $a_n = 2a_{n-1} - 12$

4. $a_n = n^2 - 2n$

Write a possible explicit rule for the *n*th term of each sequence.

5. 8, 11, 14, 17, 20, ...

6. −2, −8, −18, −32, −50, ...

7. 1000, 200, 40, 8, $\frac{8}{5}$, ...

8. 437, 393, 349, 305, 261, ...

9. A car traveling at 55 mi/h passes a mile marker that reads mile 18. If the car maintains this speed for 4 hours, what mile marker should the car pass? Graph the sequence for n hours, and describe its pattern.

12-2 Series and Summation Notation

Expand each series and evaluate.

10. $\sum_{k=1}^{4}(-14 - 2k)$

11. $\sum_{k=1}^{4}\left(\frac{k}{k+2}\right)$

12. $\sum_{k=1}^{5}(-1)^k\left(k^2 - 2\right)$

Evaluate each series.

13. $\sum_{k=1}^{5}\frac{1}{2}$

14. $\sum_{k=1}^{40}k^2$

15. $\sum_{k=1}^{15}k$

16. The first row of a theater has 20 seats, and each of the following rows has 3 more seats than the preceding row. How many seats are in the first 12 rows?

12-3 Arithmetic Sequences and Series

Find the 8th term of each arithmetic sequence.

17. 10.00, 10.11, 10.22, 10.33, ...

18. −5, −13, −21, −29, ...

19. $a_2 = 57.5$ and $a_5 = 80$

20. $a_{10} = 141$ and $a_{13} = 186$

Find the missing terms in each arithmetic sequence.

21. −23, ▒, ▒, −89

22. 31, ▒, ▒, ▒, 79

Find the indicated sum for each arithmetic series

23. S_{10} for $40 + 30 + 20 + 10 + \cdots$

24. $\sum_{k=5}^{8}4k$

25. $\sum_{k=1}^{11}(0.5k + 5.5)$

26. S_{14} for $-6 - 1 + 4 + 9 + \cdots$

27. Suppose that you make a bank deposit of \$1 the first week, \$1.50 the second week, \$2 the third week, and so on. How much will you contribute to the account on the last week of the year (52nd week)? What is the total amount that you have deposited in the bank after one year?

12-4 Geometric Sequences and Series

Objectives

Find terms of a geometric sequence, including geometric means.

Find the sums of geometric series.

Vocabulary

geometric sequence
geometric mean
geometric series

Who uses this?

Sporting-event planners can use geometric sequences and series to determine the number of matches that must be played in a tournament. (See Example 6.)

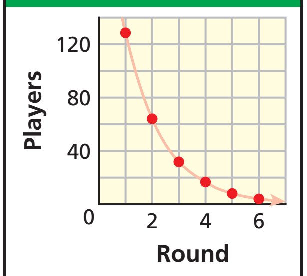
Serena Williams was the winner out of 128 players who began the 2003 Wimbledon Ladies' Singles Championship. After each match, the winner continues to the next round and the loser is eliminated from the tournament. This means that after each round only half of the players remain.

The number of players remaining after each round can be modeled by a *geometric sequence*. In a **geometric sequence**, the ratio of successive terms is a constant called the *common ratio* r $(r \neq 1)$. For the players remaining, r is $\frac{1}{2}$.

Term	a_1	a_2	a_3	a_4
Value	128	64	32	16

Ratios $\frac{64}{128} = \frac{1}{2}$ $\frac{32}{64} = \frac{1}{2}$ $\frac{16}{32} = \frac{1}{2}$

Players in Each Round of Wimbledon



Recall that exponential functions have a common ratio. When you graph the ordered pairs (n, a_n) of a geometric sequence, the points lie on an exponential curve as shown. Thus, you can think of a geometric sequence as an exponential function with sequential natural numbers as the domain.

EXAMPLE 1 Identifying Geometric Sequences

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

A 8, 12, 18, 27, ...

8, 12, 18, 27

Diff. 4 6 9

Ratio $\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$

It could be geometric, with $r = \frac{3}{2}$.

B 8, 16, 24, 32, ...

8, 16 24, 32

Diff. 8 8 8

Ratio 2 $\frac{3}{2}$ $\frac{4}{3}$

It could be arithmetic, with $d = 8$.

C 6, 10, 15, 21, ...

6, 10, 15, 21

Diff. 4 5 6

Ratio $\frac{5}{3}$ $\frac{3}{2}$ $\frac{7}{5}$

It is neither.

CHECK IT OUT! **Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.**

1a. $\frac{1}{4}, \frac{1}{12}, \frac{1}{36}, \frac{1}{108}, \ldots$

1b. 1.7, 1.3, 0.9, 0.5, ...

1c. −50, −32, −18, −8, ...

Each term in a geometric sequence is the product of the previous term and the common ratio, giving the recursive rule for a geometric sequence.

$$n\text{th term} \longrightarrow a_n = \underbrace{a_{n-1}}_{\text{First term}} r \longleftarrow \text{Common ratio}$$

You can also use an explicit rule to find the *n*th term of a geometric sequence. Each term is the product of the first term and a power of the common ratio as shown in the table.

Tennis Players in Each Round of Wimbledon					
Round	1	2	3	4	n
Players	128	64	32	16	a_n
Formula	$a_1 = 128\left(\frac{1}{2}\right)^0$	$a_2 = 128\left(\frac{1}{2}\right)^1$	$a_3 = 128\left(\frac{1}{2}\right)^2$	$a_4 = 128\left(\frac{1}{2}\right)^3$	$a_n = 128\left(\frac{1}{2}\right)^{n-1}$

This pattern can be generalized into a rule for all geometric sequences.

Know it! Note

General Rule for Geometric Sequences

The *n*th term a_n of a geometric sequence is

$$a_n = a_1 r^{n-1},$$

where a_1 is the first term and r is the common ratio.

EXAMPLE 2 Finding the *n*th Term Given a Geometric Sequence

Find the 9th term of the geometric sequence −5, 10, −20, 40, −80, ...

Step 1 Find the common ratio.

$$r = \frac{a_2}{a_1} = \frac{10}{-5} = -2$$

Step 2 Write a rule, and evaluate for $n = 9$.

$a_n = a_1 r^{n-1}$ — *General rule*

$a_9 = -5(-2)^{9-1}$ — *Substitute −5 for a_1, 9 for n, and −2 for r.*

$= -5(256) = -1280$

The 9th term is −1280.

Check Extend the sequence.

$a_5 = -80$ — *Given*

$a_6 = -80(-2) = 160$

$a_7 = 160(-2) = -320$

$a_8 = -320(-2) = 640$

$a_9 = 640(-2) = -1280$ ✔

CHECK IT OUT!

Find the 9th term of each geometric sequence.

2a. $\frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, -\frac{3}{32}, \frac{3}{64}, \ldots$

2b. $0.001, 0.01, 0.1, 1, 10, \ldots$

EXAMPLE 3 Finding the nth Term Given Two Terms

Find the 10th term of the geometric sequence with $a_5 = 96$ and $a_7 = 384$.

Step 1 Find the common ratio.

$a_7 = a_5 r^{(7-5)}$ *Use the given terms.*

$a_7 = a_5 r^2$ *Simplify.*

$384 = 96r^2$ *Substitute 384 for a_7 and 96 for a_5.*

$4 = r^2$ *Divide both sides by 96.*

$\pm 2 = r$ *Take the square root of both sides.*

Step 2 Find a_1.

Consider both the positive and negative values for r.

$a_n = a_1 r^{n-1}$		$a_n = a_1 r^{n-1}$	*General rule*
$96 = a_1(2)^{5-1}$	or	$96 = a_1(-2)^{5-1}$	*Use $a_5 = 96$ and $r = \pm 2$.*
$6 = a_1$		$6 = a_1$	

Step 3 Write the rule and evaluate for a_{10}.

Consider both the positive and negative values for r.

$a_n = a_1 r^{n-1}$		$a_n = a_1 r^{n-1}$	*General rule*
$a_n = 6(2)^{n-1}$	or	$a_n = 6(-2)^{n-1}$	*Substitute for a_1 and r.*
$a_{10} = 6(2)^{10-1}$		$a_{10} = 6(-2)^{10-1}$	*Evaluate for $n = 6$.*
$a_{10} = 3072$		$a_{10} = -3072$	

The 10th term is 3072 or -3072.

Caution!

When given two terms of a sequence, be sure to consider positive and negative values for r when necessary.

CHECK IT OUT! **Find the 7th term of the geometric sequence with the given terms.**

3a. $a_4 = -8$ and $a_5 = -40$ **3b.** $a_2 = 768$ and $a_4 = 48$

Geometric means are the terms between any two nonconsecutive terms of a geometric sequence.

Geometric Mean

If a and b are positive terms of a geometric sequence with exactly one term between them, the geometric mean is given by the following expression.

$$\sqrt{ab}$$

EXAMPLE 4 Finding Geometric Means

Find the geometric mean of $\frac{1}{2}$ and $\frac{1}{32}$.

$$\sqrt{ab} = \sqrt{\left(\frac{1}{2}\right)\left(\frac{1}{32}\right)}$$

$$= \sqrt{\frac{1}{64}} = \frac{1}{8} \quad \textit{Use the formula.}$$

CHECK IT OUT! **4.** Find the geometric mean of 16 and 25.

The indicated sum of the terms of a geometric sequence is called a **geometric series**. You can derive a formula for the partial sum of a geometric series by subtracting the product of S_n and r from S_n as shown.

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} \\ -rS_n &= \quad\;\; - a_1r - a_1r^2 - \cdots - a_1r^{n-1} - a_1r^n \\ \hline S_n - rS_n &= a_1 \qquad\qquad\qquad\qquad\qquad\qquad - a_1r^n \\ S_n(1-r) &= a_1(1-r^n) \\ S_n &= a_1\left(\frac{1-r^n}{1-r}\right) \end{aligned}$$

Know it! Note

Sum of the First n Terms of a Geometric Series

The partial sum S_n of the first n terms of a geometric series $a_1 + a_2 + \cdots + a_n$ is given by

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right),\ r \neq 1$$

where a_1 is the first term and r is the common ratio.

EXAMPLE 5 Finding the Sum of a Geometric Series

Find the indicated sum for each geometric series.

A S_7 for $3 - 6 + 12 - 24 + \cdots$

Step 1 Find the common ratio.

$$r = \frac{a_2}{a_1} = \frac{-6}{3} = -2$$

Step 2 Find S_7 with $a_1 = 3$, $r = -2$, and $n = 7$.

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \quad \textit{Sum formula}$$

$$S_7 = 3\left(\frac{1-(-2)^7}{1-(-2)}\right) \quad \textit{Substitute.}$$

$$= 3\left(\frac{1-(-128)}{3}\right)$$

$$= 129$$

Check Use a graphing calculator.

✔

B $\displaystyle\sum_{k=1}^{5}\left(\frac{1}{3}\right)^{k-1}$

Step 1 Find the first term.

$$a_1 = \left(\frac{1}{3}\right)^{1-1} = \left(\frac{1}{3}\right)^0 = 1$$

Step 2 Find S_5.

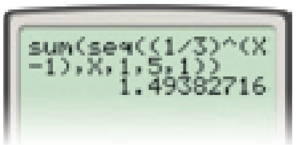
$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) \quad \textit{Sum formula}$$

$$S_5 = 1\left(\frac{1-\left(\frac{1}{3}\right)^5}{1-\left(\frac{1}{3}\right)}\right) \quad \textit{Substitute.}$$

$$= \left(\frac{1-\left(\frac{1}{243}\right)}{\frac{2}{3}}\right)$$

$$= \frac{242}{243}\cdot\frac{3}{2} = \frac{121}{81} \approx 1.49$$

Check Use a graphing calculator.



✔

Remember!

These sums are partial sums because you are finding the sum of a finite number of terms. In Lesson 12-5, you will find the sums of infinite geometric series.

CHECK IT OUT! **Find the indicated sum for each geometric series.**

5a. S_6 for $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ **5b.** $\displaystyle\sum_{k=1}^{6} -3(2)^{k-1}$

EXAMPLE 6 ***Sports Application***

The Wimbledon Ladies' Singles Championship begins with 128 players. The players compete until there is 1 winner. How many matches must be scheduled in order to complete the tournament?

Step 1 Write a sequence.

Let n = the number of rounds,

a_n = the number of matches played in the nth round, and

S_n = the total number of matches played through n rounds.

$a_n = 64\left(\frac{1}{2}\right)^{n-1}$ *The first round requires 64 matches, so $a_1 = 64$. Each successive match requires $\frac{1}{2}$ as many, so $r = \frac{1}{2}$.*

Step 2 Find the number of rounds required.

$1 = 64\left(\frac{1}{2}\right)^{n-1}$ *The final round will have 1 match, so substitute 1 for a_n.*

$\frac{1}{64} = \left(\frac{1}{2}\right)^{n-1}$ *Isolate the exponential expression by dividing by 64.*

$\left(\frac{1}{2}\right)^{6} = \left(\frac{1}{2}\right)^{n-1}$ *Express $\frac{1}{64}$ as a power of $\frac{1}{2}$: $\frac{1}{64} = \left(\frac{1}{2}\right)^6$.*

$6 = n - 1$ *Equate the exponents.*

$7 = n$ *Solve for n.*

Step 3 Find the total number of matches after 7 rounds.

$$S_7 = 64\left(\frac{1-\left(\frac{1}{2}\right)^7}{1-\left(\frac{1}{2}\right)}\right) = 127$$ *Sum function for geometric series.*

127 matches must be scheduled to complete the tournament.

Remember!

For a review of solving exponential equations, see Lesson 7-5.

CHECK IT OUT! **6. Real Estate** A 6-year lease states that the annual rent for an office space is $84,000 the first year and will increase by 8% each additional year of the lease. What will the total rent expense be for the 6-year lease?

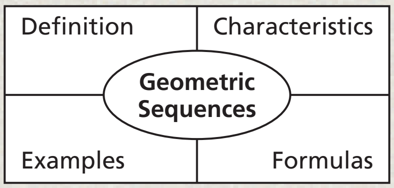
THINK AND DISCUSS

1. Find the next three terms of the geometric sequence that begins 3, 6, ... Then find the next three terms of the arithmetic sequence that begins 3, 6, ...

2. Compare the geometric mean of 4 and 16 with the mean, or average.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. In each box, summarize your understanding of geometric sequences.



12-4 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 12-4
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** The term between two given terms in a geometric sequence is the __?__. (*geometric mean* or *geometric series*)

SEE EXAMPLE 1 p. 890

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

2. $-10, -12, -14, -16, \ldots$
3. $\frac{1}{2}, 1, 2, 3, \ldots$
4. $-320, -80, -20, -5, \ldots$

SEE EXAMPLE 2 p. 891

Find the 10th term of each geometric sequence.

5. $2, 6, 18, 54, 162, \ldots$
6. $5000, 500, 50, 5, 0.5, \ldots$
7. $-0.125, 0.25, -0.5, 1, -2, \ldots$

SEE EXAMPLE 3 p. 892

Find the 6th term of the geometric sequence with the given terms.

8. $a_4 = -12, a_5 = -4$
9. $a_2 = 4, a_5 = 108$
10. $a_3 = 3, a_5 = 12$

SEE EXAMPLE 4 p. 892

Find the geometric mean of each pair of numbers.

11. 6 and $\frac{3}{8}$
12. 2 and 32
13. 12 and 192

SEE EXAMPLE 5 p. 893

Find the indicated sum for each geometric series.

14. S_6 for $2 + 0.2 + 0.02 + \cdots$
15. $\sum_{k=1}^{5} (-3)^{k-1}$
16. S_5 for $12 - 24 + 48 - 96 + \cdots$
17. $\sum_{k=1}^{9} 256\left(\frac{1}{2}\right)^{k-1}$

SEE EXAMPLE 6 p. 894

18. **Salary** In his first year, a math teacher earned \$32,000. Each successive year, he earned a 5% raise. How much did he earn in his 20th year? What were his total earnings over the 20-year period?

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
19–21	1
22–25	2
26–28	3
29–31	4
32–35	5
36	6

Extra Practice

Skills Practice p. S27
Application Practice p. S43

Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference.

19. $-36, -49, -64, -81, \ldots$
20. $-2, -6, -18, -54, \ldots$
21. $2, 7, 12, 17, \ldots$

Find the 9th term of each geometric sequence.

22. $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \frac{1}{250}, \frac{1}{1250}, \ldots$
23. $3, -6, 12, -24, 48, \ldots$
24. $3200, 1600, 800, 400, 200, \ldots$
25. $8, 24, 72, 216, 648, \ldots$

Find the 7th term of the geometric sequence with the given terms.

26. $a_4 = 54, a_5 = 162$
27. $a_5 = 13.5, a_6 = 20.25$
28. $a_4 = -4, a_6 = -100$

Find the geometric mean of each pair of numbers.

29. 9 and $\frac{1}{9}$
30. 18 and 2
31. $\frac{1}{5}$ and 45

Find the indicated sum for each geometric series.

32. S_6 for $1 + 5 + 25 + 125 + \cdots$
33. S_8 for $10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots$
34. $\sum_{k=1}^{6} -1\left(\frac{1}{3}\right)^{k-1}$
35. $\sum_{k=1}^{7} 8(10)^{k-1}$

36. **Genealogy** You have 2 biological parents, 4 biological grandparents, and 8 biological great grandparents.

   a. How many direct ancestors do you have in the 6 generations before you? 12 generations?

   b. **What if...?** How does the explicit rule change if you are considered the first generation?

Given each geometric sequence, (a) write an explicit rule for the sequence, (b) find the 10th term, and (c) find the sum of the first 10 terms.

37. $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$

38. $4, 0.4, 0.04, 0.004, \ldots$

39. $8, 16, 32, 64, \ldots$

40. $-22, -11, -\frac{11}{2}, -\frac{11}{4}, \ldots$

41. $162, -54, 18, -6, \ldots$

42. $12.5, 62.5, 312.5, 1562.5, \ldots$

LINK

Collectibles

The most expensive rock'n'roll memorabilia ever purchased was John Lennon's Rolls Royce, with a price tag of $2.2 million.

43. **Collectibles** Louis bought a vintage Rolling Stones concert shirt for $20. He estimates that the shirt will increase in value by 15% per year.

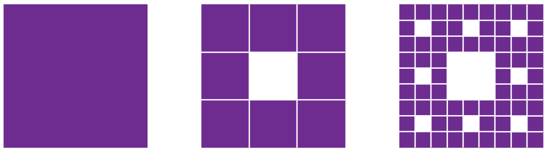
   a. How much is the shirt worth after 4 years? after 8 years?

   b. Does the shirt increase more in value during the first 4 years or the second 4 years? Explain.

44. **College Tuition** New grandparents decide to pay for their granddaughter's college education. They give the girl a penny on her first birthday and double the gift on each subsequent birthday. How much money will the girl receive when she is 18? 21? Will the money pay for her college education? Explain.

45. **Technology** You receive an e-mail asking you to forward it to 5 other people to ensure good luck. Assume that no one breaks the chain and that there are no duplications among the recipients. How many e-mails will have been sent after 10 generations, including yours, have received and sent the e-mail?

46. **Fractals** The Sierpinski carpet is a fractal based on a square. In each iteration, the center of each shaded square is removed.



   a. Given that the area of the original square is 1 square unit, write a sequence for the area of the nth iteration of the Sierpinski carpet.

   b. In which iteration will the area be less than $\frac{1}{2}$ of the original area?

47. **Paper** A piece of paper is 0.1 mm thick. When folded, the paper is twice as thick.

   a. Studies have shown that you can fold a piece of paper a maximim of 7 times. How thick will the paper be if it is folded on top of itself 7 times?

   b. Assume that you could fold the paper as many times as you want. How many folds would be required for the paper to be taller than Mount Everest (8850 m)?

48. **Measurement** Several common U.S. paper sizes are shown in the table.

   a. Examine the length and width measures for the different paper sizes. What interrelationships do you observe?

   b. How are the areas for each paper size (from A to E) mathematically related and what name is this relationship given?

Common U.S. Paper Sizes

U.S. Paper Size	Dimensions (in.)
A (letter)	$8\frac{1}{2} \times 11$
B (ledger)	11×17
C	17×22
D	22×34
E	34×44

MULTI-STEP TEST PREP

49. This problem will prepare you for the Multi-Step Test Prep on page 908.
A movie earned \$60 million in its first week of release and \$9.6 million in the third week of release. The sales each week can be modeled by a geometric sequence.
a. Estimate the movie's sales in its second week of release.
b. By what percent did the sales decrease each week?
c. In what week would you expect sales to be less than \$1 million?
d. Estimate the movie's total sales during its 8-week release period.

50. Biology The population growth of bacteria in a petri dish each hour creates a geometric sequence. After 1 hour there were 4 bacteria cells, and after 5 hours there were 324 cells. How many cells were found at hours 2, 3, and 4?

51. Critical Thinking Find an arithmetic sequence, a geometric sequence, and a sequence that is neither arithmetic nor geometric that begins 1, 4, ….

52. Finance Suppose that you pay \$750 in rent each month. Suppose also that your rent is increased by 10% each year thereafter.
a. Write a series that describes the total rent paid each year over the first 5 years, and find its sum.
b. Use sigma notation to represent the series for the total rent paid each year over the first 10 years, and evaluate it.

53. Music The frequencies produced by playing C notes in ascending octaves make up a geometric sequence. C0 is the lowest C note audible to the human ear.
a. The note commonly called middle C is C4. Find the frequency of middle C.
b. Write a geometric sequence for the frequency of C notes in hertz where $n = 1$ represents C1.
c. Humans cannot hear sounds with frequencies greater than 20,000 Hz. What is the first C note that humans cannot hear?

Scale of C's

Note	Frequency (Hz)
C0	16.24
C1	32.7
C2	65.4
C3	130.8
C4	

54. Medicine During a flu outbreak, a hospital recorded 16 cases the first week, 56 cases the second week, and 196 cases the third week.
a. Write a geometric sequence to model the flu outbreak.
b. If the hospital did nothing to stop the outbreak, in which week would the total number infected exceed 10,000?

55. Graphing Calculator Use the **SEQ** and **SUM** features to find each indicated sum of the geometric series $8 + 6 + 4.5 + \cdots$ to the nearest thousandth.
a. S_{10} **b.** S_{20} **c.** S_{30} **d.** S_{40}
e. Does the series appear to be approaching any particular value? Explain.

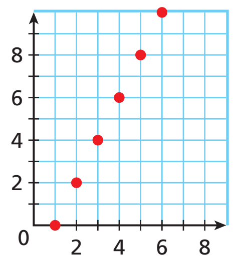
56. Critical Thinking If a geometric sequence has $r > 1$, what happens to the terms as n increases? What happens if $0 < r < 1$?

57. Write About It What happens to the terms of a geometric sequence when the first term is tripled? What happens to the sum of this geometric sequence?

TEST PREP

58. Find the sum of the first 6 terms for the geometric series $4.5 + 9 + 18 + 36 + \cdots$.
Ⓐ 67.5 Ⓑ 144 Ⓒ 283.5 Ⓓ 445.5

59. Which graph might represent the terms of a geometric sequence?

Ⓕ

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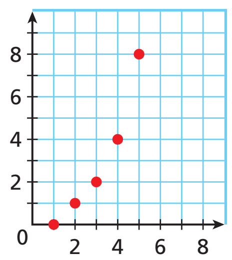
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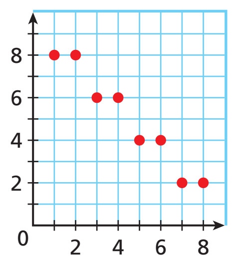
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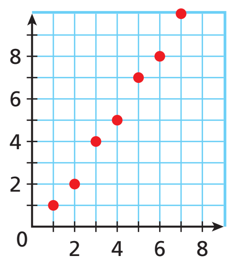


60. Find the first 3 terms of the geometric sequence with $a_7 = -192$ and $a_9 = -768$.

Ⓐ 3, −6, 12

Ⓑ −3, 12, −48

Ⓒ −3, 6, −12 or −3, −6, −12

Ⓓ 3, −12, 48 or −3, −12, −48

61. Which represents the sum of the series 10 − 15 + 22.5 − 33.75 + 50.625?

Ⓕ $\sum_{k=1}^{5} 10\left(\frac{3}{2}\right)^{k-1}$

Ⓖ $\sum_{k=1}^{5} 10\left(-\frac{3}{2}\right)^{k-1}$

Ⓗ $\sum_{k=1}^{5} -10\left(\frac{3}{2}\right)^{k-1}$

Ⓙ $\sum_{k=1}^{5} 10\left(-\frac{3}{2}\right)^{k}$

62. **Short Response** Why does the general rule for a geometric sequence use $n - 1$ instead of n? Explain.

CHALLENGE AND EXTEND

Graphing Calculator For each geometric sequence, find the first term with a value greater than 1,000,000.

63. $a_1 = 10$ and $r = 2$

64. $a_1 = \frac{1}{4}$ and $r = 4$

65. $a_1 = 0.01$ and $r = 3.2$

66. The sum of three consecutive terms of a geometric sequence is 73.5. If the product of these terms is 2744, what are the terms?

67. Consider the geometric sequence whose first term is 55 with the common ratio $\frac{1+\sqrt{5}}{2}$.
   a. Find the next 5 terms rounded to the nearest integer.
   b. Add each pair of successive terms together. What do you notice?
   c. Make a conjecture about this sequence.

SPIRAL REVIEW

Identify the zeros and asymptotes of each function. *(Lesson 8-2)*

68. $f(x) = \frac{x^2 + 2x - 3}{x + 1}$

69. $f(x) = \frac{x + 5}{x^2 - x - 6}$

70. $f(x) = \frac{x^2 - 16}{4x}$

71. **Shopping** During a summer sale, a store gives a 20% discount on all merchandise. On Mondays, the store takes another 10% off of the sale price. *(Lesson 9-4)*
   a. Write a composite function to represent the cost on Monday of an item with an original price of x dollars.
   b. Find the cost on Monday of an item originally priced at \$275.

Find the 10th term of each arithmetic sequence. *(Lesson 12-3)*

72. 78, 65, 52, 39, 26, …

73. 1.7, 7.3, 12.9, 18.5, 24.1, …

74. 9.42, 9.23, 9.04, 8.85, 8.66, …

75. 16.4, 26.2, 36, 45.8, 55.6, …

12-5 Algebra LAB

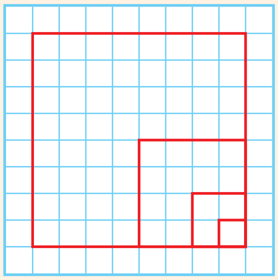
Use with Lesson 12-5

Explore Infinite Geometric Series

You can explore infinite geometric series by using a sequence of squares.

Activity

1. On a piece of graph paper, draw a 16×16 unit square. Note that its perimeter is 64 units.
2. Starting at one corner of the original square, draw a new square with side lengths half as long, or in this case, 8×8 units. Note that its perimeter is 32 units.
3. Create a table as shown at right. Fill in the perimeters and the cumulative sum of the perimeters that you have found so far.



Square	Perimeter	Sum
16×16	64	64
8×8	32	96
4×4	■	■
2×2	■	■
1×1	■	■
$\frac{1}{2} \times \frac{1}{2}$	■	■

Try This

1. Copy the table, and complete the first 6 rows.
2. Use summation notation to write a geometric series for the perimeters.
3. Use a graphing calculator to find the sum of the first 20 terms of the series.
4. **Make a Conjecture** Make a conjecture about the sum of the perimeter series if it were to continue indefinitely.
5. Evaluate $\frac{64}{1 - \frac{1}{2}}$. How does this relate to your answer to Problem 4?
6. Copy and complete the table by finding the area of each square and the cumulative sums.
7. Use summation notation to write a geometric series for the areas.
8. Use a graphing calculator to find the sum of the first 10 terms of the series.
9. **Make a Conjecture** Make a conjecture about the sum of the area series if it were to continue indefinitely.
10. Evaluate $\frac{256}{1 - \frac{1}{4}}$. How does this relate to your answer to Problem 9?
11. **Draw a Conclusion** Write a formula for the sum of an infinite geometric sequence.

Square	Area	Sum
16×16	■	■
8×8	■	■
4×4	■	■
2×2	■	■
1×1	■	■
$\frac{1}{2} \times \frac{1}{2}$	■	■

12-5 Mathematical Induction and Infinite Geometric Series

Objectives

Find sums of infinite geometric series.

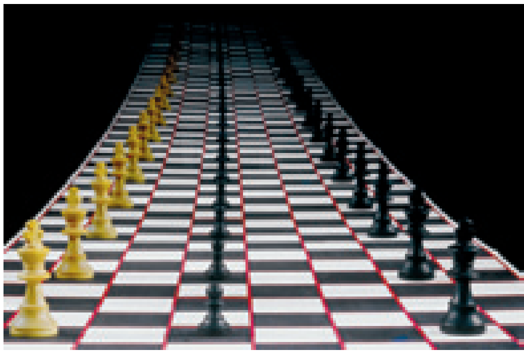
Use mathematical induction to prove statements.

Vocabulary

infinite geometric series
converge
limit
diverge
mathematical induction

Why learn this?

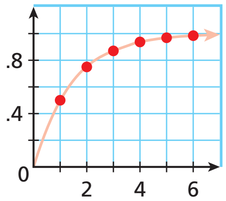
You can use infinite geometric series to explore repeating patterns. (See Exercise 58.)



In Lesson 12-4, you found partial sums of geometric series. You can also find the sums of some infinite geometric series. An **infinite geometric series** has infinitely many terms. Consider the two infinite geometric series below.

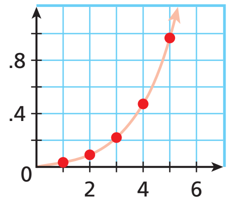
$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

Partial Sums						
n	1	2	3	4	5	6
S_n	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$



$$R_n = \frac{1}{32} + \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots$$

Partial Sums						
n	1	2	3	4	5	6
R_n	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{7}{32}$	$\frac{15}{32}$	$\frac{31}{32}$	$\frac{63}{32}$



Notice that the series S_n has a common ratio of $\frac{1}{2}$ and the partial sums get closer and closer to 1 as n increases. When $|r| < 1$ and the partial sum approaches a fixed number, the series is said to **converge**. The number that the partial sums approach, as n increases, is called a **limit**.

For the series R_n, the opposite applies. Its common ratio is 2, and its partial sums increase toward infinity. When $|r| \geq 1$ and the partial sum does not approach a fixed number, the series is said to **diverge**.

EXAMPLE 1 Finding Convergent or Divergent Series

Determine whether each geometric series converges or diverges.

A $20 + 24 + 28.8 + 34.56 + \cdots$

$$r = \frac{24}{20} = 1.2, |r| \geq 1$$

The series diverges and does not have a sum.

B $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

$$r = \frac{\frac{1}{3}}{1} = \frac{1}{3}, |r| < 1$$

The series converges and has a sum.

CHECK IT OUT! **Determine whether each geometric series converges or diverges.**

1a. $\frac{2}{3} + 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$ **1b.** $32 + 16 + 8 + 4 + 2 + \cdots$

If an infinite series converges, we can find the sum. Consider the series $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$ from the previous page. Use the formula for the partial sum of a geometric series with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$.

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right) = \frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{2\left(\frac{1}{2}\right)} = \frac{1-\left(\frac{1}{2}\right)^n}{1} = 1-\left(\frac{1}{2}\right)^n$$

Graph the simplified equation on a graphing calculator. Notice that the sum levels out and converges to 1.

As n approaches infinity, the term $\left(\frac{1}{2}\right)^n$ approaches zero. Therefore, the sum of the series is 1. This concept can be generalized for all convergent geometric series and proved by using calculus.

Know it! Note

Sum of an Infinite Geometric Series

The sum of an infinite geometric series S with common ratio r and $|r| < 1$ is

$$S = \frac{a_1}{1-r},$$

where a_1 is the first term.

EXAMPLE 2 Finding the Sums of Infinite Geometric Series

Helpful Hint

You can graph a geometric series by using the sum formula from Lesson 12-4: $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$ with S_n as y and n as x and with values substituted for r and a_1.

Find the sum of each infinite geometric series, if it exists.

A $5 + 4 + 3.2 + 2.56 + \cdots$

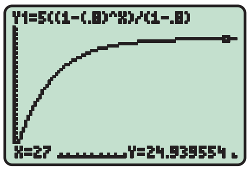
$r = 0.8$ *Converges: $|r| < 1$*

$S = \frac{a_1}{1-r}$ *Sum formula*

$= \frac{5}{1-0.8} = \frac{5}{0.2} = 25$

Check Graph $y = 5\left(\frac{1-(0.8)^x}{1-0.8}\right)$ on a graphing calculator. The graph approaches $y = 25$. ✔

31



0 28.2

0

B $\sum_{k=1}^{\infty} \frac{2}{3^{k-1}}$

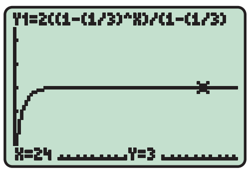
$\sum_{k=1}^{\infty} \frac{2}{3^{k-1}} = \frac{2}{1} + \frac{2}{3} + \frac{2}{9} + \cdots$ *Evaluate.*

$r = \frac{\frac{2}{3}}{2} = \frac{2}{6} = \frac{1}{3}$ *Converges: $|r| < 1$*

$S = \frac{a_1}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = \frac{6}{2} = 3$

Check Graph $y = 2\left(\frac{1-\left(\frac{1}{3}\right)^x}{1-\frac{1}{3}}\right)$ on a graphing calculator. The graph approaches $y = 3$. ✔

6.2



0 28.2

0

CHECK IT OUT! **Find the sum of each infinite geometric series, if it exists.**

2a. $25 - 5 + 1 - \frac{1}{5} + \frac{1}{25} - \cdots$ **2b.** $\sum_{k=1}^{\infty} \left(\frac{2}{5}\right)^k$

You can use infinite series to write a repeating decimal as a fraction.

EXAMPLE 3 Writing Repeating Decimals as Fractions

Write 0.232323… as a fraction in simplest form.

Step 1 Write the repeating decimal as an infinite geometric series.

$$0.232323\ldots = 0.23 + 0.0023 + 0.000023 + \cdots$$ *Use the pattern for the series.*

Step 2 Find the common ratio.

$$r = \frac{0.0023}{0.23}$$

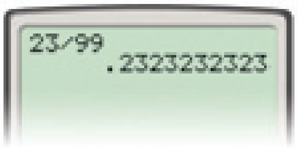
$$= \frac{1}{100}, \text{ or } 0.01$$ *$|r| < 1$; the series converges to a sum.*

Step 3 Find the sum.

$$S = \frac{a_1}{1 - r}$$ *Apply the sum formula.*

$$= \frac{0.23}{1 - 0.01} = \frac{0.23}{0.99} = \frac{23}{99}$$

Check Use a calculator to divide the fraction $\frac{23}{99}$. ✔



Remember!

Recall that every repeating decimal, such as 0.232323…, or $0.\overline{23}$, is a rational number and can be written as a fraction.

CHECK IT OUT! **3.** Write 0.111… as a fraction in simplest form.

You have used series to find the sums of many sets of numbers, such as the first 100 natural numbers. The formulas that you used for such sums can be proved by using a type of mathematical proof called **mathematical induction**.

Know it! Note

Proof by Mathematical Induction

To prove that a statement is true for all natural numbers *n*,

Step 1 The base case: Show that the statement is true for $n = 1$.

Step 2 Assume that the statement is true for a natural number *k*.

Step 3 Prove that the statement is true for the natural number $k + 1$.

EXAMPLE 4 Proving with Mathematical Induction

Use mathematical induction to prove that the sum of the first *n* natural numbers is $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Step 1 Base case: Show that the statement is true for $n = 1$.

$$1 = \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$ *The base case is true.*

Step 2 Assume that the statement is true for a natural number *k*.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$ *Replace n with k.*

Step 3 Prove that it is true for the natural number $k + 1$.

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \quad \textit{Add the next term } (k+1) \textit{ to each side.}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \quad \textit{Find the common denominator.}$$

$$= \frac{k(k+1) + 2(k+1)}{2} \quad \textit{Add numerators.}$$

$$= \frac{(k+1)(k+2)}{2} \quad \textit{Factor out } k+1.$$

$$= \frac{(k+1)[(k+1)+1]}{2} \quad \textit{Write with } k+1.$$

Therefore, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

CHECK IT OUT!

4. Use mathematical induction to prove that the sum of the first n odd numbers is $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Mathematical statements that seem to be true may in fact be false. By finding a counterexample, you can disprove a statement.

Helpful Hint

Often counterexamples can be found using special numbers like 1, 0, negative numbers, or fractions.

EXAMPLE 5 Using Counterexamples

Identify a counterexample to disprove $2^n \geq n^2$, where n is a real number.

$2^0 \geq (0)^2$	$2^1 \geq (1)^2$	$2^4 \geq (4)^2$	$2^{-1} \geq (-1)^2$
$1 \geq 0$ ✔	$2 \geq 1$ ✔	$24 \geq 16$ ✔	$\frac{1}{2} \geq 1$ ✘

$2^n \geq n^2$ is not true for $n = -1$, so it is not true for all real numbers.

CHECK IT OUT!

5. Identify a counterexample to disprove $\frac{a^2}{2} \leq 2a + 1$, where a is a real number.

THINK AND DISCUSS

1. Explain how to determine whether a geometric series converges or diverges.

2. Explain how to represent the repeating decimal $0.8\overline{3}$ as an infinite geometric series.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. Summarize the different infinite geometric series.

	Example	Common Ratio	Sum
Convergent Series			
Divergent Series			

12-5 Exercises

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Homework Help Online
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Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** An infinite geometric series whose sum approaches a fixed number is said to __?__. (*converge* or *diverge*)

SEE EXAMPLE 1 p. 900

Determine whether each geometric series converges or diverges.

2. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$
3. $1 - 5 + 25 - 125 + 625 + \cdots$
4. $27 + 18 + 12 + 8 + \cdots$

SEE EXAMPLE 2 p. 901

Find the sum of each infinite geometric series, if it exists.

5. $\frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots$
6. $\sum_{k=1}^{\infty} 4(0.25)^k$
7. $800 + 200 + 50 + \cdots$

SEE EXAMPLE 3 p. 902

Write each repeating decimal as a fraction in simplest form.

8. $0.888\ldots$
9. $0.\overline{56}$
10. $0.131313\ldots$

SEE EXAMPLE 4 p. 902

11. Use mathematical induction to prove that the sum of the first n even numbers is $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

SEE EXAMPLE 5 p. 903

Identify a counterexample to disprove each statement, where n is a real number.

12. $n^4 \geq 1$
13. $\log n > 0$
14. $n^3 \leq 3n^2$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
15–17	1
18–20	2
21–23	3
24	4
25–27	5

Extra Practice
Skills Practice p. S27
Application Practice p. S43

Determine whether each geometric series converges or diverges.

15. $3 + \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} + \cdots$
16. $5 + 10 + 20 + 40 + \cdots$
17. $2 - 4 + 8 - 16 + 32 + \cdots$

Find the sum of each infinite geometric series, if it exists.

18. $\sum_{k=1}^{\infty} 60\left(\frac{1}{10}\right)^k$
19. $\frac{8}{5} - \frac{4}{5} + \frac{2}{5} - \frac{1}{5} + \cdots$
20. $\sum_{k=1}^{\infty} 3.5^k$

Write each repeating decimal as a fraction in simplest form.

21. $0.\overline{6}$
22. $0.90909\ldots$
23. $0.541541541\ldots$

24. Use mathematical induction to prove
$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

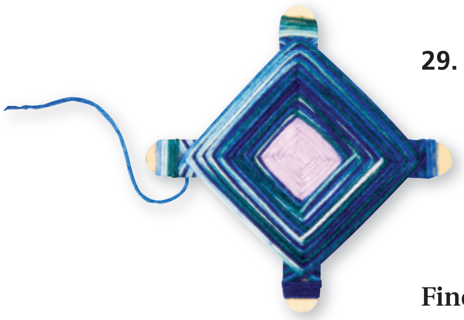
Identify a counterexample to disprove each statement, where a is a real number.

25. $a^3 \neq -a^2$
26. $a^4 > 0$
27. $5a^2 > 2^a$

28. **ERROR ANALYSIS** Two possible sums for the series $\frac{1}{5} + \frac{2}{5} + \frac{4}{5} + \cdots$ are shown. Which is incorrect? Explain the error.

A
$S = \frac{\frac{1}{5}}{1 - 2} = -\frac{1}{5}$

B
no finite sum



29. **Art** Ojos de Dios are Mexican holiday decorations. They are made of yarn, which is wrapped around sticks in a repeated square pattern. Suppose that the side length of the outer square is 8 inches. The side length of each inner square is 90% of the previous square's length. How much yarn will be required to complete the decoration? (Assume that the pattern is represented by an infinite geometric series.)

Find the sum of each infinite geometric series, if it exists.

30. $215 - 86 + 34.4 - 13.76 + \cdots$

31. $500 + 400 + 320 + \cdots$

32. $8 - 10 + 12.5 - 15.625 + \cdots$

33. $\sum_{k=1}^{\infty} -5\left(\frac{1}{8}\right)^{k-1}$

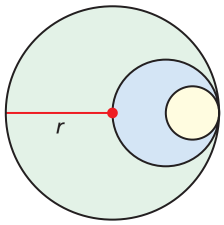
34. $\sum_{k=1}^{\infty} 2\left(\frac{1}{4}\right)^{k-1}$

35. $\sum_{k=1}^{\infty} \left(\frac{5}{3}\right)^{k-1}$

36. $-25 - 30 - 36 - 43.2 + \cdots$

37. $\sum_{k=1}^{n} 200(0.6)^{k-1}$

38. **Geometry** A circle of radius r has smaller circles drawn inside it as shown. Each smaller circle has half the radius of the previous circle.



   a. Write an infinite geometric series in terms of r that expresses the circumferences of the circles, and find its sum.
   b. Find the sum of the circumferences for the infinite set of circles if the first circle has a radius of 3 cm.

Write each repeating decimal as a fraction in simplest form.

39. $0.\overline{4}$

40. $0.\overline{9}$

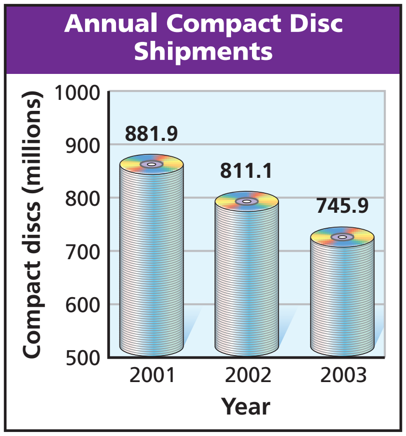
41. $0.\overline{123}$

42. $0.\overline{18}$

43. $0.\overline{5}$

44. $0.\overline{054}$

45. **Music** Due to increasing online downloads, CD sales have declined in recent years. Starting in 2001, the number of CDs shipped each year can be modeled by a geometric sequence.



   a. Estimate the number of CDs that will be shipped in 2010.
   b. Estimate the total number of CDs shipped from 2001 through 2010.
   c. Suppose that the geometric series continued indefinitely. Find the total number of CDs shipped from 2001.

Use mathematical induction to prove each statement.

46. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

47. $1 + 4 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

48. $1(2) + 2(3) + 2(4) + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

49. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^n$

MULTI-STEP TEST PREP

50. This problem will prepare you for the Multi-Step Test Prep on page 908.

A movie earned \$80 million in the first week that it was released. In each successive week, sales declined by about 40%.

a. Write a general rule for a geometric sequence that models the movie's sales each week.

b. Estimate the movie's total sales in the first 6 weeks.

c. If this pattern continued indefinitely, what would the movie's total sales be?

51. **Game Shows** Imagine that you have just won the grand prize on a game show. You can choose between two payment options as shown. Which would you choose, and why?

PRIZE PAYMENT OPTIONS

A.	B.
\$1 million the first year and half of the previous year's amount for eternity	\$100,000 a year for 20 years

Identify a counterexample to disprove each statement, where x is a real number.

52. $\frac{x^4}{x^3} \le 2x$

53. $x^4 - 1 \ge 0$

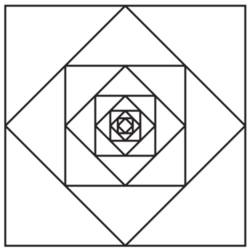
54. $\ln x^5 > \ln x$

55. $2x^2 \le 3x^3$

56. $2x^2 - x \ge 0$

57. $12x - x^2 > 25$

58. **Geometry** The midpoints of the sides of a 12-inch square are connected to form another concentric square as shown. Suppose that this process is continued without end to form a sequence of concentric squares.



a. Find the perimeter of the 2nd square.

b. Find the sum of the perimeters of the squares.

c. Find the sum of the areas of the squares.

d. Write the sum of the perimeters in summation notation for the general case of a square with side length s. Then write the sum of the areas for the general case.

e. Which series decreases faster, the sum of the perimeters or the sum of the areas? How do you know?

59. **Critical Thinking** Compare the partial-sum S_n with the sum S for an infinite geometric series when $a_1 > 0$ and $r = \frac{4}{5}$. Which is greater? What if $a_1 < 0$?

60. **Write About It** Why might the notation for a partial sum S_n change for the sum S for an infinite geometric series?

TEST PREP

61. Which infinite geometric series converges?

Ⓐ $\sum_{k=1}^{\infty}\left(\frac{5}{4}\right)^k$ Ⓑ $\sum_{k=1}^{\infty}5\left(\frac{1}{4}\right)^k$ Ⓒ $\sum_{k=1}^{\infty}\frac{1}{4}(5)^k$ Ⓓ $\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k 5^k$

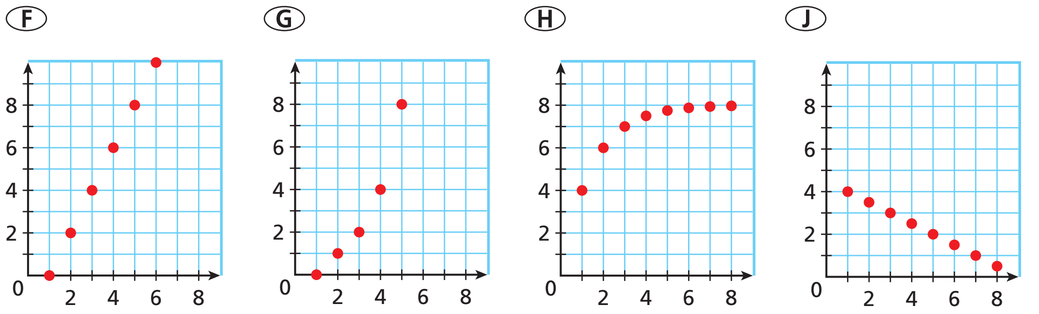
62. What is the sum of the infinite geometric series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots$?

Ⓕ 2 Ⓖ $\frac{2}{3}$ Ⓗ $\frac{1}{2}$ Ⓙ $\frac{1}{3}$

63. An infinite geometric series has a sum of 180 and a common ratio of $\frac{2}{3}$. What is the first term of the series?

Ⓐ 60 Ⓑ 120 Ⓒ 270 Ⓓ 540

64. Which graph represents a converging infinite geometric series?



65. **Extended Response** Use mathematical induction to prove $3 + 5 + \cdots + (2n + 1) = n(n + 2)$. Show all of your work.

CHALLENGE AND EXTEND

Write each repeating decimal as a fraction in simplest form.

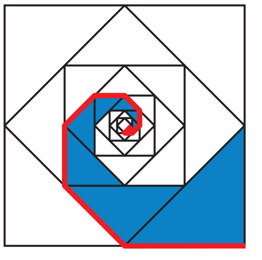
66. $0.1\overline{6}$

67. $0.41\overline{6}$

68. $0.5\overline{286}$

69. **Critical Thinking** Can an infinite arithmetic series approach a limit like an infinite geometric series? Explain why or why not.

70. **Geometry** Consider the construction that starts with a 12-inch square and contains concentric squares as indicated. Notice that a spiral is formed by the sequence of segments starting at a corner and moving inward as each midpoint is reached. A second similar spiral determines the area shown in blue.



a. Use the sum of a series to find the length of the spiral indicated in red.

b. Use the sum of a series to find the polygonal area indicated in blue.

c. Is your answer to the sum of the polygonal area in part **b** reasonable? Explain.

SPIRAL REVIEW

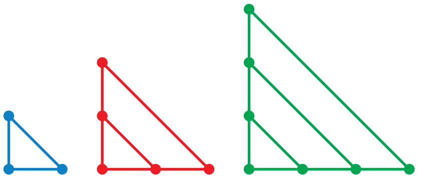
71. **Football** A kickoff specialist kicks 80% of his kickoffs into the end zone. What is the probability that he kicks at least 4 out of 5 of his next kickoffs into the end zone? *(Lesson 11-6)*

72. **Geometry** Consider the pattern of figures shown. *(Lesson 12-3)*

a. Find the number of dots in each of the next 3 figures in the pattern.

b. Write a general rule for the sequence of the number of dots in the nth figure.

c. How many dots will be in the 22nd figure in the pattern?



Determine whether each sequence could be geometric or arithmetic. If possible, find the common ratio or difference. *(Lesson 12-4)*

73. 297, 99, 33, 11, …

74. $\frac{4}{3}, \frac{8}{3}, 4, \frac{16}{3}, \ldots$

75. 25, 100, 250, 1000, …

76. 4, 4.8, 5.76, 6.912, …

CHAPTER 12

SECTION 12B

MULTI-STEP TEST PREP

Exploring Geometric Sequences and Series

Sticky Business Big-budget movies often have their greatest sales in the first weekend, and then weekend sales decrease with each passing week. After a movie has been released for a few weeks, movie studios may try to predict the total sales that the movie will generate.

Weekend Box Office Sales (million $)		
Weekend	***Spider-Man***	***Spider-Man 2***
1	114.9	115.8
2	71.4	45.2
3	45.0	24.8

1. Find the ratios of the sequences of weekend sales for *Spider-Man* and *Spider-Man 2*.
2. Write the rule for a geometric sequence that could be used to estimate the sales for *Spider-Man* in a given weekend.
3. Use the sequence from Problem 2 to predict *Spider-Man's* weekend sales for weeks 4 and 5.
4. Write and evaluate a series in summation notation to find *Spider-Man's* total weekend sales for the first 5 weekends of its release.
5. Suppose that the series from Problem 4 continued infinitely. Estimate the total weekend sales for *Spider-Man*. The actual total weekend sales for *Spider-Man* were about $311.1 million. How does this compare with your estimate?
6. Would a geometric sequence be a good model for the weekend sales of *Spider-Man 2*? Justify your answer.

READY TO GO ON?

CHAPTER 12

SECTION 12B

Quiz for Lessons 12-4 Through 12-5

12-4 Geometric Sequences and Series

Find the 8th term of each geometric sequence.

1. $\frac{2}{5}, \frac{6}{5}, \frac{18}{5}, \frac{54}{5}, \ldots$

2. $-16, -40, -100, -250, \ldots$

3. $-1, 11, -121, 1331, \ldots$

4. $2, 20, 200, 2000, \ldots$

Find the 10th term of each geometric sequence with the given terms.

5. $a_1 = 3.3$ and $a_2 = 33$

6. $a_4 = -1$ and $a_6 = -4$

7. $a_6 = 20.25$ and $a_8 = 9$

8. $a_3 = 57$ and $a_5 = 513$

Find the geometric mean of each pair of numbers.

9. $\frac{1}{3}$ and $\frac{1}{27}$

10. 4.5 and 450

11. 32 and $\frac{1}{8}$

Find the indicated sum for each geometric series.

12. S_6 for $8 - 16 + 32 - 64 + \cdots$

13. S_5 for $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$

14. $\sum_{k=1}^{7} (8)^k$

15. $\sum_{k=1}^{5} 18\left(\frac{1}{6}\right)^{k-1}$

16. The cost for electricity is expected to rise at an annual rate of 8%. In its first year, a business spends \$3000 for electricity.

a. How much will the business pay for electricity in the 6th year?

b. How much in total will be paid for electricity over the first 6 years?

12-5 Mathematical Induction and Infinite Geometric Series

Find the sum of each infinite series, if it exists.

17. $25 + 20 + 16 + 12.8 + \cdots$

18. $15 - 18 + 21.6 - 25.92 + \cdots$

19. $\sum_{k=1}^{\infty} (-1)^k \left(\frac{2}{3}\right)^k$

20. $\sum_{k=1}^{\infty} 4(0.22)^k$

Use mathematical induction to prove $4 + 8 + 12 + \cdots + 4n = 2n(n+1)$.

21. Step 1

22. Step 2

23. Step 3

24. A table-tennis ball is dropped from a height of 5 ft. The ball rebounds to 60% of its previous height after each bounce.

a. Write an infinite geometric series to represent the distance that the ball travels after it initially hits the ground. (*Hint:* The ball travels up and down on each bounce.)

b. What is the total distance that the ball travels after it initially hits the ground?

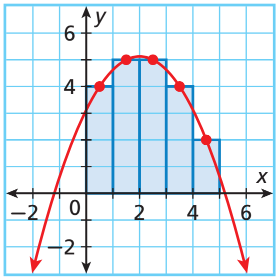
EXTENSION Area Under a Curve

Objective
Approximate area under a curve by using rectangles.

Finding the area under a curve is an important topic in higher mathematics, such as calculus. You can approximate the area under a curve by using a series of rectangles as shown in Example 1.

EXAMPLE 1 Finding Area Under a Curve

Estimate the area under the curve $f(x) = -\frac{1}{2}x^2 + 2x + 3\frac{1}{8}$ over $0 \le x \le 5$.



x	$f(x)$
0.5	4
1.5	5
2.5	5
3.5	4
4.5	2

Graph the function. Divide the area into 5 rectangles, each with a **width of 1 unit**.

Find the **height** of each rectangle by evaluating the function at the center of each rectangle, as shown in the table.

Remember!
Area is measured in square units.

Approximate the area by finding the sum of the areas of the rectangles.

$A \approx 1(4) + 1(5) + 1(5) + 1(4) + 1(2) = 20$

The estimate of 20 square units is very close to the actual area of $19\frac{19}{24}$ square units, which can be found by using calculus.

CHECK IT OUT!

1. Estimate the area under the curve $f(x) = -x^2 + 5x + 5.75$ over $0 \le x \le 6$. Use 6 intervals.

You can formalize the procedure for finding the area under the curve of a function by using the sum of a series.

Number of rectangles

Area under the curve

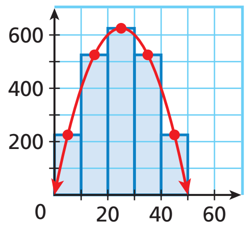
Middle x-value of each rectangle

$$A = w\sum_{k=1}^{n} \underbrace{f(a_k)}_{\text{Height of each rectangle}}$$

Width of each rectangle

EXAMPLE 2 Finding Area Under a Curve by Using a Series

Use the sum of a series to estimate the area under the curve $f(x) = -x^2 + 50x$ over $0 \le x \le 50$.



a_k	$f(a_k)$
$a_1 = 5$	225
$a_2 = 15$	525
$a_3 = 25$	625
$a_4 = 35$	525
$a_5 = 45$	225

Step 1 Graph the function.

Step 2 Divide the area into 5 rectangles, each with a width of **10** units.

Step 3 Find the value of the function at the center of each rectangle, as shown in the table.

Step 4 Write the sum that approximates the area.

$$A \approx 10\sum_{k=1}^{5} f(a_k) = 10\left[f(a_1) + f(a_2) + f(a_3) + f(a_4) + f(a_5)\right]$$
$$= 10\left[(225) + (525) + (625) + (525) + (225)\right]$$
$$= 10(2125) = 21{,}250$$

The estimated area is 21,250 square units.

CHECK IT OUT! **2.** Use the sum of a series to estimate the area under the curve $f(x) = -x^2 + 9x + 3$ over $0 \le x \le 9$. Use 3 intervals.

EXTENSION Exercises

Estimate the area under each curve. Use 4 intervals.

1. $f(x) = -\frac{1}{2}x^2 + 12x + 2$ over $0 \le x \le 24$

2. $f(x) = -x^2 + 8x + 4$ over $0 \le x \le 8$

3. $f(x) = -\frac{1}{4}x^2 + 5x$ over $0 \le x \le 16$

4. $f(x) = -0.1x^2 + 20x$ over $0 \le x \le 200$

Use the sum of a series to estimate the area under each curve. Use 5 intervals.

5. $f(x) = -x^2 + 10x + 5$ over $0 \le x \le 10$

6. $f(x) = -x^2 + 30x$ over $0 \le x \le 20$

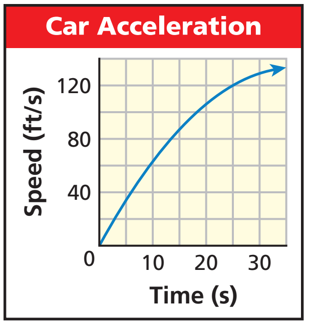
7. $f(x) = -\frac{1}{10}x^2 + 400$ over $0 \le x \le 50$

8. $f(x) = -0.2x^2 + 28x + 300$ over $0 \le x \le 150$

9. Physics The graph shows a car's speed versus time as the car accelerates. This realistic curve can be approximated by $v(t) = -0.1t^2 + 7.3t$, where v is the velocity in feet per second and t is the time in seconds.

a. Estimate the area under the curve for $0 \le t \le 35$.

b. What does the area under the curve represent? Explain. (*Hint:* Consider the units of your answer to part **a.**)



10. Energy Conservation Daily electricity use peaks in the early afternoon and can be approximated by a parabola. Suppose that the rate of electricity use in kilowatts (kW) is modeled by the function $f(x) = -1.25x^2 + 30x + 700$, where x represents the time in hours.

a. Write a sum to represent the area under the curve for a domain of $0 \le x \le 24$.

b. Estimate the area under the curve.

11. Write About It Explain how the units of the values on the x-axis and the units of the values on the y-axis can be used to find the units that apply to the area under a curve.

12. Critical Thinking For a given function and domain, how would increasing the number of rectangles affect the approximation of the area under the curve?

CHAPTER 12

Study Guide: Review

Vocabulary

Complete the sentences below with vocabulary words from the list above.

1. A(n) __?__ has a common difference, and a(n) __?__ has a common ratio.

2. A series that has no limit __?__ , whereas a series that approaches a limit __?__ .

3. A(n) __?__ defines the nth term. A(n) __?__ defines the next term by using one or more of the previous terms.

4. A(n) __?__ continues without end, and a(n) __?__ has a last term.

5. Each step in a repeated process is called a(n) __?__ .

12-1 Introduction to Sequences *(pp. 862–868)*

EXAMPLES

■ **Find the first 5 terms of the sequence with $a_1 = -52$; $a_n = 0.5a_{n-1} + 2$.**

Evaluate the rule using each term to find the next term.

n	1	2	3	4	5
a_n	−52	−24	−10	−3	0.5

■ **Write an explicit rule for the nth term of 100, 72, 44, 16, −12, ...**

Examine the differences or ratios.

Terms	100	72	44	16	−12
1st differences	28	28	28	28	

The first differences are constant, so the sequence is linear.

The first term is 100, and each term is 28 less than the previous term.

The explicit rule is $a_n = 100 - 28(n - 1)$.

EXERCISES

Find the first 5 terms of each sequence.

6. $a_n = n - 9$

7. $a_n = \frac{1}{2}n^2$

8. $a_n = \left(-\frac{3}{2}\right)^{n-1}$

9. $a_1 = 55$ and $a_n = a_{n-1} - 2$

10. $a_1 = 200$ and $a_n = \frac{1}{5}a_{n-1}$

11. $a_1 = -3$ and $a_n = -3a_{n-1} + 1$

Write a possible explicit rule for the nth term of each sequence.

12. −4, −8, −12, −16, −20, . . .

13. 5, 20, 80, 320, 1280, . . .

14. −24, −19, −14, −9, −4, . . .

15. 27, 18, 12, 8, $\frac{16}{3}$, . . .

16. **Sports** Suppose that a basketball is dropped from a height of 3 ft. If the ball rebounds to 70% of its height after each bounce, how high will the ball reach after the 4th bounce? the 9th bounce?

12-2 Series and Summation Notation *(pp. 870–877)*

EXAMPLES

- **Expand $\sum_{k=1}^{5}(-1)^{n+1}(11-2n)$, and evaluate.**

$$\sum_{k=1}^{5}(-1)^{n+1}(11-2n) = (-1)^2(11-2) + (-1)^3(11-4) + (-1)^4(11-6) + (-1)^5(11-8) + (-1)^6(11-10)$$

$$= 9 - 7 + 5 - 3 + 1$$

$$= 5 \quad \textit{Simplify.}$$

- **Evaluate $\sum_{k=1}^{8} k^2$.**

Use summation formula for a quadratic series.

$$\sum_{k=1}^{8} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{8(8+1)(2\cdot 8+1)}{6} = \frac{72(17)}{6} = 204$$

EXERCISES

Expand each series and evaluate.

17. $\sum_{k=1}^{4} k^2(-1)^k$
18. $\sum_{k=1}^{5} (0.5k+4)$
19. $\sum_{k=1}^{5} (-1)^{k+1}(2k-1)$
20. $\sum_{k=1}^{4} \frac{5k}{k^2}$

Evaluate each series.

21. $\sum_{k=1}^{8} -5$
22. $\sum_{k=1}^{10} k^2$
23. $\sum_{k=1}^{12} k$
24. **Finance** A household has a monthly mortgage payment of \$1150. How much is paid by the household after 2 years? 15 years?

12-3 Arithmetic Sequences and Series *(pp. 879–887)*

EXAMPLES

- **Find the 12th term for the arithmetic sequence 85, 70, 55, 40, 25, ….**

Find the common difference:
$d = 70 - 85 = -15$.

$a_n = a_1 + (n-1)d$ *General rule*

$a_{12} = 85 + (12-1)(-15)$ *Substitute.*

$= -80$ *Simplify.*

- **Find $\sum_{k=1}^{11}(-2-33k)$.**

Find the 1st and 11th terms.

$a_1 = -2 - 33(1) = -35$

$a_{11} = -2 - 33(11) = -365$

Find S_{11}.

$S_n = n\left(\frac{a_1 + a_n}{2}\right)$ *Sum formula*

$S_{11} = 11\left(\frac{-35-365}{2}\right)$ *Substitute.*

$= -2200$

EXERCISES

Find the 11th term of each arithmetic sequence.

25. 23, 19, 15, 11, …
26. $\frac{1}{5}, \frac{3}{5}, 1, \frac{7}{5}, \frac{9}{5}, \ldots$
27. −9.2, −8.4, −7.6, −6.8, …
28. $a_3 = 1.5$ and $a_4 = 5$
29. $a_6 = 47$ and $a_8 = 21$
30. $a_5 = -7$ and $a_9 = 13$

Find the indicated sum for each arithmetic series.

31. S_{18} for $-1 - 5 - 9 - 13 + \cdots$
32. S_{12} for $\frac{1}{3} + \frac{1}{6} + 0 - \frac{1}{6} + \cdots$
33. $\sum_{k=1}^{15}(-14+3k)$
34. $\sum_{k=1}^{15}\left(\frac{3}{2}k+10\right)$
35. **Savings** Kelly has \$50 and receives \$8 a week for allowance. He wants to save all of his money to buy a new mountain bicycle that costs \$499. Write an arithmetic sequence to represent the situation. Then find whether Kelly will be able to buy the new bicycle after one year (52 weeks).

12-4 Geometric Sequences and Series *(pp. 890–898)*

EXAMPLES

■ **Find the 8th term of the geometric sequence 6, 24, 96, 384, ….**

Find the common ratio. $r = \frac{24}{6} = 4$

Write a rule, and evaluate for $n = 8$.

$a_n = a_1 r^{n-1}$ *General rule*

$a_8 = 6(4)^{8-1} = 98{,}304$

■ **Find the 8th term of the geometric sequence with $a_4 = -1000$ and $a_6 = -40$.**

Step 1 Find the common ratio.

$a_6 = a_4 r^{(6-4)}$ *Use the given terms.*

$-40 = -1000r^2$ *Substitute.*

$\frac{1}{25} = r^2$ *Simplify.*

$\pm\frac{1}{5} = r$

Step 2 Find a_1 using both possible values for r.

$-1000 = a_1\left(\frac{1}{5}\right)^{4-1}$ or $-1000 = a_1\left(-\frac{1}{5}\right)^{4-1}$

$a_1 = -125{,}000$ or $a_1 = 125{,}000$

Step 3 Write the rule and evaluate for a_8 by using both possible values for r.

$a_n = a_1 r^{n-1}$ $a_n = a_1 r^{n-1}$

$a_n = -125{,}000\left(\frac{1}{5}\right)^{n-1}$ or $a_n = 125{,}000\left(-\frac{1}{5}\right)^{n-1}$

$a_8 = -125{,}000\left(\frac{1}{5}\right)^{8-1}$ $a_8 = 125{,}000\left(-\frac{1}{5}\right)^{8-1}$

$a_8 = -1.6$ $a_8 = -1.6$

■ **Find $\sum_{k=1}^{7} -2(5)^{k-1}$.**

Find the common ratio. $r = \frac{a_2}{a_1} = \frac{-6}{3} = -2$

Find S_7

$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$ *Sum formula*

$S_7 = 3\left(\frac{1 - (-2)^7}{1 - (-2)}\right)$ *Substitute.*

$= 3\left(\frac{1 - (-128)}{3}\right) = 129$

EXERCISES

Find the 8th term of each geometric sequence.

36. 40, 4, 0.4, 0.04, 0.004, …

37. $\frac{1}{18}, \frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \ldots$

38. $-16, -8, -4, -2, \ldots$

39. $-6, 12, -24, 48, \ldots$

Find the 9th term of the geometric sequence with the given terms.

40. $a_3 = 24$ and $a_4 = 96$

41. $a_1 = \frac{2}{3}$ and $a_2 = -\frac{4}{3}$

42. $a_4 = -1$ and $a_6 = -4$

43. $a_3 = 4$ and $a_6 = 500$

Find the geometric mean of each pair of numbers.

44. 10 and 2.5

45. $\frac{1}{2}$ and 8

46. $\frac{\sqrt{3}}{96}$ and $\frac{\sqrt{3}}{6}$

47. $\frac{5}{12}$ and $\frac{125}{108}$

Find the indicated sum for each geometric series.

48. S_5 for $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

49. S_6 for $-\frac{4}{5} + 8 - 80 + 800 + \cdots$

50. $\sum_{k=1}^{8} (4)^{k-1}$

51. $\sum_{k=1}^{7} -2(5)^{k-1}$

52. $\sum_{k=1}^{6} 60\left(-\frac{1}{2}\right)^{k-1}$

53. $\sum_{k=1}^{5} 18\left(\frac{1}{2}\right)^{k-1}$

54. Depreciation A new photocopier costs $9000 and depreciates each year such that it retains only 65% of its preceding year's value. What is the value of the photocopier after 5 years?

55. Rent A one-bedroom apartment rents for $650 a month. The rent is expected to increase by 6% per year.

a. What will be the annual rent expense on the apartment after 5 years?

b. What will be the total amount spent on rent if a person rents the apartment for the entire 5-year period?

EXAMPLES

Find the sum of each infinite series, if it exists.

■ $-9261 + 441 - 21 + 1 + \cdots$

$r = \frac{441}{-9261} = -\frac{1}{21}$ *Converges: $|r| < 1$*

$S = \frac{a_1}{1 - r}$ *Sum formula*

$= \frac{-9261}{1 - \left(-\frac{1}{21}\right)} = \frac{-9261}{\frac{22}{21}}$

$= -\frac{194{,}481}{22}$, or $-8840.0\overline{45}$

■ $\sum_{k=1}^{\infty} -5\left(\frac{7}{10}\right)^{k-1}$

$= -5 - \frac{35}{10} - \frac{245}{100} + \cdots$ *Evaluate.*

$r = \frac{-\frac{35}{10}}{-5} = \frac{7}{10}$ *Converges: $|r| < 1$*

$S = \frac{a_1}{1 - r} = \frac{-5}{1 - \frac{7}{10}} = \frac{-5}{\frac{3}{10}} = -\frac{50}{3}$, or $-16.\overline{6}$

■ **Use mathematical induction to prove**

$$2 + 5 + \cdots + (3n - 1) = \frac{n}{2}(3n + 1).$$

Step 1 Base case: Show that the statement is true for $n = 1$.

$2 = \frac{n}{2}(3n + 1) = \frac{1}{2}(3 \cdot 1 + 1) = 2$ *True*

Step 2 Assume that the statement is true for a natural number k.

$2 + 5 + \cdots + (3k - 1) = \frac{k}{2}(3k + 1)$ *Replace n with k.*

Step 3 Prove that it is true for the natural number $k + 1$.

$2 + 5 + \ldots + (3k - 1) + 3(k + 1) - 1$ *Add to both sides.*

$= \frac{k}{2}(3k + 1) + 3(k + 1) - 1$

$= \frac{k(3k + 1)}{2} + (3k + 3 - 1)$ *Multiply.*

$= \frac{3k^2 + k}{2} + \frac{2(3k + 2)}{2}$ *Simplify and rewrite with like denominators.*

$= \frac{3k^2 + 7k + 4}{2}$ *Add.*

$= \frac{(k + 1)(3k + 4)}{2}$ *Factor.*

$= \frac{(k + 1)}{2}\left(3(k + 1) + 1\right)$ *Write with k + 1.*

EXERCISES

Find the sum of each infinite series, if it exists.

56. $-2700 + 900 - 300 + 100 + \cdots$

57. $-1.2 - 0.12 - 0.012 - 0.0012 + \cdots$

58. $-49 - 42 - 36 - \frac{216}{7} + \cdots$

59. $4 + \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \cdots$

60. $\sum_{k=1}^{\infty} \frac{9}{3^k}$

61. $\sum_{k=1}^{\infty} -7\left(\frac{3}{5}\right)^k$

62. $\sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{1}{8^k}\right)$

63. $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k$

Use mathematical induction to prove each statement.

64. $2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2$

65. $1 + 5 + 25 + \cdots + 5^{n-1} = \frac{5^n - 1}{4}$

66. $\frac{1}{3} + \frac{1}{15} + \cdots + \frac{1}{4n^2 - 1} = \frac{n}{2n + 1}$

67. Recreation A child on a swing is let go from a vertical height so that the distance that he travels in the first back-and-forth swing is exactly 9 feet.

a. If each swing decreases the distance by 85%, write an infinite geometric series that expresses the distance that the child travels in feet.

b. What is the total distance that the child in the swing travels before the swing stops?

CHAPTER 12

CHAPTER TEST

Find the first 5 terms of each sequence.

1. $a_n = n^2 - 4$
2. $a_1 = 48$ and $a_n = \frac{1}{2}a_{n-1} - 8$

Write a possible explicit rule for the *n*th term of each sequence.

3. $-4, -2, 0, 2, 4, \ldots$
4. $54, 18, 6, 2, \frac{2}{3}, \ldots$

Expand each series and evaluate.

5. $\sum_{k=1}^{4} 5k^3$
6. $\sum_{k=1}^{7} (-1)^{k+1}(k)$

Find the 9th term of each arithmetic sequence.

7. $-19, -13, -7, -1, \ldots$
8. $a_2 = 11.6$ and $a_5 = 5$
9. Find 2 missing terms in the arithmetic sequence 125, ▭, ▭, 65.

Find the indicated sum for each arithmetic series.

10. S_{20} for $4 + 7 + 10 + 13 + \ldots$
11. $\sum_{k=1}^{12} (-9k + 8)$
12. The front row of a theater has 16 seats and each subsequent row has 2 more seats than the row that precedes it. How many seats are in the 12th row? How many seats in total are in the first 12 rows?

Find the 10th term of each geometric sequence.

13. $\frac{3}{256}, \frac{3}{64}, \frac{3}{16}, \frac{3}{4}, \ldots$
14. $a_4 = 2$ and $a_5 = 8$
15. Find the geometric mean of 4 and 25.

Find the indicated sum for each geometric series.

16. S_6 for $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$
17. $\sum_{k=1}^{6} 250\left(-\frac{1}{5}\right)^{k-1}$
18. You invest \$1000 each year in an account that pays 5% annual interest. How much is the first \$1000 you invested worth after 10 full years of interest payments? How much in total do you have in your account after 10 full years?

Find the sum of each infinite geometric series, if it exists.

19. $200 - 100 + 50 - 25 + \ldots$
20. $\sum_{k=1}^{\infty} 2\left(\frac{7}{8}\right)^k$

Use mathematical induction to prove $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \cdots + \frac{2n-1}{2} = \frac{n^2}{2}$.

21. Step 1
22. Step 2
23. Step 3

College Entrance Exam Practice

FOCUS ON SAT

When you get your SAT scores, you are given the percentile in which your scores fall. This tells you the percentage of students that scored lower than you did on the same test. You'll see your percentile score at the national and state levels. They are usually not the same.

HOT TIP! Read each problem carefully, and make sure that you understand what the question is asking. Before marking your final answer on the answer sheet, check that your answer makes sense in the context of the question.

You may want to time yourself as you take this practice test. It should take you about 8 minutes to complete.

1. The first term of a sequence is 6, and each successive term is 3 less than twice the preceding term. What is the sum of the first four terms of the sequence?

(A) 27

(B) 30

(C) 51

(D) 57

(E) 123

2. The first term of a sequence is 2, and the nth term is defined to be $3n - 1$. What is the average of the 7th, 10th, and 12th terms?

(A) 24.5

(B) 28

(C) 29

(D) 32

(E) 84

3. The first term of an arithmetic sequence is -5. If the common difference is 4, what is the 7th term of the sequence?

(A) $-20{,}480$

(B) -29

(C) 19

(D) 20

(E) 23

4. A population of 50 grows exponentially by doubling every 4 years. After how many years will the population have 1600 members?

(A) 20

(B) 16

(C) 10

(D) 6

(E) 5

5. Which of the following sequences can be expressed by the rule $a_n = \frac{n-1}{n+1}$?

(A) $3, 2, \frac{5}{3}, \frac{3}{2}, \frac{7}{5}, \ldots$

(B) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \ldots$

(C) $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots$

(D) $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \ldots$

(E) $0, \frac{3}{2}, 3, \frac{5}{3}, 5, \ldots$

6. Which of the following sequences is a geometric sequence?

(A) $-7, 14, -28, 56, -112, \ldots$

(B) $-4, -6, -8, -10, -12, \ldots$

(C) $-3, 1, -3, 1, -3, \ldots$

(D) $4, 12, 48, 144, 576, \ldots$

(E) $1, 4, 9, 16, 25, \ldots$

CHAPTER 12

TEST TACKLER

Standardized Test Strategies

Short/Extended Response: Outline Your Response

Answering short and extended response items on tests is a lot like writing essays in English class. You can use an outline to plan your response to the question. Outlines help you organize the main points and the order in which they will appear in your answer. Outlining your response will help ensure that your explanation is clearly organized and includes all necessary information.

EXAMPLE 1

Short Response Ariana is saving money for a new car. She saves \$40 the first week, \$45 the second week, \$50 the third week, and so on. Explain whether an arithmetic or geometric sequence would best represent this situation. Use a sequence or series to determine the amount that she will save in the 8th week and the total amount that she will have saved after 8 weeks.

Create an outline for your response.

Outline

1. Explain whether arithmetic or geometric.
2. Write sequence and series.
3. Find amount saved in 8th week.
4. Find total saved after 8 weeks.

Follow the outline, and write out your response.

Include evidence to explain the answer for the first step.

An arithmetic sequence would best represent this situation because Ariana is adding \$5 each week to the amount that she saves. This would be an arithmetic sequence where the first term is 40 and the common difference is 5.

Clearly indicate which is the sequence and which is the series for the second step.

The sequence for the amount saved each week is $a_n = 40 + 5(n - 1)$. The series for the total amount saved is $\sum_{k=1}^{n}\left[40 + 5(k - 1)\right]$.

Show how you found the answers for the last two steps.

The amount saved in the 8th week is $a_8 = 40 + 5(8 - 1) = \$75$.

The total saved after 8 weeks is $\sum_{k=1}^{8}\left[40 + 5(k - 1)\right] = \$460$.

When you finish your response, check it against your outline to make sure that you did not leave out any details.

Read each test item and answer the questions that follow.

Item A

Short Response Explain how to determine whether an infinite geometric series has a sum.

1. What should be included in an outline of the response for this test item?
2. Read the two different outlines below. Which outline is the most useful? Why?

   Student A

   I. Definition of an infinite geometric series and the common ratio *r*.

   II. Definition of the sum of an infinite geometric series.

   III. Explain for which values of *r* that a sum exists.

   Student B

   A. Geometric series has a common ratio.

   B. Common ratio has to be less than 1.

Item B

Extended Response

A pattern for stacking cereal boxes is shown at right.

a. Explain how many boxes are in a 9-row display.

b. If 91 boxes are to be stacked in this display, explain how many rows the display will have.

3. Read the outline below. Identify any areas that need improvement. Rewrite the outline to make it more useful.

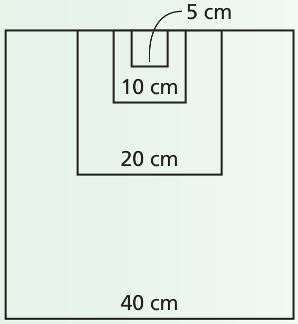
   Outline

   1. Find the number of boxes if there are 9 rows.

   2. Find the number of rows needed for 91 boxes.

Item C

Extended Response A pattern of squares is created by doubling the dimensions of the previous square. Explain how to find the sum of the perimeters of the first 8 squares if the first square is 5 cm wide.



4. A student correctly gave the following response. Write an outline for a response to this question.

   To find the perimeters of the first 8 squares, I need to determine the series.

$$4(5) + 4(10) + 4(20) + 4(40) + \cdots$$
$$4(5 + 10 + 20 + 40 + \cdots)$$
$$4 \cdot 5(1 + 2 + 4 + 8 + \cdots)$$
$$20(1 + 2 + 4 + 8 + \cdots)$$
$$20\sum_{n=1}^{8} 2^{(n-1)}$$

   Now that I know that the first term is 20 and the common ratio is 2, I can use the formula $S_n = t_1\left(\frac{1 - r^n}{1 - r}\right)$, where t_1 is the first term and r is the common ratio.

$$S_n = 20\left(\frac{1 - 2^8}{1 - 2}\right)$$
$$= 20\left(\frac{1 - 256}{-1}\right)$$
$$= 20\left(\frac{-255}{-1}\right)$$
$$= 20(255) = 5100$$

   So the sum of the perimeters of the first 8 squares is 5100 cm.

STANDARDIZED TEST PREP

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CUMULATIVE ASSESSMENT, CHAPTERS 1–12

Multiple Choice

1. Which shows the series in summation notation?

$4 + 6 + 4 + 6 + 4$

Ⓐ $\sum 24$

Ⓑ $\sum_{n=0}^{5} [(-1)^n + 5]$

Ⓒ $\sum_{n=1}^{4} [(-1)^n + 5]$

Ⓓ $\sum_{n=1}^{5} [(-1)^n + 5]$

2. What is the expanded binomial?

$(2x - y)^3$

Ⓕ $x^3 - 3x^2y + 3xy^2 - y^3$

Ⓖ $8x^3 - 12x^2y + 6xy^2 - y^3$

Ⓗ $x^3 + 3x^2y + 3xy^2 + y^3$

Ⓙ $8x^3 + 12x^2y + 6xy^2 + y^3$

3. Let $f(x) = x^3 + 2x^2 - 5x - 9$. Which function would show $f(x)$ reflected across the y-axis?

Ⓐ $g(x) = -x^3 - 2x^2 + 5x + 9$

Ⓑ $g(x) = -x^3 + 2x^2 + 5x - 9$

Ⓒ $g(x) = 2x^3 + 4x^2 - 10x - 18$

Ⓓ $g(x) = x^3 + 2x^2 - 5x - 5$

4. Which function shows exponential decay?

Ⓕ $f(x) = -5x$

Ⓖ $f(x) = 2.3(6.7)^x$

Ⓗ $f(x) = 0.49(7.9)^x$

Ⓙ $f(x) = 5.13(0.32)^x$

5. A ball is dropped from a height of 10 feet. On each bounce, the ball bounces 60% of the height of the previous bounce. Which expression represents the height in feet of the ball on the nth bounce?

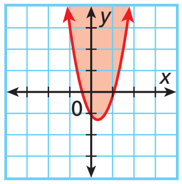
Ⓐ $10(0.6n)$

Ⓑ $10(0.6)^{n-1}$

Ⓒ $\dfrac{10 - n}{0.6}$

Ⓓ $10(0.6)^n$

6. Which is the graph of the inequality $6x + 3y \geq 9x^2 - 3$?

Ⓕ


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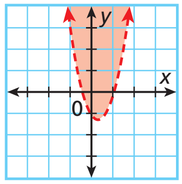

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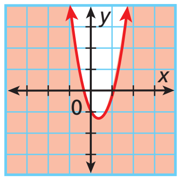
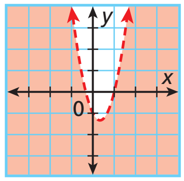


Ⓙ


7. Gina opened a new deli. Her revenues for the first 4 weeks were \$2000, \$2400, \$2880, and \$3456. If the trend continues, which is the best estimate of Gina's revenues in the 6th week?

Ⓐ \$3856

Ⓑ \$4032

Ⓒ \$4147

Ⓓ \$4980

8. What is the 9th term in the sequence?

$a_n = \frac{1}{2}(2^{n-1}) + 4$

Ⓕ 36

Ⓖ 68

Ⓗ 132

Ⓙ 260

9. Find the inverse of $f(x) = 4x - 5$.

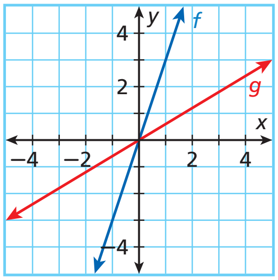
Ⓐ $f^{-1}(x) = -4x + 5$

Ⓑ $f^{-1}(x) = \frac{1}{4}x + 5$

Ⓒ $f^{-1}(x) = \dfrac{x + 5}{4}$

Ⓓ $f^{-1}(x) = 5x - 4$

10. What transformation has been applied to f to get g?



Ⓕ Horizontal compression by $\frac{1}{5}$

Ⓖ Horizontal stretch by 5

Ⓗ Vertical compression by $\frac{1}{3}$

Ⓙ Vertical stretch by 5

In Item 11, you may choose to graph, factor, complete the square, or use the Quadratic Formula to find the zeros.

11. Find the zeros of $f(x) = 2x^2 + 5x - 12$.

Ⓐ $-4, \frac{3}{2}$

Ⓑ $-2, 3$

Ⓒ $-\frac{3}{2}, 4$

Ⓓ $\frac{3}{2}, 2$

Gridded Response

12. Find the common ratio of the geometric sequence.

125, 50, 20, 8, . . .

13. A card is drawn from a deck of 52. What is the probability, to the nearest hundredth, of drawing a 10 or a diamond?

14. What is the sum of the arithmetic series?

$$\sum_{k=1}^{8} 7k - 3$$

15. What is the y-value of the point, to the nearest hundredth, that represents the solution to the given system of equations?

$$\begin{cases} 2y - 2 = 4x \\ 6 - x = 8y \end{cases}$$

Short Response

16. Use the function $f(x) = \sqrt[3]{5x}$ to answer the following questions.

a. What is the domain and range?

b. What is the inverse of $f(x)$?

c. What is the domain and range of the inverse function?

d. Graph $f(x)$ and $f^{-1}(x)$ on the same coordinate plane.

17. Use the infinite geometric series $\sum_{n=1}^{\infty} \frac{5}{4^{n-1}}$ to answer the following questions.

a. Determine if the series converges or diverges.

b. Find the sum of the infinite series, if it exists.

18. A grocery store display contains 3 cans on the top row and an additional can in each row forming a triangular shape.

a. Would you use a sequence or a series to represent the number of cans in the nth row? Explain.

b. How many cans are in the 12th row?

c. What does the series $\sum_{k=1}^{n} k + 2$ represent? Explain.

Extended Response

19. A test to be on a trivia show has two parts. 60% of contestants pass the first part, and 20% pass the second part.

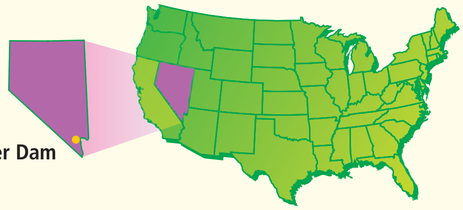
a. Draw a tree diagram that gives the probabilities for a contestant's possible outcomes on the test.

b. If a contestant must pass both parts of the test to be on the show, how many contestants out of a group of 50 would likely make the show? Show your work.

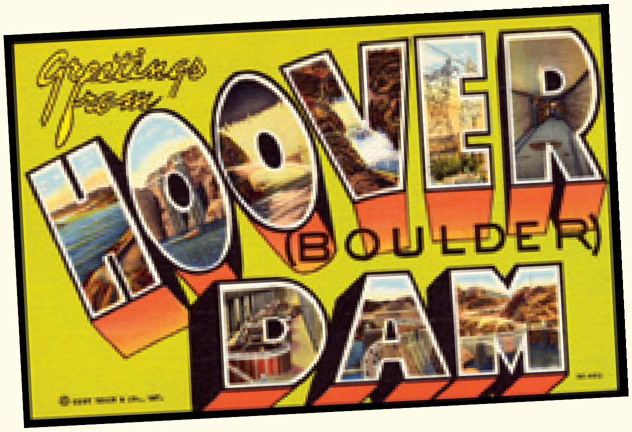
c. Is it more likely that a contestant would pass both parts or fail both parts of the test? Explain.

Problem Solving on Location

NEVADA



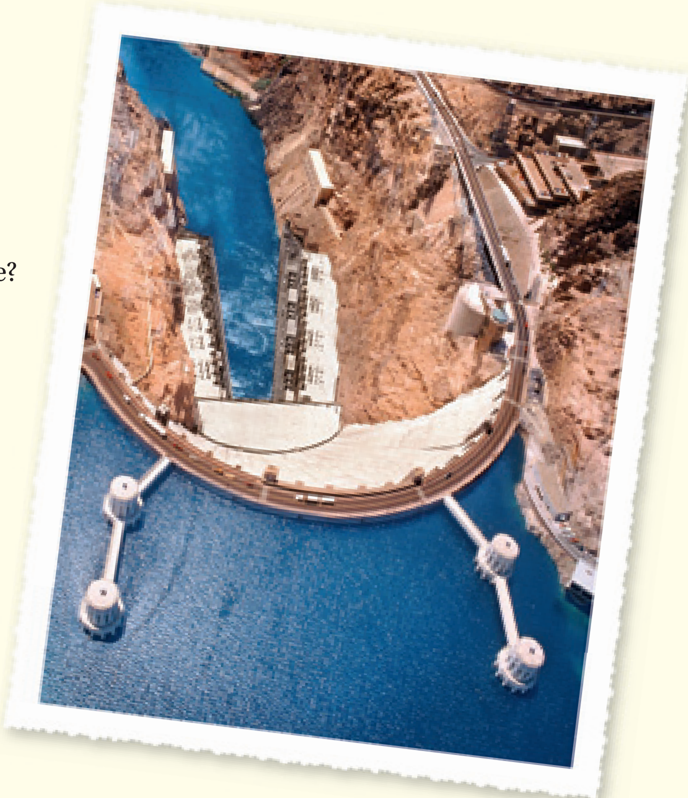
The Hoover Dam



Since its completion in 1935, the Hoover Dam has often been cited as one of the seven engineering wonders of the world. Its 6.6 million tons of concrete tame the waters of the Colorado River and form Lake Mead, the largest man-made reservoir in the United States.

Choose one or more strategies to solve each problem. For 1 and 2, use the table.

Traffic Forecasts for the Hoover Dam			
Year	2007	2008	2009
Number of Cars per Day	16,300	16,780	17,260



1. The Hoover Dam serves as a bridge between the Nevada and Arizona sides of the Colorado River. Traffic analysts project that the traffic on the dam will increase according to an arithmetic sequence. How many cars, on average, would you predict to cross the dam each day in 2017?

2. Approximately how many vehicles will cross the dam in the years 2007 through 2017, inclusive? (*Hint:* Assume 365 days per year.)

3. Small trucks make up 18% of the traffic on the dam, and RVs account for another 4%. All trucks and RVs are inspected before they are allowed to cross.

  a. Assuming that other types of vehicles are not inspected, what is the probability that three consecutive vehicles arriving at a checkpoint will be inspected?

  b. What is the probability that 3 out of 5 vehicles arriving at a checkpoint will need to be inspected?

Problem Solving Strategies

- Draw a Diagram
- Make a Model
- Guess and Test
- Work Backward
- Find a Pattern
- Make a Table
- Solve a Simpler Problem
- Use Logical Reasoning
- Use a Venn Diagram
- Make an Organized List

Silver and Gold Mining

Nevada's nickname is the Silver State, which is not surprising considering that more than 10 million ounces of the metal are mined in Nevada each year. Gold mining is also an essential part of Nevada's economy. In fact, if Nevada were a nation, it would rank third in the world in gold production behind South Africa and Australia.

Choose one or more strategies to solve each problem.

1. Nevada experienced a gold-mining boom from 1981 to 1990. During this period, the number of thousands of ounces of gold mined each year can be modeled by a geometric sequence in which $a_1 = 375$ (that is, 375,000 ounces were mined in 1981) and the common ratio is 1.35. Approximately how many ounces of gold were mined in 1990?

2. What was the total gold production in the years 1981 through 1990, inclusive?

3. In a particular mine, the probability of discovering a profitable quantity of gold in a sector is approximately 40%. What is the probability that the miners will discover a profitable quantity of gold in 3 of the next 4 sectors?

For 4, use the table.

4. Nevada experienced a silver boom during the 1990s. An industry analyst is collecting detailed data for all of the years from 1991 to 2000 in which silver production was outside of 1 standard deviation of the mean. For which years should she collect this data?

Nevada Silver Production

Year	Production (million oz)
1991	18.6
1992	19.7
1993	23.2
1994	22.8
1995	24.6
1996	20.7
1997	24.7
1998	21.5
1999	19.5
2000	23.2