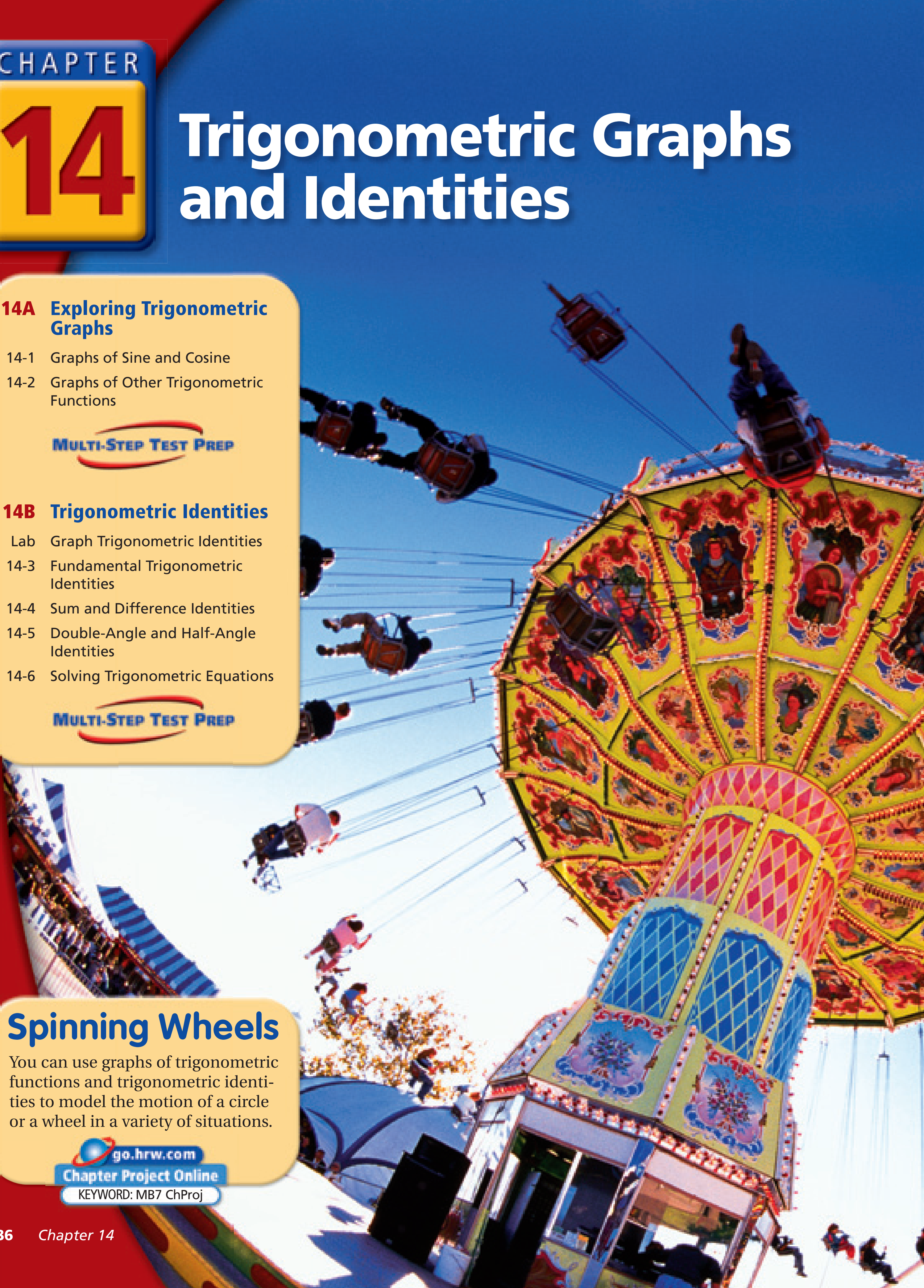
CHAPTER 14

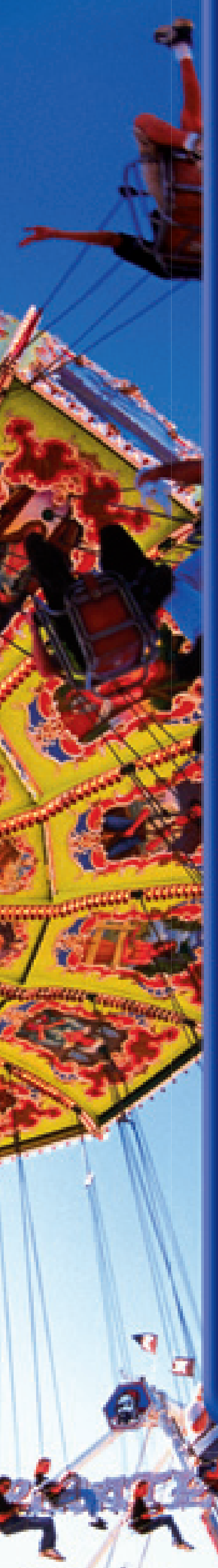
Trigonometric Graphs and Identities



Spinning Wheels

You can use graphs of trigonometric functions and trigonometric identities to model the motion of a circle or a wheel in a variety of situations.

go.hrw.com
Chapter Project Online
KEYWORD: MB7 ChProj



ARE YOU READY?

Vocabulary

Match each term on the left with a definition on the right.

1. cosecant

2. cosine

3. hypotenuse

4. tangent of an angle

A. the ratio of the length of the leg adjacent the angle to the length of the opposite leg

B. the ratio of the length of the leg adjacent the angle to the length of the hypotenuse

C. the ratio of the length of the leg opposite the angle to the length of the adjacent leg

D. the ratio of the length of the hypotenuse to the length of the leg opposite the angle

E. the side opposite the right angle

Divide Fractions

Divide.

5. $\dfrac{\frac{3}{5}}{\frac{5}{2}}$ **6.** $\dfrac{\frac{3}{4}}{\frac{1}{2}}$ **7.** $\dfrac{-\frac{3}{8}}{\frac{1}{8}}$ **8.** $\dfrac{\frac{2}{3}}{-\frac{7}{4}}$

Simplify Radical Expressions

Simplify each expression.

9. $\sqrt{6} \cdot \sqrt{2}$ **10.** $\sqrt{100 - 64}$ **11.** $\dfrac{\sqrt{9}}{\sqrt{36}}$ **12.** $\sqrt{\dfrac{4}{25}}$

Multiply Binomials

Multiply.

13. $(x + 11)(x + 7)$

14. $(y - 4)(y - 9)$

15. $(2x - 3)(x + 5)$

16. $(k + 3)(3k - 3)$

17. $(4z - 4)(z + 1)$

18. $(y + 0.5)(y - 1)$

Special Products of Binomials

Multiply.

19. $(2x + 5)^2$

20. $(3y - 2)^2$

21. $(4x - 6)(4x + 6)$

22. $(2m + 1)(2m - 1)$

23. $(s + 7)^2$

24. $(-p + 4)(-p - 4)$

CHAPTER 14

Study Guide: Preview

Where You've Been

In previous chapters, you

- solved problems involving triangles and trigonometric ratios.
- factored to solve quadratic equations.
- applied function models to solve real-world problems.
- solved equations by using algebra and graphs.

In This Chapter

You will study

- problems involving trigonometric functions.
- factoring to solve trigonometric equations.
- trigonometric function models of real-world problems.
- solving trigonometric equations by using algebra and graphs.

Where You're Going

You can use the skills in this chapter

- in your future math classes, particularly Calculus.
- in other classes, such as Physics, Biology, and Economics.
- outside of school to observe cyclical patterns and make conjectures.

Key Vocabulary/Vocabulario

amplitude	amplitud
cycle	ciclo
frequency	frecuencia
period	periodo
periodic function	función periódica
phase shift	cambio de fase
rotation matrix	matriz de rotación

Vocabulary Connections

To become familiar with some of the vocabulary terms in the chapter, consider the following. You may refer to the chapter, the glossary, or a dictionary if you like.

1. What does the word *amplify* mean? What might the **amplitude** of a pendulum swing refer to?
2. What does a **cycle** refer to in everyday language? Give examples of cyclical phenomena.
3. Give an example of something that occurs *frequently*. To describe how often something occurs, like brushing our teeth, we can say "we brush twice a day." Describe the **frequency** of your example.
4. What does **period** mean in everyday language? What might a **periodic function** refer to?
5. What result might you expect from using a **rotation matrix**?

Reading and Writing Math

Study Strategy: Prepare for Your Final Exam

Math is a cumulative subject, so your final exam will probably cover all of the material that you have learned from the beginning of the course. Preparation is essential for you to be successful on your final exam. It may help you to make a study timeline like the one below.

2 weeks before the final:

- Look at previous exams and homework to determine areas I need to focus on; rework problems that were incorrect or incomplete.
- Make a list of all formulas and theorems that I need to know for the final.
- Create a practice exam using problems from the book that are similar to problems from each exam.

1 week before the final:

- Take the practice exam and check it. For each problem I miss, find two or three similar ones and work those.
- Work with a friend in the class to quiz each other on formulas, postulates, and theorems from my list.

1 day before the final:

- Make sure I have pencils and a calculator (check batteries!).

Try This

1. Create a timeline that you will use to study for your final exam.

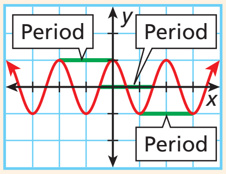
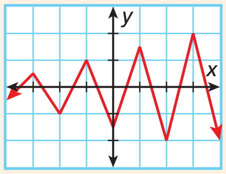
14-1 Graphs of Sine and Cosine

Objective
Recognize and graph periodic and trigonometric functions.

Vocabulary
periodic function
cycle
period
amplitude
frequency
phase shift

Why learn this?
Periodic phenomena such as sound waves can be modeled with trigonometric functions. (See Example 3.)

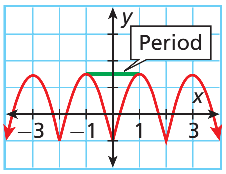
Periodic functions are functions that repeat exactly in regular intervals called **cycles**. The length of the cycle is called its **period**. Examine the graphs of the periodic function and nonperiodic function below. Notice that a cycle may begin at any point on the graph of a function.

Periodic	Not Periodic
	

EXAMPLE 1 Identifying Periodic Functions

Identify whether each function is periodic. If the function is periodic, give the period.

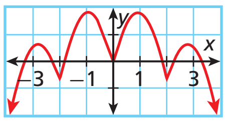
A



The pattern repeats exactly, so the function is periodic. Identify the period by using the start and finish of one cycle.

This function is periodic with period 2.

B

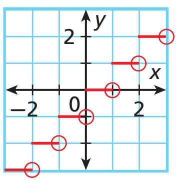


Although there is some symmetry, the pattern does not repeat exactly.

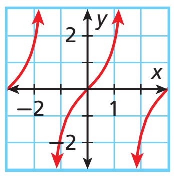
This function is not periodic.

CHECK IT OUT! **Identify whether each function is periodic. If the function is periodic, give the period.**

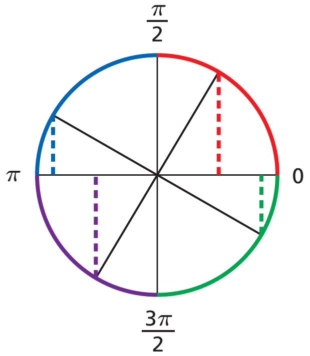
1a.



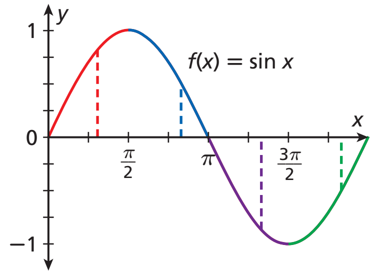
1b.



The trigonometric functions that you studied in Chapter 13 are periodic. You can graph the function $f(x) = \sin x$ on the coordinate plane by using y-values from points on the unit circle where the independent variable x represents the angle θ in standard position.



$x(=\theta)$	y
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$

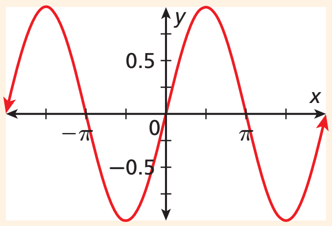
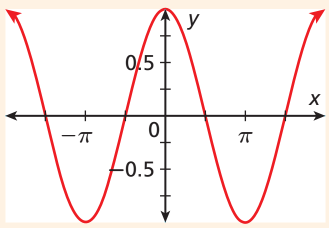


Similarly, the function $f(x) = \cos x$ can be graphed on the coordinate plane by using x-values from points on the unit circle.

The **amplitude** of sine and cosine functions is half of the difference between the maximum and minimum values of the function. The amplitude is always positive.

Know it! Note

Characteristics of the Graphs of Sine and Cosine

FUNCTION	$y = \sin x$	$y = \cos x$
GRAPH		
DOMAIN	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$
RANGE	$\{y \mid -1 \le y \le 1\}$	$\{y \mid -1 \le y \le 1\}$
PERIOD	2π	2π
AMPLITUDE	1	1

Helpful Hint

The graph of the sine function passes through the origin. The graph of the cosine function has y-intercept 1.

You can use the parent functions to graph transformations $y = a \sin bx$ and $y = a \cos bx$. Recall that a indicates a vertical stretch $(|a| > 1)$ or compression $(0 < |a| < 1)$, which changes the amplitude. If a is less than 0, the graph is reflected across the x-axis. The value of b indicates a horizontal stretch or compression, which changes the period.

Know it! Note

Transformations of Sine and Cosine Graphs

For the graphs of $y = a \sin bx$ or $y = a \cos bx$ where $a \neq 0$ and x is in radians,

- the amplitude is $|a|$.
- the period is $\frac{2\pi}{|b|}$.

EXAMPLE 2 Stretching or Compressing Sine and Cosine Functions

Using $f(x) = \sin x$ as a guide, graph the function $g(x) = 3\sin 2x$. Identify the amplitude and period.

Step 1 Identify the amplitude and period.

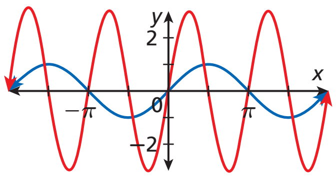
Because $a = 3$, the amplitude is $|a| = |3| = 3$.

Because $b = 2$, the period is $\frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \pi$.

Step 2 Graph.

The curve is vertically stretched by a factor of 3 and horizontally compressed by a factor of $\frac{1}{2}$.

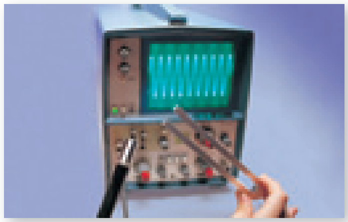
The parent function f has x-intercepts at multiples of π and g has x-intercepts at multiples of $\frac{\pi}{2}$.



The maximum value of g is 3, and the minimum value is -3.

CHECK IT OUT! **2.** Using $f(x) = \cos x$ as a guide, graph the function $h(x) = \frac{1}{3}\cos 2x$. Identify the amplitude and period.

Sine and cosine functions can be used to model real-world phenomena, such as sound waves. Different sounds create different waves. One way to distinguish sounds is to measure *frequency*. **Frequency** is the number of cycles in a given unit of time, so it is the reciprocal of the period of a function.

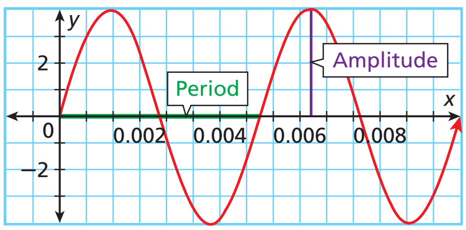


Hertz (Hz) is the standard measure of frequency and represents one cycle per second. For example, the sound wave made by a tuning fork for middle A has a frequency of 440 Hz. This means that the wave repeats 440 times in 1 second.

EXAMPLE 3 *Sound Application*

Use a sine function to graph a sound wave with a period of 0.005 second and an amplitude of 4 cm. Find the frequency in hertz for this sound wave.

Use a horizontal scale where one unit represents 0.001 second. The period tells you that it takes 0.005 seconds to complete one full cycle. The maximum and minimum values are given by the amplitude.



$$\text{frequency} = \frac{1}{\text{period}}$$

$$= \frac{1}{0.005} = 200 \text{ Hz}$$

The frequency of the sound wave is 200 Hz.

CHECK IT OUT! **3.** Use a sine function to graph a sound wave with a period of 0.004 second and an amplitude of 3 cm. Find the frequency in hertz for this sound wave.

Sine and cosine can also be translated as $y = \sin(x - h) + k$ and $y = \cos(x - h) + k$. Recall that a vertical translation by k units moves the graph up $(k > 0)$ or down $(k < 0)$.

A **phase shift** is a horizontal translation of a periodic function. A phase shift of h units moves the graph left $(h < 0)$ or right $(h > 0)$.

EXAMPLE 4 Identifying Phase Shifts for Sine and Cosine Functions

Using $f(x) = \sin x$ as a guide, graph $g(x) = \sin\left(x + \frac{\pi}{2}\right)$. Identify the x-intercepts and phase shift.

Step 1 Identify the amplitude and period.

Amplitude is $|a| = |1| = 1$.

The period is $\frac{2\pi}{|b|} = \frac{2\pi}{|1|} = 2\pi$.

Step 2 Identify the phase shift.

$x + \frac{\pi}{2} = x - \left(-\frac{\pi}{2}\right)$ *Identify h.*

Because $h = -\frac{\pi}{2}$, the phase shift is $\frac{\pi}{2}$ radians to the left.

All x-intercepts, maxima, and minima of $f(x)$ are shifted $\frac{\pi}{2}$ units to the left.

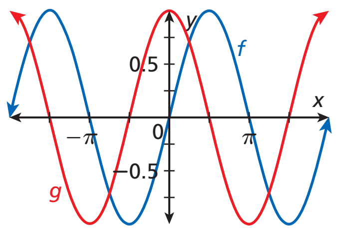
Step 3 Identify the x-intercepts.

The first x-intercept occurs at $-\frac{\pi}{2}$. Because $\sin x$ has two x-intercepts in each period of 2π, the x-intercepts occur at $-\frac{\pi}{2} + n\pi$, where n is an integer.

Step 4 Identify the maximum and minimum values.

The maximum and minimum values occur between the x-intercepts. The maxima occur at $2\pi n$ and have a value of 1. The minima occur at $\pi + 2\pi n$ and have a value of -1.

Step 5 Graph using all of the information about the function.



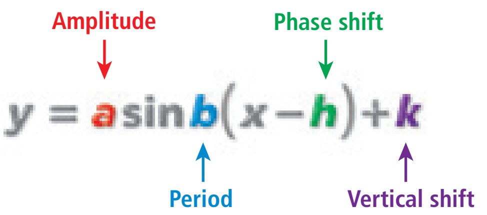
Helpful Hint

The repeating pattern is maximum, intercept, minimum, intercept,.... So intercepts occur twice as often as maximum or minimum values.

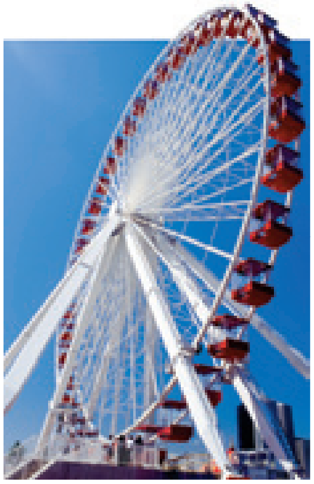
CHECK IT OUT!

4. Using $f(x) = \cos x$ as a guide, graph $g(x) = \cos(x - \pi)$. Identify the x-intercepts and phase shift.

You can combine the transformations of trigonometric functions. Use the values of a, b, h, and k to identify the important features of a sine or cosine function.



EXAMPLE 5 *Entertainment Application*



The Ferris wheel at the landmark Navy Pier in Chicago takes 7 minutes to make one full rotation. The height H in feet above the ground of one of the six-person gondolas can be modeled by $H(t) = 70\sin\frac{2\pi}{7}(t - 1.75) + 80$, where t is time in minutes.

a. Graph the height of a cabin for two complete periods.

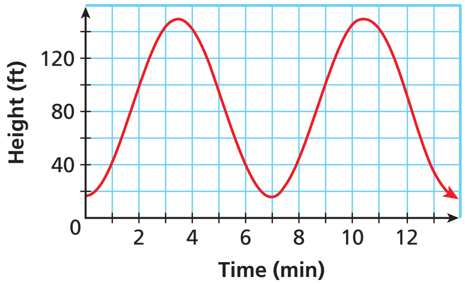
$$H(t) = 70\sin\frac{2\pi}{7}(t - 1.75) + 80 \qquad a = 70,\ b = \frac{2\pi}{7},\ h = 1.75,\ k = 80$$

Step 1 Identify the important features of the graph.

Amplitude: 70

Period: $\frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{2\pi}{7}\right|} = 7$

The period is equal to the time required for one full rotation.



Phase shift: 1.75 minutes right

Vertical shift: 80

There are no x-intercepts.

Maxima: $80 + 70 = 150$ at 3.5 and 10.5

Minima: $80 - 70 = 10$ at 0, 7, and 14

Step 2 Graph using all of the information about the function.

b. What is the maximum height of a cabin?

The maximum height is $80 + 70 = 150$ feet above the ground.

CHECK IT OUT!

5. **What if...?** Suppose that the height H of a Ferris wheel can be modeled by $H(t) = -16\cos\frac{\pi}{45}t + 24$, where t is the time in seconds.

   a. Graph the height of a cabin for two complete periods.

   b. What is the maximum height of a cabin?

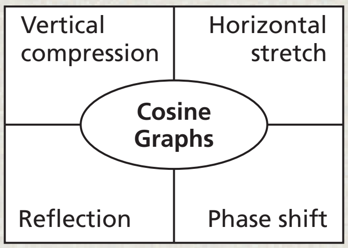
THINK AND DISCUSS

1. **DESCRIBE** how the frequency and period of a periodic function are related. How does this apply to the graph of $f(x) = \cos x$?

2. **EXPLAIN** how the maxima and minima are related to the amplitude and period of sine and cosine functions.

3. **GET ORGANIZED** Copy and complete the graphic organizer. For each type of transformation, give an example and state the period.

Know it! Note



14-1 Exercises

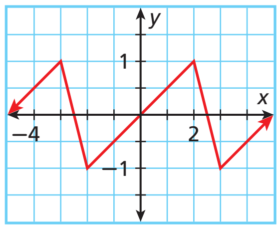
go.hrw.com
Homework Help Online
KEYWORD: MB7 14-1
Parent Resources Online
KEYWORD: MB7 Parent

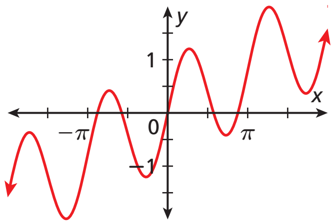
GUIDED PRACTICE

1. **Vocabulary** Periodic functions repeat in regular intervals called __?__. *(cycles* or *periods)*

SEE EXAMPLE 1
p. 990

Identify whether each function is periodic. If the function is periodic, give the period.

2. 

3. 

SEE EXAMPLE 2
p. 992

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the amplitude and the period.

4. $f(x) = 2\sin\frac{1}{2}x$
5. $h(x) = \frac{1}{4}\cos x$
6. $k(x) = \sin \pi x$

SEE EXAMPLE 3
p. 992

7. **Sound** Use a sine function to graph a sound wave with a period of 0.01 second and an amplitude of 6 in. Find the frequency in hertz for this sound wave.

SEE EXAMPLE 4
p. 993

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the x-intercepts and the phase shift.

8. $f(x) = \sin\left(x + \frac{3\pi}{2}\right)$
9. $g(x) = \cos\left(x - \frac{\pi}{2}\right)$
10. $h(x) = \sin\left(x - \frac{\pi}{4}\right)$

SEE EXAMPLE 5
p. 994

11. **Recreation** The height H in feet above the ground of the seat of a playground swing can be modeled by $H(\theta) = -4\cos\theta + 6$, where θ is the angle that the swing makes with a vertical extended to the ground. Graph the height of a swing's seat for $0° \leq \theta \leq 90°$. How high is the swing when $\theta = 60°$?

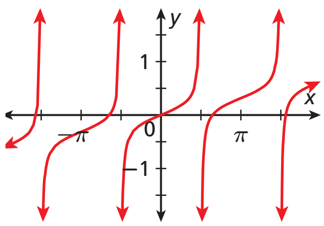
PRACTICE AND PROBLEM SOLVING

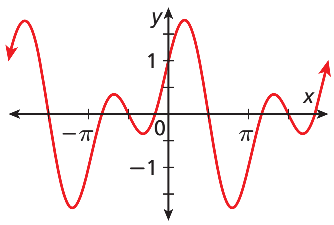
Independent Practice

For Exercises	See Example
12–13	1
14–17	2
18	3
19–22	4
23	5

Extra Practice
Skills Practice p. S30
Application Practice p. S45

Identify whether each function is periodic. If the function is periodic, give the period.

12. 

13. 

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the amplitude and period.

14. $f(x) = 4\cos x$
15. $g(x) = \frac{3}{2}\sin x$
16. $g(x) = -\cos 4x$
17. $j(x) = 6\sin\frac{1}{3}x$

18. **Sound** Use a sine function to graph a sound wave with a period of 0.025 seconds and an amplitude of 5 in. Find the frequency in hertz for this sound wave.

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the x-intercepts and phase shift.

19. $f(x) = \sin(x + \pi)$

20. $h(x) = \cos(x - 3\pi)$

21. $g(x) = \sin\left(x + \frac{3\pi}{4}\right)$

22. $j(x) = \cos\left(x + \frac{\pi}{4}\right)$

23. Oceanography The depth d in feet of the water in a bay at any time is given by $d(t) = \frac{3}{2}\sin\left(\frac{5\pi}{31}t\right) + 23$, where t is the time in hours. Graph the depth of the water. What are the maximum and minimum depths of the water?

LINK

Medicine

An EKG measures the electrical signals that control the rhythm of a beating heart. EKGs are used to diagnose and monitor heart disease.

24. Medicine The figure shows a normal adult electrocardiogram, known as an EKG. Each cycle in the EKG represents one heartbeat.

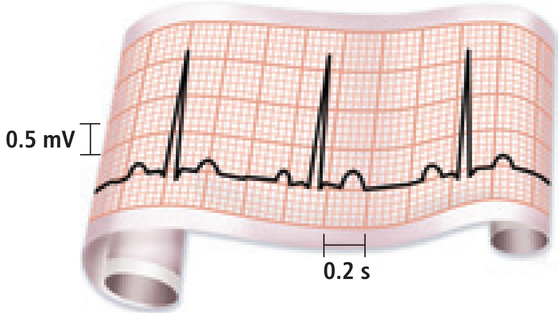
a. What is the period of one heartbeat?

b. The pulse rate is the number of beats in one minute. What is the pulse rate indicated by the EKG?

c. What is the frequency of the EKG?

d. How does the pulse rate relate to the frequency in hertz?

Adult EKG



Determine the amplitude and period for each function. Then describe the transformation from its parent function.

25. $f(x) = \sin\left(x + \frac{\pi}{4}\right) - 1$

26. $h(x) = \frac{3}{4}\cos\frac{\pi}{4}x$

27. $h(x) = \cos(2\pi x) - 2$

28. $j(x) = -3\sin 3x$

Estimation Use a graph of sine or cosine to estimate each value.

29. $\sin 160°$

30. $\cos 50°$

31. $\sin 15°$

32. $\cos 95°$

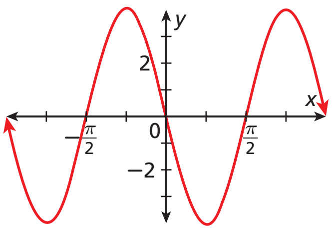
Write both a sine and a cosine function for each set of conditions.

33. amplitude of 6, period of π

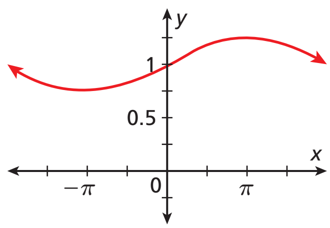
34. amplitude of $\frac{1}{4}$, phase shift of $\frac{2}{3}\pi$ left

Write both a sine and a cosine function that could be used to represent each graph.

35.



36.



MULTI-STEP TEST PREP

37. This problem will prepare you for the Multi-Step Test Prep on page 1004.

The tide in a bay has a maximum height of 3 m and a minimum height of 0 m. It takes 6.1 hours for the tide to go out and another 6.1 hours for it to come back in. The height of the tide h is modeled as a function of time t.

a. What are the period and amplitude of h? What are the maximum and minimum values?

b. Assume that high tide occurs at $t = 0$. What are $h(0)$ and $h(6.1)$?

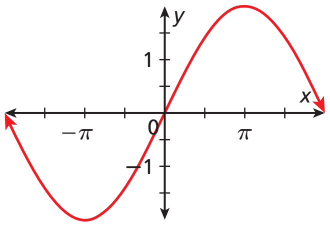
c. Write h in the form $h(t) = a\cos bt + k$.

38. **Critical Thinking** Given the amplitude and period of a sine function, can you find its maximum and minimum values and their corresponding x-values? If not, what information do you need and how would you use it?

39. **Write About It** What happens to the period of $f(x) = \sin b\theta$ when $b > 1$? $b < 1$? Explain.

TEST PREP

40. Which trigonometric function best matches the graph?



(A) $y = \frac{1}{2}\sin x$ (C) $y = \frac{1}{2}\sin 2x$

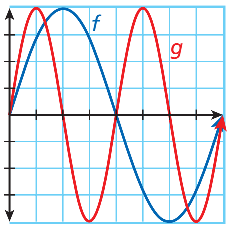
(B) $y = 2\sin x$ (D) $y = 2\sin\frac{1}{2}x$

41. What is the amplitude for $y = -4\cos 3\pi x$?

(F) -4 (H) 4

(G) 3 (J) 3π

42. Based on the graphs, what is the relationship between f and g?



(A) f has twice the amplitude of g.

(B) f has twice the period of g.

(C) f has twice the frequency of g.

(D) f has twice the cycle of g.

43. **Short Response** Using $y = \sin x$ as a guide, graph $y = -4\sin 2(x - \pi)$ on the interval $[0, 2\pi]$ and describe the transformations.

CHALLENGE AND EXTEND

44. Graph $f(x) = \text{Sin}^{-1} x$ and $g(x) = \text{Cos}^{-1} x$. (*Hint:* Use what you learned about graphs of inverse functions in Lesson 9-5 and inverse trigonometric functions in Lesson 13-4.)

Consider the functions $f(\theta) = \frac{1}{2}\sin\theta$ and $g(\theta) = 2\cos\theta$ for $0° \le \theta \le 360°$.

45. On the same set of coordinate axes, graph $f(\theta)$ and $g(\theta)$.

46. What are the approximate coordinates of the points of intersection of $f(\theta)$ and $g(\theta)$?

47. When is $f(\theta) > g(\theta)$?

SPIRAL REVIEW

Use interval notation to represent each set of numbers. *(Lesson 1-1)*

48. $-7 < x \le 5$

49. $x \le -2$ or $1 \le x < 13$

50. $0 \le x \le 9$

51. **Flowers** Adam has \$100 to purchase a combination of roses, lilies, and carnations. Roses cost \$6 each, lilies cost \$2 each, and carnations cost \$4 each. *(Lesson 3-5)*

Roses	6		3	7
Lilies		8	5	
Carnations	11	15		13

a. Write a linear equation in three variables to represent this situation.

b. Complete the table.

Use the given measurements to solve △*ABC*. Round to the nearest tenth. *(Lesson 13-6)*

52. $b = 20, c = 11, A = 165°$

53. $a = 11.9, b = 14.7, c = 26.1$

14-2 Graphs of Other Trigonometric Functions

Objective

Recognize and graph trigonometric functions.

Why learn this?

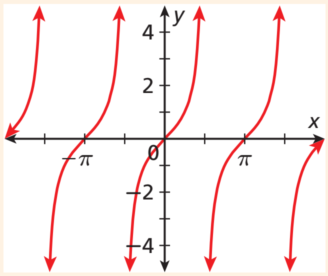
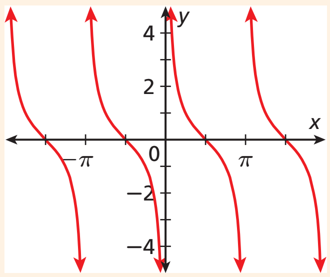
You can use the graphs of reciprocal trigonometric functions to model rotating objects such as lights. (See Exercise 25.)

The tangent and cotangent functions can be graphed on the coordinate plane. The tangent function is undefined when $\theta = \frac{\pi}{2} + \pi n$, where n is an integer. The cotangent function is undefined when $\theta = \pi n$. These values are excluded from the domain and are represented by vertical asymptotes on the graph. Because tangent and cotangent have no maximum or minimum values, amplitude is undefined.

To graph tangent and cotangent, let the variable x represent the angle θ in standard position.

Know it! Note

Characteristics of the Graphs of Tangent and Cotangent

FUNCTION	$y = \tan x$	$y = \cot x$
GRAPH		
DOMAIN	$\left\{x \mid x \neq \frac{\pi}{2} + \pi n,\right.$ where n is an integer$\left.\right\}$	$\left\{x \mid x \neq \pi n,\right.$ where n is an integer$\left.\right\}$
RANGE	$\{y \mid -\infty < y < \infty\}$	$\{y \mid -\infty < y < \infty\}$
PERIOD	π	π
AMPLITUDE	undefined	undefined

Like sine and cosine, you can transform the tangent function.

Know it! Note

Transformations of Tangent Graphs

For the graph of $y = a \tan bx$, where $a \neq 0$ and x is in radians,

- the period is $\frac{\pi}{|b|}$.
- the asymptotes are located at $x = \frac{\pi}{2|b|} + \frac{\pi n}{|b|}$, where n is an integer.

EXAMPLE 1 Transforming Tangent Functions

Using $f(x) = \tan x$ as a guide, graph $g(x) = \tan 2x$. Identify the period, x-intercepts, and asymptotes.

Step 1 Identify the period.

Because $b = 2$, the period is $\frac{\pi}{|b|} = \frac{\pi}{|2|} = \frac{\pi}{2}$.

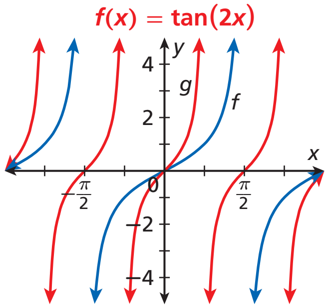
Step 2 Identify the x-intercepts.

The first x-intercept occurs at $x = 0$. Because the period is $\frac{\pi}{2}$, the x-intercepts occur at $\frac{\pi}{2}n$, where n is an integer.

Step 3 Identify the asymptotes.

Because $b = 2$, the asymptotes occur at $x = \frac{\pi}{2|2|} + \frac{\pi n}{|2|}$, or $x = \frac{\pi}{4} + \frac{\pi n}{2}$.

Step 4 Graph using all of the information about the function.



CHECK IT OUT! **1.** Using $f(x) = \tan x$ as a guide, graph $g(x) = 3\tan\frac{1}{2}x$. Identify the period, x-intercepts, and asymptotes.

Know it! Note: Transformations of Cotangent Graphs

For the graph of $y = a\cot bx$, where $a \neq 0$ and x is in radians,

- the period is $\frac{\pi}{|b|}$.
- the asymptotes are located at $x = \frac{\pi n}{|b|}$, where n is an integer.

EXAMPLE 2 Graphing the Cotangent Function

Using $f(x) = \cot x$ as a guide, graph $g(x) = \cot 0.5x$. Identify the period, x-intercepts, and asymptotes.

Step 1 Identify the period.

Because $b = 0.5$, the period is $\frac{\pi}{|b|} = \frac{\pi}{|0.5|} = 2\pi$.

Step 2 Identify the x-intercepts.

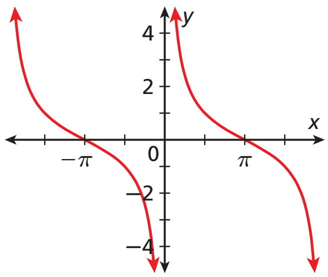
The first x-intercept occurs at $x = \pi$. Because the period is 2π, the x-intercepts occur at $x = \pi + 2\pi n$, where n is an integer.

Step 3 Identify the asymptotes.

Because $b = 0.5$, the asymptotes occur at

$$x = \frac{\pi n}{|0.5|} = 2\pi n.$$

Step 4 Graph using all of the information about the function.

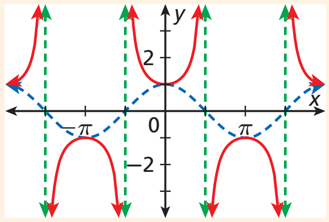
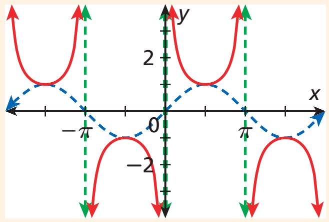


2. Using $f(x) = \cot x$ as a guide, graph $g(x) = -\cot 2x$. Identify the period, x-intercepts, and asymptotes.

Recall that $\sec\theta = \frac{1}{\cos\theta}$. So, secant is undefined where cosine equals zero and the graph will have vertical asymptotes at those locations. Secant will also have the same period as cosine. Sine and cosecant have a similar relationship. Because secant and cosecant have no absolute maxima or minima, amplitude is undefined.

Know it! Note

Characteristics of the Graphs of Secant and Cosecant

FUNCTION	$y = \sec x$	$y = \csc x$
GRAPH		
DOMAIN	$\left\{x \mid x \neq \frac{\pi}{2} + \pi n,\right.$ where n is an integer$\left.\right\}$	$\left\{x \mid x \neq \pi n,\right.$ where n is an integer$\left.\right\}$
RANGE	$\{y \mid y \leq -1, \text{ or } y \geq 1\}$	$\{y \mid y \leq -1, \text{ or } y \geq 1\}$
PERIOD	2π	2π
AMPLITUDE	undefined	undefined

You can graph transformations of secant and cosecant by using what you learned in Lesson 14-1 about transformations of graphs of cosine and sine.

EXAMPLE 3 Graphing Secant and Cosecant Functions

Using $f(x) = \cos x$ as a guide, graph $g(x) = \sec 2x$. Identify the period and asymptotes.

Step 1 Identify the period.

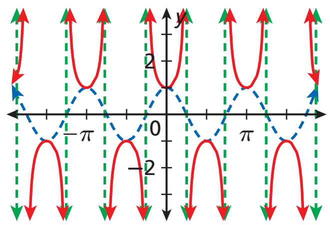
Because $\sec 2x$ is the reciprocal of $\cos 2x$, the graphs will have the same period.

Because $b = 2$ for $\cos 2x$, the period is $\frac{2\pi}{|b|} = \frac{2\pi}{|2|} = \pi$.

Step 2 Identify the asymptotes.

Because the period is π, the asymptotes occur at $x = \frac{\pi}{2|2|} + \frac{\pi}{|2|}n = \frac{\pi}{4} + \frac{\pi}{2}n$, where n is an integer.

Step 3 Graph using all of the information about the function.



3. Using $f(x) = \sin x$ as a guide, graph $g(x) = 2\csc x$. Identify the period and asymptotes.

THINK AND DISCUSS

1. **EXPLAIN** why $f(x) = \sin x$ can be used to graph $g(x) = \csc x$.
2. **EXPLAIN** how the zeros of the cosine function relate to the vertical asymptotes of the graph of the tangent function.
3. **GET ORGANIZED** Copy and complete the graphic organizer.

Know it! Note

Function	Zeros	Asymptotes	Period
$y = \sec x$			
$y = \csc x$			
$y = \cot x$			
$y = \tan x$			

14-2 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 14-2
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

SEE EXAMPLE 1
p. 999

Using $f(x) = \tan x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

1. $k(x) = 2\tan(3x)$
2. $g(x) = \tan\frac{1}{4}x$
3. $h(x) = \tan 2\pi x$

SEE EXAMPLE 2
p. 999

Using $f(x) = \cot x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

4. $j(x) = 0.25\cot x$
5. $p(x) = \cot 2x$
6. $g(x) = \frac{3}{2}\cot x$

SEE EXAMPLE 3
p. 1000

Using $f(x) = \cos x$ or $f(x) = \sin x$ as a guide, graph each function. Identify the period and asymptotes.

7. $g(x) = \frac{1}{2}\sec x$
8. $q(x) = \sec 4x$
9. $h(x) = 3\csc x$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
10–13	1
14–16	2
17–19	3

Extra Practice
Skills Practice p. S30
Application Practice p. S45

Using $f(x) = \tan x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

10. $p(x) = \tan\frac{3}{2}x$
11. $g(x) = \tan\left(x + \frac{\pi}{4}\right)$
12. $h(x) = \frac{1}{2}\tan 4x$
13. $j(x) = -2\tan\frac{\pi}{2}x$

Using $f(x) = \cot x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

14. $h(x) = 4\cot x$
15. $g(x) = \cot\frac{1}{4}x$
16. $j(x) = 0.1\cot x$

Using $f(x) = \cos x$ or $f(x) = \sin x$ as a guide, graph each function. Identify the period and asymptotes.

17. $g(x) = -\sec x$
18. $k(x) = \frac{1}{2}\csc x$
19. $h(x) = \csc(-x)$

MULTI-STEP TEST PREP

20. This problem will prepare you for the Multi-Step Test Prep on page 1004.

Between 1:00 P.M. $(t = 1)$ and 6:00 P.M. $(t = 6)$, the height (in meters) of the tide in a bay is modeled by $h(t) = 0.4 \csc \frac{5\pi}{31} t$.

a. Graph the function for the range $1 \le t \le 6$.

b. At what time does low tide occur?

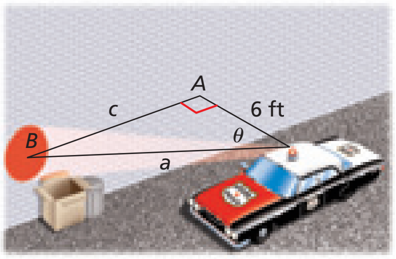
c. What is the height of the tide at low tide?

d. What is the maximum height of the tide during this time span? When does this occur?

Find four values for which each function is undefined.

21. $f(\theta) = \tan\theta$ **22.** $g(\theta) = \cot\theta$ **23.** $h(\theta) = \sec\theta$ **24.** $j(\theta) = \csc\theta$

25. Law Enforcement A police car is parked on the side of the road next to a building. The flashing light on the car is 6 feet from the wall and completes one full rotation every 3 seconds. As the light rotates, it shines on the wall. The equation representing the distance a in feet is $a(t) = 6\sec\left(\frac{2}{3}\pi t\right)$.



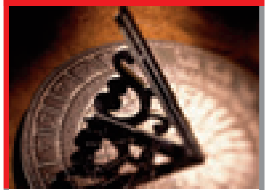
a. What is the period of $a(t)$?

b. Graph the function for $0 \le t \le 3$.

c. Critical Thinking Identify the location of any asymptotes. What do the asymptotes represent?

LINK

Math History

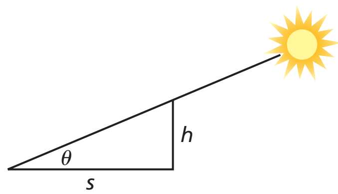


The Greek gnomon was a tall staff, but gnomon is also the part of a sundial that casts a shadow. Based on the variation of shadows at high noon, a gnomon can be used to determine the day of the year, in addition to the time of day.

26. Math History The ancient Greeks used a *gnomon,* a type of tall staff, to tell the time of day based on the lengths of shadows and the altitude θ of the sun above the horizon.

a. Use the figure to write a cotangent function that can be used to find the length of the shadow s in terms of the height of the gnomon h and the angle θ.

b. Graph your answer to part **a** for a gnomon of height 6 ft.



Complete the table by labeling each function as increasing or decreasing.

		$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \pi$	$\pi < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
27.	$\sin x$				
28.	$\csc x$				
29.	$\cos x$				
30.	$\sec x$				
31.	$\tan x$				
32.	$\cot x$				

33. Critical Thinking Based on the table above, what do you observe about the increasing/decreasing relationship between reciprocal pairs of trigonometric functions?

34. **Critical Thinking** How do the signs (whether a function is positive or negative) of reciprocal pairs of trigonometric functions relate?

35. **Write About It** Describe how to graph $f(x) = 3\sec 4x$ by using the graph of $g(x) = 3\cos 4x$.

TEST PREP

36. Which is NOT in the domain of $y = \cot x$?

Ⓐ $-\frac{\pi}{2}$ Ⓑ 0 Ⓒ $\frac{\pi}{2}$ Ⓓ $\frac{3\pi}{2}$

37. What is the range of $f(x) = 3\csc 2\theta$?

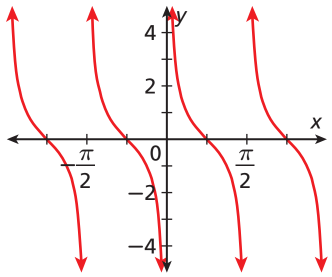
Ⓕ $\{y \mid y \le -1 \text{ or } y \ge 1\}$ Ⓗ $\{y \mid y \le -2 \text{ or } y \ge 2\}$

Ⓖ $\{y \mid y \le -3 \text{ or } y \ge 3\}$ Ⓙ $\left\{y \mid y \le -\frac{1}{2} \text{ or } y \ge \frac{1}{2}\right\}$

38. Which could be the equation of the graph?

Ⓐ $y = \tan 2x$ Ⓒ $y = 2\tan x$

Ⓑ $y = \cot 2x$ Ⓓ $y = 2\cot x$



39. What is the period of $y = \tan\frac{1}{2}x$?

Ⓕ $\frac{\pi}{2}$ Ⓗ 2π

Ⓖ π Ⓙ 4π

40. The graph of which function has a period of $\frac{2\pi}{3}$ and an asymptote at $x = \frac{\pi}{2}$?

Ⓐ $y = \sec\frac{3}{2}x$ Ⓒ $y = \csc\frac{3}{2}x$

Ⓑ $y = \sec 3x$ Ⓓ $y = \csc 3x$

CHALLENGE AND EXTEND

Describe the period, local maximum and minimum values, and phase shift.

41. $f(x) = 4 - 3\csc \pi(x - 1)$ 42. $g(x) = 4\cot\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ 43. $h(x) = 0.5\sec 2\left(x + \frac{\pi}{4}\right)$

44. $f(x) = 9 + 2\tan 3(x + \pi)$ 45. $g(x) = 0.62 + 0.76\sec x$ 46. $h(x) = \csc\frac{\pi}{2}\left(x + \frac{5}{7}\right)$

Graph each trigonometric function and its inverse. Identify the domain and range of the corresponding inverse function.

47. $f(x) = \mathrm{Sec}\,x$ for $0 \le x \le \pi$ and $x \ne \frac{\pi}{2}$ 48. $f(x) = \mathrm{Tan}\,x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

49. $g(x) = \mathrm{Csc}\,x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $x \ne 0$ 50. $g(x) = \mathrm{Cot}\,x$ for $0 < x < \pi$

SPIRAL REVIEW

Find the additive and multiplicative inverse for each number. *(Lesson 1-2)*

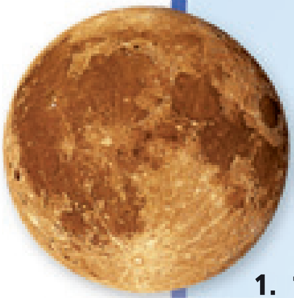
51. $-\frac{1}{10}$ 52. 0.2 53. $-3\sqrt{5}$ 54. $\frac{4}{9}$

55. **Technology** Marjorie's printer prints 30 pages per minute. How many pages does Marjorie's printer print in 22 seconds? *(Lesson 2-2)*

Convert each measure from degrees to radians or from radians to degrees. *(Lesson 13-3)*

56. $45°$ 57. $\frac{3\pi}{4}$ radians 58. $225°$ 59. $-\frac{\pi}{3}$ radians

MULTI-STEP TEST PREP



Trigonometric Graphs

The Tide Is Turning Tides are caused by several factors, but the main factor is the gravitational pull of the Moon. As the Moon revolves around Earth, the Moon causes large bodies of water to swell toward it resulting in rising and falling tides. You can use trigonometric functions to develop a model of a simplified tide.

Tides at the Bay of Fundy		
	Time (h)	Height (m)
High Tide	$t = 0$	16.3
Low Tide	$t = 6.25$	0

1. The highest tides in the world have been measured at the Bay of Fundy, in Nova Scotia, Canada. As shown in the table, high tides in the bay can reach heights of 16.3 m. Assume that it takes 6.25 hours for the tide to completely retreat and then another 6.25 hours for the tide to come back in. Write a periodic function based on the cosine function that models the height of the tide over time.
2. What are the amplitude, period, maximum and minimum values, and phase shift of the function?
3. Graph the function.
4. At time $t = 0$, the tide is at 16.3 m. What is the tide's height after 3 hours? after 9 hours?
5. Will a high tide occur at the same time each day at the Bay of Fundy? Why or why not?
6. It is possible to write a function that models the height of the tide based on the sine function. What is the function? What is the phase shift?

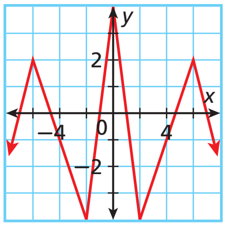
READY TO GO ON?

Quiz for Lessons 14-1 Through 14-2

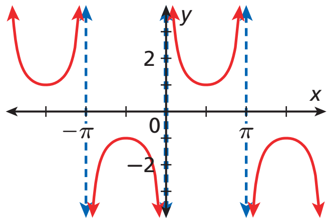
14-1 Graphs of Sine and Cosine

Identify whether each function is periodic. If the function is periodic, give the period.

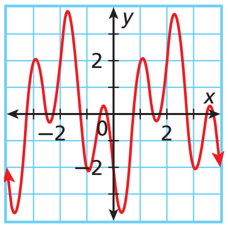
1.



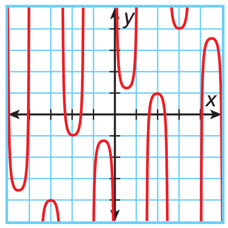
2.



3.



4.



Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the amplitude and period.

5. $f(x) = \sin 4x$ **6.** $g(x) = -3\sin x$ **7.** $h(x) = 0.25\cos \pi x$

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the x-intercepts and phase shift.

8. $f(x) = \cos\left(x - \frac{3\pi}{2}\right)$ **9.** $g(x) = \sin\left(x - \frac{3\pi}{4}\right)$ **10.** $h(x) = \cos\left(x + \frac{5\pi}{4}\right)$

11. The torque τ applied to a bolt is given by $\tau(x) = Fr\sin x$, where r is the length of the wrench in meters, F is the applied force in newtons, and x is the angle between F and r in radians. Graph the torque for a 0.5 meter wrench and a force of 500 newtons for $0 \le x \le \frac{\pi}{2}$. What is the torque for an angle of $\frac{\pi}{3}$?

14-2 Graphs of Other Trigonometric Functions

Using $f(x) = \tan x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

12. $f(x) = \frac{1}{2}\tan 4x$ **13.** $g(x) = -2\tan\frac{1}{2}x$ **14.** $h(x) = \tan\frac{1}{2}\pi x$

Using $f(x) = \cot x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

15. $g(x) = -2\cot x$ **16.** $h(x) = \cot 0.5x$ **17.** $j(x) = \cot 4x$

Using $f(x) = \cos x$ or $f(x) = \sin x$ as a guide, graph each function. Identify the period and asymptotes.

18. $f(x) = -2\sec x$ **19.** $g(x) = \frac{1}{4}\csc x$ **20.** $h(x) = \sec \pi x$

14-3 Technology LAB

Use with Lesson 14-3

Graph Trigonometric Identities

You can use a graphing calculator to compare graphs and make conjectures about trigonometric identities.

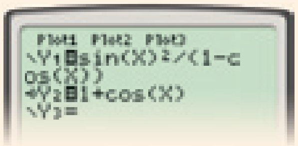
go.hrw.com
Lab Resources Online
KEYWORD: MB7 Lab14

Activity

Determine whether $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$ is a possible identity.

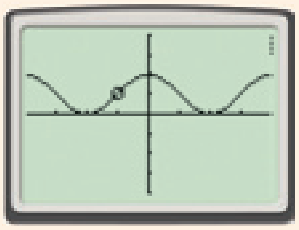
If the equation is an identity, there should be no visible difference in the graphs of the left- and right-hand sides of the equation.

1. Enter $\frac{\sin^2 x}{1 - \cos x}$ as **Y1** and $1 + \cos x$ as **Y2**. For **Y2**, select the mode represented by the 0 with a line through it. This will help you see the path of the graph.



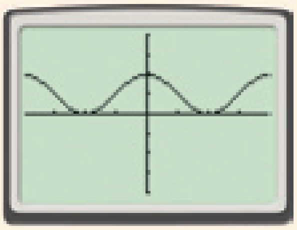
2. Set the graphing window by using ZOOM and **7:ZTrig.**

3. Watch the calculator as the graphs are generated. As **Y2** is being graphed, a circle will move along the path of the graph.



4. The path of the circle, **Y2**, traced the graph of **Y1**. The graphs appear to be the same.

   Because the graphs appear to be identical, $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$ is most likely an identity. Use algebra to confirm.



Try This

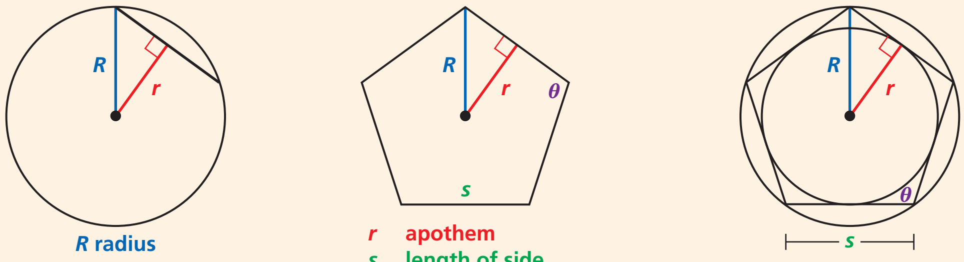
1. **Make a Conjecture** Determine whether $\sec x - \tan x \sin x = \cos x$ is a possible identity.
2. Prove or disprove your answer to Problem 1 by using algebra.
3. **Make a Conjecture** Determine whether $\frac{1 + \tan x}{1 + \cot x} = \tan x$ is a possible identity.
4. Prove or disprove your answer to Problem 3 by using algebra.

Connecting Algebra to **Geometry**

Angle Relationships

Angle relationships in circles and polygons can be used to solve problems.

See Skills Bank page S62



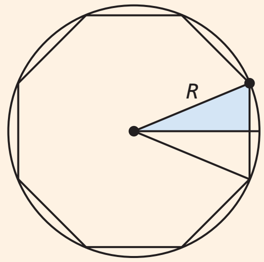
The figures show regular polygons. A **regular polygon** has sides of equal length and equal interior angles. Here are some useful relationships for regular polygons.

R bisects θ.	$\theta = \left(\frac{n-2}{n}\right)180^\circ$	$r = R\cos\left(\frac{180}{n}\right)$	$s = 2r\tan\left(\frac{180}{n}\right) = 2R\sin\left(\frac{180}{n}\right)$

Example

A regular octagon is inscribed in a circle with a radius of 5 cm. What is the length of each side of the octagon?

Make a sketch of the problem.



$s = 2R\sin\left(\frac{180}{n}\right)$ — *Choose a formula relating the radius of the circumscribed circle to the side length of the polygon.*

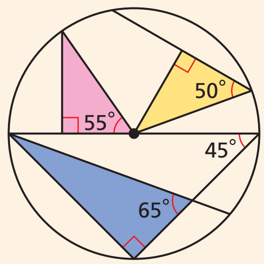
$s = 2(5)\sin\left(\frac{180}{8}\right)$ — *Substitute 5 for R and 8 for n.*

$s = 10\sin 22.5^\circ \approx 3.83 \text{ cm}$

Try This

Solve each problem. Round each answer to the nearest hundredth.

1. A circle is inscribed in an equilateral triangle with 8 in. sides. What is the diameter of the circle? What is the altitude of the triangle?
2. An isosceles right triangle is inscribed in a semicircle with a radius of 20 cm. What are the lengths of the three sides of the triangle?
3. The interior angles of a regular polygon each measure 150°. If this polygon is inscribed in a circle with a 10 in. diameter, how long is each side of the polygon?
4. Use the figure to find the side lengths of all three shaded triangles if the diameter of the circle is 10 cm.



14-3 Fundamental Trigonometric Identities

Objective
Use fundamental trigonometric identities to simplify and rewrite expressions and to verify other identities.

Who uses this?
Ski supply manufacturers can use trigonometric identities to determine the type of wax to use on skis. (See Example 3.)

You can use trigonometric identities to simplify trigonometric expressions. Recall that an identity is a mathematical statement that is true for all values of the variables for which the statement is defined.

A derivation for a Pythagorean identity is shown below.

$$x^2 + y^2 = r^2$$ *Pythagorean Theorem*

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$ *Divide both sides by r^2.*

$$\cos^2\theta + \sin^2\theta = 1$$ *Substitute $\cos\theta$ for $\frac{x}{r}$ and $\sin\theta$ for $\frac{y}{r}$.*

Know it! Note

Fundamental Trigonometric Identities

Reciprocal Identities	Tangent and Cotangent Ratio Identities	Pythagorean Identities	Negative-Angle Identities
$\csc\theta = \frac{1}{\sin\theta}$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cos^2\theta + \sin^2\theta = 1$	$\sin(-\theta) = -\sin\theta$
$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$	$1 + \tan^2\theta = \sec^2\theta$	$\cos(-\theta) = \cos\theta$
$\cot\theta = \frac{1}{\tan\theta}$		$\cot^2\theta + 1 = \csc^2\theta$	$\tan(-\theta) = -\tan\theta$

To prove that an equation is an identity, alter one side of the equation until it is the same as the other side. Justify your steps by using the fundamental identities.

EXAMPLE 1 **Proving Trigonometric Identities**

Prove each trigonometric identity.

A $\sec\theta = \csc\theta\tan\theta$

$\sec\theta = \csc\theta\tan\theta$ *Choose the right-hand side to modify.*

$= \left(\frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)$ *Reciprocal and ratio identities*

$= \frac{1}{\cos\theta}$ *Simplify.*

$= \sec\theta$ *Reciprocal identity*

Helpful Hint
You may start with either side of the given equation. It is often easier to begin with the more complicated side and simplify it to match the simpler side.

Prove each trigonometric identity.

B $\csc(-\theta) = -\csc\theta$

$$\csc(-\theta) = -\csc\theta$$ *Choose the left-hand side to modify.*

$$\frac{1}{\sin(-\theta)} =$$ *Reciprocal identity*

$$\frac{1}{-\sin\theta} =$$ *Negative-angle identity*

$$-\left(\frac{1}{\sin\theta}\right) = -\csc\theta$$

$$-\csc\theta = -\csc\theta$$ *Reciprocal identity*

CHECK IT OUT! **Prove each trigonometric identity.**

1a. $\sin\theta\cot\theta = \cos\theta$ **1b.** $1 - \sec(-\theta) = 1 - \sec\theta$

You can use the fundamental trigonometric identities to simplify expressions.

EXAMPLE 2 Using Trigonometric Identities to Rewrite Trigonometric Expressions

Rewrite each expression in terms of cos *θ*, and simplify.

A $\dfrac{\sin^2\theta}{1 - \cos\theta}$

$$\frac{1 - \cos^2\theta}{1 - \cos\theta}$$ *Pythagorean identity*

$$\frac{(1 + \cos\theta)(1 - \cos\theta)}{1 - \cos\theta}$$ *Factor the difference of two squares.*

$$\frac{(1 + \cos\theta)\cancel{(1 - \cos\theta)}}{\cancel{1 - \cos\theta}}$$ *Simplify.*

$$1 + \cos\theta$$

B $\sec\theta - \tan\theta\sin\theta$

$$\frac{1}{\cos\theta} - \left(\frac{\sin\theta}{\cos\theta}\right)\cdot\sin\theta$$ *Substitute.*

$$\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}$$ *Multiply.*

$$\frac{1 - \sin^2\theta}{\cos\theta}$$ *Subtract fractions.*

$$\frac{\cos^2\theta}{\cos\theta}$$ *Pythagorean identity*

$$\cos\theta$$ *Simplify.*

Helpful Hint

If you get stuck, try converting all of the trigonometric functions into sine and cosine functions.

CHECK IT OUT! **Rewrite each expression in terms of sin *θ*, and simplify.**

2a. $\dfrac{\cos^2\theta}{1 - \sin\theta}$ **2b.** $\cot^2\theta$

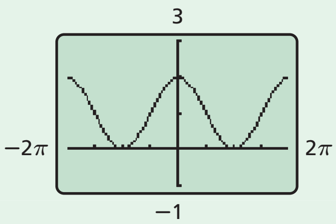
Student to Student

Graphing to Check for Equivalent Expressions

Julia Zaragoza
Oak Ridge
High School

I like to use a graphing calculator to check for equivalent expressions.

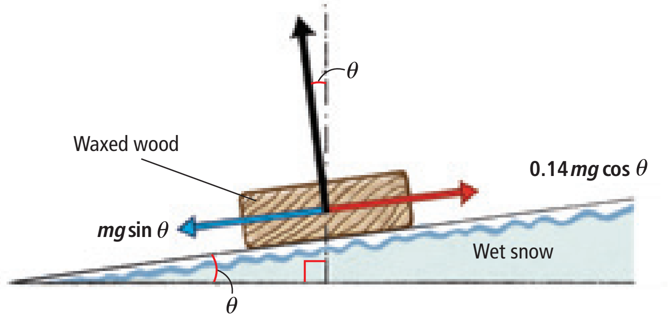
For Example 2A, enter $y = \dfrac{\sin^2\theta}{(1 - \cos\theta)}$ *and* $y = 1 + \cos\theta$*. Graph both functions in the same viewing window.*



The graphs appear to coincide, so the expressions are most likely equivalent.

EXAMPLE 3 Sports Application

A ski supply company is testing the friction of a new ski wax by placing a waxed wood block on an inclined plane of wet snow. The incline plane is slowly raised until the wood block begins to slide.



Reading Math

The symbol μ is read as "mu."

At the instant the block starts to slide, the component of the weight of the block parallel to the incline, $mg\sin\theta$, and the resistive force of friction, $\mu mg\cos\theta$, are equal. μ is the coefficient of friction. At what angle will the block start to move if $\mu = 0.14$?

Set the expression for the weight component equal to the expression for the force of friction.

$mg\sin\theta = \mu mg\cos\theta$

$\sin\theta = \mu\cos\theta$	*Divide both sides by mg.*
$\sin\theta = 0.14\cos\theta$	*Substitute 0.14 for μ.*
$\frac{\sin\theta}{\cos\theta} = 0.14$	*Divide both sides by $\cos\theta$.*
$\tan\theta = 0.14$	*Ratio identity*
$\theta \approx 8°$	*Evaluate inverse tangent.*

The wood block will start to move when the wet snow incline is raised to an angle of about 8°.

CHECK IT OUT!

3. Use the equation $mg\sin\theta = \mu mg\cos\theta$ to determine the angle at which a waxed wood block on a wood incline with $\mu = 0.4$ begins to slide.

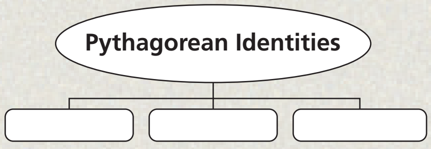
THINK AND DISCUSS

1. DESCRIBE how you prove that an equation is an identity.

2. EXPLAIN which identity can be used to prove that $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer by writing the three Pythagorean identities.



14-3 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 14-3
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

SEE EXAMPLE 1
p. 1008

Prove each trigonometric identity.

1. $\sin\theta\sec\theta = \tan\theta$
2. $\cot(-\theta) = -\cot\theta$
3. $\cos^2\theta\left(\sec^2\theta - 1\right) = \sin^2\theta$

SEE EXAMPLE 2
p. 1009

Rewrite each expression in terms of cos θ, and simplify.

4. $\csc\theta\tan\theta$
5. $\left(1 + \sec^2\theta\right)\left(1 - \sin^2\theta\right)$
6. $\sin^2\theta + \cos^2\theta + \tan^2\theta$

SEE EXAMPLE 3
p. 1010

7. **Physics** Use the equation $mg\sin\theta = \mu mg\cos\theta$ to determine the angle at which a glass-top table can be tilted before a glass plate on the table begins to slide. Assume $\mu = 0.94$.

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
8–11	1
12–15	2
16	3

Extra Practice

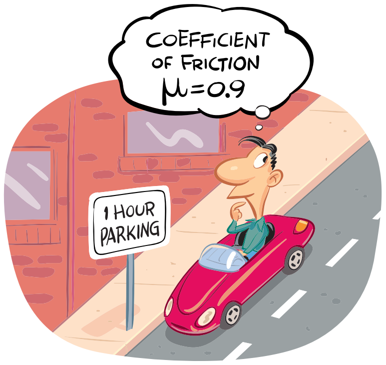
Skills Practice p. S31
Application Practice p. S45

Prove each trigonometric identity.

8. $\sec\theta\cot\theta = \csc\theta$
9. $\dfrac{\sin\theta - \cos\theta}{\sin\theta} = 1 - \cot\theta$
10. $\tan\theta\sin\theta = \sec\theta - \cos\theta$
11. $\sec^2\theta\left(1 - \cos^2\theta\right) = \tan^2\theta$

Rewrite each expression in terms of sin θ, and simplify.

12. $\dfrac{\cos^2\theta}{1 + \sin\theta}$
13. $\dfrac{\tan\theta}{\cot\theta}$
14. $\cos\theta\cot\theta + \sin\theta$
15. $\dfrac{\sec^2\theta - 1}{1 + \tan^2\theta}$



16. **Physics** Use the equation $mg\sin\theta = \mu mg\cos\theta$ to determine the steepest slope of the street shown on which a car with rubber tires can park without sliding.

Multi-Step Rewrite each expression in terms of a single trigonometric function.

17. $\tan\theta\cot\theta$
18. $\sin\theta\cot\theta\tan\theta$
19. $\cos\theta + \sin\theta\tan\theta$
20. $\sin\theta\csc\theta - \cos^2\theta$
21. $\cos^2\theta\sec\theta\csc\theta$
22. $\cos\theta\left(\tan^2\theta + 1\right)$
23. $\csc\theta\left(1 - \cos^2\theta\right)$
24. $\csc\theta\cos\theta\tan\theta$
25. $\dfrac{\sin\theta}{1 - \cos^2\theta}$
26. $\dfrac{\sin^2\theta}{1 - \cos^2\theta}$
27. $\dfrac{\tan\theta}{\sin\theta\sec\theta}$
28. $\dfrac{\cos\theta}{\sin\theta\cot\theta}$
29. $\tan\theta\left(\tan\theta + \cot\theta\right)$
30. $\sin^2\theta + \cos^2\theta + \cot^2\theta$
31. $\sin^2\theta\sec\theta\csc\theta$

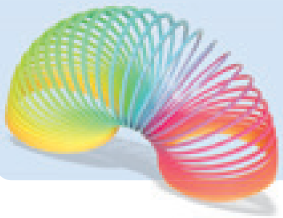
Verify each identity.

32. $\dfrac{\cos\theta - 1}{\cos^2\theta} = \sec\theta - \sec^2\theta$
33. $\sin^2\theta\left(\csc^2\theta - 1\right) = \cos^2\theta$
34. $\tan\theta + \cot\theta = \sec\theta\csc\theta$
35. $\dfrac{\cos\theta}{1 - \sin^2\theta} = \sec\theta$
36. $\dfrac{1 - \cos^2\theta}{\tan\theta} = \sin\theta\cos\theta$
37. $\dfrac{\csc^2\theta}{1 + \tan^2\theta} = \cot^2\theta$

Prove each fundamental identity without using any of the other fundamental identities. (*Hint:* Use the trigonometric ratios with *x*, *y*, and *r*.)

38. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
39. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
40. $1 + \cot^2\theta = \csc^2\theta$
41. $\csc\theta = \dfrac{1}{\sin\theta}$
42. $\sec\theta = \dfrac{1}{\cos\theta}$
43. $1 + \tan^2\theta = \sec^2\theta$

MULTI-STEP TEST PREP



44. This problem will prepare you for the Multi-Step Test Prep on page 1034.

The displacement y of a mass attached to a spring is modeled by $y(t) = 5 \sin t$, where t is the time in seconds. The displacement z of another mass attached to a spring is modeled by $z(t) = 2.6 \cos t$.

a. The two masses are set in motion at $t = 0$. When do the masses have the same displacement for the first time?

b. What is the displacement at this time?

c. At what other times will the masses have the same displacement?

Graphing Calculator **Use a graphing calculator to determine whether each of the following equations represents an identity. (*Hint:* You may need to rewrite the equations in terms of sine, cosine, and tangent.)**

45. $(\csc\theta - 1)(\csc\theta + 1) = \tan^2\theta$

46. $\sec\theta - \cos\theta = \sin\theta$

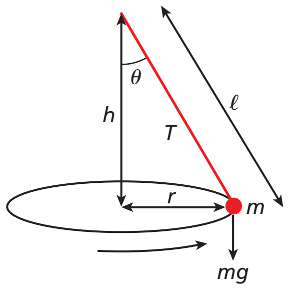
47. $\cos\theta(\sec\theta + \cos\theta\csc^2\theta) = \csc^2\theta$

48. $\cot\theta(\cos\theta + \sin\theta\tan\theta) = \csc\theta$

49. $\cos\theta = 0.99\cos\theta$

50. $\sin\theta\cos\theta = \tan\theta - \tan\theta\sin^2\theta$

51. Physics A conical pendulum is created by a pendulum that travels in a circle rather than side to side and traces out the shape of a cone. The radius r of the base of the cone is given by the formula $r = \frac{g\tan\theta}{\omega^2}$, where g represents the force of gravity and ω represents the angular velocity of the pendulum.



a. Use $\omega = \sqrt{\frac{g}{\ell\cos\theta}}$ and fundamental trigonometric identities to rewrite the formula for the radius.

b. Find a formula for ℓ in terms of g, ω, and a single trigonometric function.

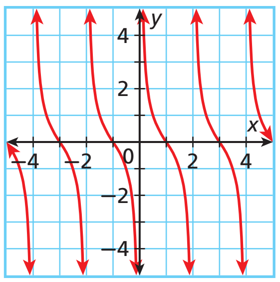
Critical Thinking **A function is called odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$.**

52. Which of the six trigonometric functions are odd? Which are even?

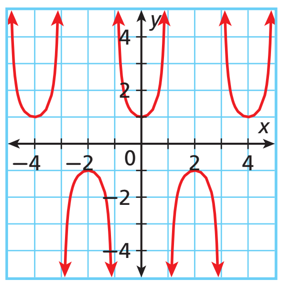
53. What distinguishes the graph of an odd function from an even function or a function that is neither odd nor even?

54. Determine whether the following functions are odd, even, or neither.

a.



b.



55. Critical Thinking In how many equivalent forms can $\tan\theta = \frac{\sin\theta}{\cos\theta}$ be expressed? Write at least three of its forms.

56. Write About It Use the fact that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ to explain why $\tan(-\theta) = -\tan\theta$.

TEST PREP

57. Which expression is equivalent to $\sec\theta \sin\theta$?

Ⓐ $\sin\theta$ Ⓑ $\cos\theta$ Ⓒ $\csc\theta$ Ⓓ $\tan\theta$

58. Which expression is NOT equivalent to the other expressions?

Ⓕ $\sec\theta \csc\theta$ Ⓖ $\frac{1}{\sin\theta\cos\theta}$ Ⓗ $\frac{\tan\theta}{\sin^2\theta}$ Ⓙ $\frac{\cos^2\theta}{\cot\theta}$

59. Which trigonometric statement is NOT an identity?

Ⓐ $1 + \cos^2\theta = \sin^2\theta$ Ⓒ $1 + \tan^2\theta = \sec^2\theta$

Ⓑ $\csc^2\theta - 1 = \cot^2\theta$ Ⓓ $1 - \sin^2\theta = \cos^2\theta$

60. Which is equivalent to $1 - \sec^2\theta$?

Ⓕ $\tan^2\theta$ Ⓖ $-\tan^2\theta$ Ⓗ $\cot^2\theta$ Ⓙ $-\cot^2\theta$

61. **Short Response** Verify that $\sin\theta + \cot\theta\cos\theta = \csc\theta$ is an identity. Write the justification for each step.

CHALLENGE AND EXTEND

Write each expression as a single fraction.

62. $\frac{1}{\cos\theta} + \frac{1}{\cos^2\theta}$

63. $\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$

64. $1 - \frac{\cos\theta}{\sin\theta}$

65. $\frac{1}{1-\cos\theta} - \frac{\cos\theta}{1-\cos^2\theta}$

Simplify.

66. $\dfrac{\frac{1}{\sin^2\theta} - 1}{\frac{\cos^2\theta}{\sin^2\theta}}$

67. $\dfrac{\frac{1}{\sin\theta} + \frac{1}{\cos\theta}}{\frac{1}{\sin\theta\cos\theta}}$

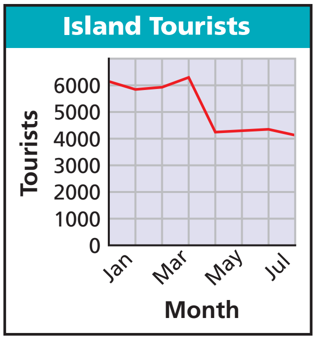
68. $\dfrac{\frac{1}{\sin\theta} - \frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}}$

69. $\dfrac{1 - \frac{1}{\sin\theta}}{1 - \frac{1}{\sin^2\theta}}$

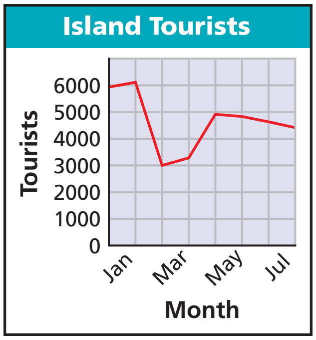
SPIRAL REVIEW

70. **Travel** A statistician kept a record of the number of tourists in Hawaii for six months. Match each situation to its corresponding graph. *(Lesson 9-1)*

A



B



a. There were predictions of hurricanes in March and April.

b. High airfares and high temperatures cause tourism to drop off in the summer.

Find each probability. *(Lesson 11-3)*

71. rolling a 4 on a number cube and a 4 on another number cube

72. getting heads on both tosses when a coin is tossed 2 times

Find four values for which each function is undefined. *(Lesson 14-2)*

73. $y = -\tan\theta$ **74.** $y = \sec(0.5\theta)$ **75.** $y = -\csc\theta$

14-4 Sum and Difference Identities

Objectives

Evaluate trigonometric expressions by using sum and difference identities.

Use matrix multiplication with sum and difference identities to perform rotations.

Vocabulary

rotation matrix

Why learn this?

You can use sum and difference identities and matrices to form images made from rotations. (See Example 4.)

Matrix multiplication and sum and difference identities are tools to find the coordinates of points rotated about the origin on a plane.

Sum and Difference Identities

Sum Identities	Difference Identities
$\sin(A+B) = \sin A\cos B + \cos A\sin B$	$\sin(A-B) = \sin A\cos B - \cos A\sin B$
$\cos(A+B) = \cos A\cos B - \sin A\sin B$	$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$	$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

EXAMPLE 1 Evaluating Expressions with Sum and Difference Identities

Find the exact value of each expression.

A $\sin 75°$

$\sin 75° = \sin(30° + 45°)$ *Write 75° as the sum 30° + 45° because trigonometric values of 30° and 45° are known.*

$= \sin 30°\cos 45° + \cos 30°\sin 45°$ *Apply identity for sin(A + B).*

$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$ *Evaluate.*

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$ *Simplify.*

B $\cos\left(-\frac{\pi}{12}\right)$

$\cos\left(-\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$ *Write $-\frac{\pi}{12}$ as the difference $\frac{\pi}{6} - \frac{\pi}{4}$.*

$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} + \sin\frac{\pi}{6}\sin\frac{\pi}{4}$ *Apply the identity for cos(A − B).*

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$ *Evaluate.*

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$ *Simplify.*

Helpful Hint

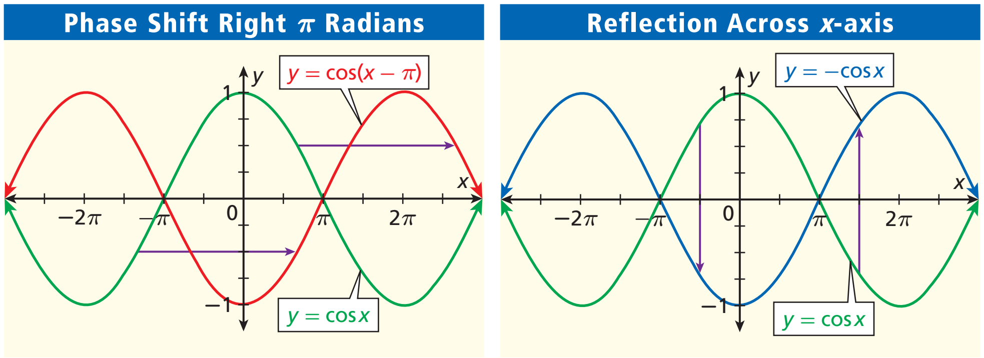
In Example 1B, there is more than one way to get $-\frac{\pi}{12}$. For example, $\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$ or $\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$.

CHECK IT OUT! **Find the exact value of each expression.**

1a. $\tan 105°$

1b. $\sin\left(-\frac{11\pi}{12}\right)$

Shifting the cosine function right π radians is equivalent to reflecting it across the x-axis. A proof of this is shown in Example 2 by using a difference identity.



EXAMPLE 2 Proving Identities with Sum and Difference Identities

Prove the identity $\cos(x - \pi) = -\cos x$.

$\cos(x - \pi) = -\cos x$	*Choose the left-hand side to modify.*
$\cos x \cos \pi + \sin x \sin \pi =$	*Apply the identity for cos(A − B).*
$-1 \cdot \cos x + 0 \cdot \sin x =$	*Evaluate.*
$-\cos x = -\cos x$	*Simplify.*

CHECK IT OUT!

2. Prove the identity $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$.

EXAMPLE 3 Using the Pythagorean Theorem with Sum and Difference Identities

Find $\tan(A + B)$ if $\sin A = -\frac{7}{25}$ with $180° < A < 270°$ and if $\cos B = \frac{8}{17}$ with $0° < B < 180°$.

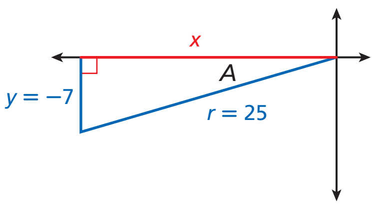
Remember!

Refer to Lessons 13-2 and 13-3 to review reference angles.

Step 1 Find $\tan A$ and $\tan B$.

Use reference angles and the ratio definitions $\sin A = \frac{y}{r}$ and $\cos B = \frac{x}{r}$. Draw a triangle in the appropriate quadrant and label x, y, and r for each angle.

In Quadrant III (QIII), $180° < A < 270°$ and $\sin A = -\frac{7}{25}$.

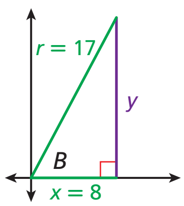


$$x^2 + (-7)^2 = 25^2$$

$$x = -\sqrt{625 - 49} = -24$$

Thus, $\tan A = \frac{y}{x} = \frac{7}{24}$.

In Quadrant I (QI), $0° < B < 180°$ and $\cos B = \frac{8}{17}$.



$$8^2 + y^2 = 17^2$$

$$y = \sqrt{289 - 64} = 15$$

Thus, $\tan B = \frac{y}{x} = \frac{15}{8}$.

Step 2 Use the angle-sum identity to find $\tan(A + B)$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$ *Apply identity for tan(A + B).*

$$= \frac{\left(\frac{7}{24}\right) + \left(\frac{15}{8}\right)}{1 - \left(\frac{7}{24}\right)\left(\frac{15}{8}\right)}$$ *Substitute $\frac{7}{24}$ for tan A and $\frac{15}{8}$ for tan B.*

$$\tan(A+B) = \frac{\frac{52}{24}}{1 - \frac{35}{64}}, \text{ or } \frac{416}{87}$$ *Simplify.*

CHECK IT OUT!

3. Find $\sin(A - B)$ if $\sin A = \frac{4}{5}$ with $90° < A < 180°$ and if $\cos B = \frac{3}{5}$ with $0° < B < 90°$.

To rotate a point $P(x, y)$ through an angle θ, use a **rotation matrix**.

The sum identities for sine and cosine are used to derive the system of equations that yields the rotation matrix.

Know it! Note

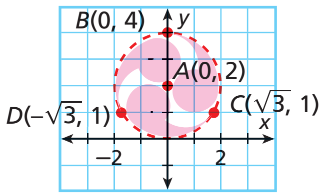
Using a Rotation Matrix

If $P(x, y)$ is any point in a plane, then the coordinates $P'(x', y')$ of the image after a rotation of θ degrees counterclockwise about the origin can be found by using the rotation matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

EXAMPLE 4 Using a Rotation Matrix

Find the coordinates, to the nearest hundredth, of the points in the figure shown after a 30° rotation about the origin.



Step 1 Write matrices for a 30° rotation and for the points in the figure.

$$R_{30°} = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$$ *Rotation matrix*

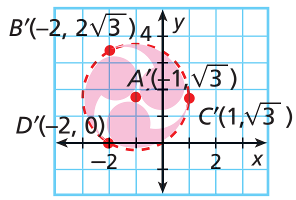
$$S = \begin{bmatrix} 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 2 & 4 & 1 & 1 \end{bmatrix}$$ *Matrix of point coordinates*

Step 2 Find the matrix product.

$$R_{30°} \times S = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 0 & 0 & \sqrt{3} & -\sqrt{3} \\ 2 & 4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 1 & -2 \\ \sqrt{3} & 2\sqrt{3} & \sqrt{3} & 0 \end{bmatrix}$$

Step 3 The approximate coordinates of the points after a 30° rotation are $A'(-1, \sqrt{3})$, $B'(-2, 2\sqrt{3})$, $C'(1, \sqrt{3})$, and $D'(-2, 0)$.

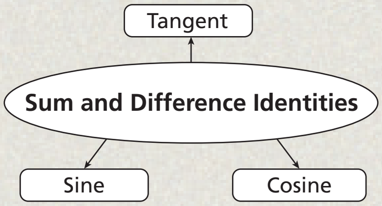


CHECK IT OUT!

4. Find the coordinates, to the nearest hundredth, of the points in the original figure after a 60° rotation about the origin.

THINK AND DISCUSS

1. **DESCRIBE** three different ways that you can use the difference identity to find the exact value of $\sin 15°$.
2. **EXPLAIN** the similarities and differences between the identity formulas for sine and cosine. How do the signs of the terms relate to whether the identity is a sum or a difference?
3. **GET ORGANIZED** Copy and complete the graphic organizer. For each type of function, give the sum and difference identity and an example.



Know it! Note

14-4 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 14-4
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

1. **Vocabulary** A geometric rotation requires that a center point of rotation be defined. Which point and which direction does a rotation matrix such as R_θ assume?

SEE EXAMPLE 1 p. 1014

Find the exact value of each expression.

2. $\cos 105°$
3. $\sin \frac{11\pi}{12}$
4. $\tan \frac{\pi}{12}$
5. $\cos(-75°)$

SEE EXAMPLE 2 p. 1015

Prove each identity.

6. $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
7. $\tan(\pi + x) = \tan x$
8. $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$

SEE EXAMPLE 3 p. 1015

Find each value if $\sin A = -\frac{12}{13}$ with $180° < A < 270°$ and if $\sin B = \frac{4}{5}$ with $90° < B < 180°$.

9. $\sin(A + B)$
10. $\cos(A - B)$
11. $\tan(A + B)$
12. $\tan(A - B)$

SEE EXAMPLE 4 p. 1016

13. Find the coordinates, to the nearest hundredth, of the vertices of triangle ABC with $A(0, 2)$, $B(0, -1)$, and $C(3, 0)$ after a 120° rotation about the origin.

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
14–17	1
18–20	2
21–24	3
25	4

Extra Practice
Skills Practice p. S31
Application Practice p. S45

Find the exact value of each expression.

14. $\sin \frac{7\pi}{12}$
15. $\tan 165°$
16. $\sin 195°$
17. $\cos \frac{11\pi}{12}$

Prove each identity.

18. $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$
19. $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$
20. $\tan(x - 2\pi) = \tan x$

Find each value if $\cos A = -\frac{12}{13}$ with $90° < A < 180°$ and if $\sin B = -\frac{4}{5}$ with $270° < B < 360°$.

21. $\sin(A + B)$
22. $\tan(A - B)$
23. $\cos(A + B)$
24. $\cos(A - B)$

25. Find the coordinates, to the nearest hundredth, of the vertices of figure *ABC* with $A(0, 2)$, $B(1, 2)$, and $C(0, 1)$ after a 45° rotation about the origin.

Find the exact value of each expression.

26. $\sin 165°$

27. $\tan(-105°)$

28. $\cos 195°$

29. $\sin(-15°)$

30. $\cos \frac{19\pi}{12}$

31. $\tan \frac{5\pi}{12}$

32. $\sin 255°$

33. $\tan 195°$

34. $\cos \frac{\pi}{12}$

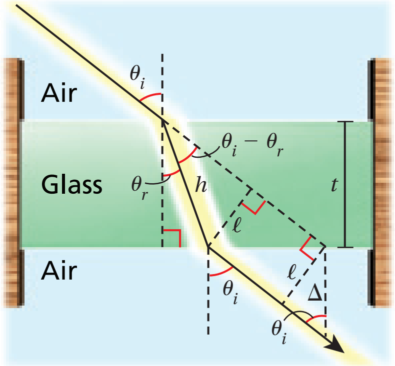
Find the value for each unknown angle given that $0° \le \theta \le 180°$.

35. $\cos(\theta - 30°) = \frac{1}{2}$

36. $\cos(20° + \theta) = \frac{\sqrt{2}}{2}$

37. $\sin(180° - \theta) = \frac{1}{2}$

38. Physics Light enters glass of thickness t at an angle θ_i and leaves the glass at the same angle θ_i. However, the exiting ray of light is offset from the initial ray by a distance $\Delta = \left(\frac{\sin(\theta_i - \theta_r)}{\sin\theta_i \cos\theta_r}\right)t$, indicated in the figure shown.



a. Write the formula for Δ in terms of tangent and cotangent by using the difference identities and other trigonometric identities.

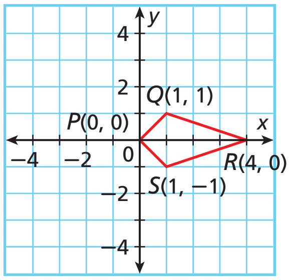
b. Use the figure to write a ratio for $\sin(\theta_i - \theta_r)$.

Multi-Step **Find $\tan(A + B)$, $\cos(A + B)$, and $\sin(A - B)$ for each situation.**

39. $\sin A = -\frac{7}{25}$ with $180° < A < 270°$ and $\cos B = \frac{12}{13}$ with $0° < B < 90°$

40. $\sin A = -\frac{1}{3}$ with $270° < A < 360°$ and $\sin B = \frac{4}{5}$ with $0° < B < 90°$

41. The figure *PQRS* will be rotated about the origin repeatedly to create the logo for a new product.



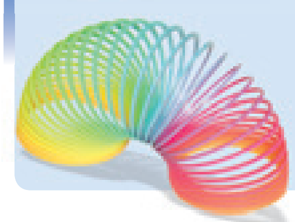
a. Write the rotation matrices for 90°, 180°, and 270° rotations.

b. Use your answers to part **a** to find the coordinates of the vertices of the figure after each of the three rotations.

c. Graph the three rotations on the same graph as *PQRS* to create the logo.

42. Critical Thinking Is it possible to find the exact value of $\sin\left(\frac{11\pi}{24}\right)$ by using sum or difference identities? Explain.

MULTI-STEP TEST PREP



43. This problem will prepare you for the Multi-Step Test Prep on page 1034.

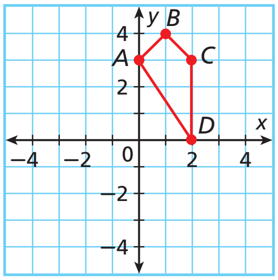
The displacement y of a mass attached to a spring is modeled by $y(t) = 4.2\sin\left(\frac{2\pi}{3}t - \frac{\pi}{2}\right)$, where t is the time in seconds.

a. What are the amplitude and period of the function?

b. Use a trigonometric identity to write the displacement, using only the cosine function.

c. What is the displacement of the mass when $t = 8$ s?

Geometry Find the coordinates, to the nearest hundredth, of the vertices of figure $ABCD$ with $A(0, 3)$, $B(1, 4)$, $C(2, 3)$, and $D(2, 0)$ after each rotation about the origin.



44. 45°

45. 60°

46. 120°

47. −30°

48. **Write About It** In general, does $\sin(A + B) = \sin A + \sin B$? Give an example to support your response.

TEST PREP

49. Which is the value of $\cos 15° \cos 45° - \sin 15° \sin 45°$?

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $-\frac{\sqrt{2}}{2}$ (D) $\frac{2 + \sqrt{2}}{2}$

50. Which gives the value for x if $\sin\left(\frac{\pi}{2} + x\right) = \frac{1}{2}$?

(F) $\frac{\pi}{6}$ (G) $\frac{\pi}{4}$ (H) $\frac{\pi}{3}$ (J) $\frac{\pi}{2}$

51. Given $\sin A = \frac{1}{2}$ with $0° < A < 90°$ and $\cos B = \frac{3}{5}$ with $0° < B < 90°$, which expression gives the value of $\cos(A - B)$?

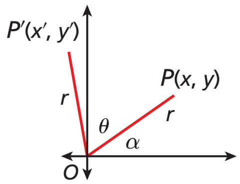
(A) $\frac{3\sqrt{3} + 4}{10}$ (B) $\frac{3\sqrt{3} - 4}{10}$ (C) $\frac{3 + 4\sqrt{3}}{10}$ (D) $\frac{3 - 4\sqrt{3}}{10}$

52. **Short Response** Find the exact value for $\sin(-15°)$. Show your work.

CHALLENGE AND EXTEND

53. Verify that the rotation matrix for θ is the inverse of the rotation matrix for $-\theta$.

54. Derive the identity for $\tan(A + B)$.



55. Derive the rotation matrix by using the sum identities for sine and cosine and recalling from Lesson 13-2 that any point $P(x, y)$ can be represented as $(r\cos\alpha, r\sin\alpha)$ by using a reference angle.

Find the angle by which a figure ABC with vertices $A(1, 0)$, $B(0, 2)$, and $C(-1, 0)$ was rotated to get $A'B'C'$.

56. $A'(0, 1)$, $B'(-2, 0)$, $C'(0, -1)$

57. $A'\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $B'\left(-\sqrt{2}, \sqrt{2}\right)$, $C'\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

58. $A'(-1, 0)$, $B'(0, -2)$, $C'(1, 0)$

59. $A'\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $B'\left(-1, \sqrt{3}\right)$, $C'\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

SPIRAL REVIEW

Divide. Assume that all expressions are defined. *(Lesson 8-2)*

60. $\frac{3x^2}{7y^3} \div \frac{6x}{21y}$

61. $\frac{x^2 + x - 2}{x^2 - 2x - 8} \div \frac{x^2 + 3x + 2}{x^2 - 3x - 4}$

62. $\frac{9x^3y^2}{15xy^4} \div \frac{6x^4y}{3x^2y^5}$

Identify the conic section that each equation represents. *(Lesson 10-6)*

63. $x^2 + 2xy + y^2 + 12x - 25 = 0$

64. $5x^2 + 5y^2 + 20x - 15y = 0$

Rewrite each expression in terms of a single trigonometric function. *(Lesson 14-3)*

65. $\frac{\cot\theta \sec\theta}{\sin\theta \cos\theta}$

66. $\cot\theta \tan\theta \csc\theta$

67. $\frac{\tan\theta}{\sec\theta}\sin\theta$

14-5 Double-Angle and Half-Angle Identities

Objective
Evaluate and simplify expressions by using double-angle and half-angle identities.

Who uses this?
Double-angle formulas can be used to find the horizontal distance for a projectile such as a golf ball. (See Exercise 49.)

You can use sum identities to derive the *double-angle identities*.

$$\sin 2\theta = \sin(\theta + \theta)$$
$$= \sin\theta\cos\theta + \cos\theta\sin\theta$$
$$= 2\sin\theta\cos\theta$$

You can derive the double-angle identities for cosine and tangent in the same way. There are three forms of the identity for $\cos 2\theta$, which are derived by using $\sin^2\theta + \cos^2\theta = 1$. It is common to rewrite expressions as functions of θ only.

Know it! Note

Double-Angle Identities		
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$ $\cos 2\theta = 1 - 2\sin^2\theta$	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

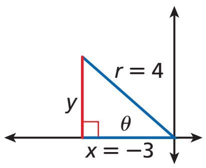
EXAMPLE 1 **Evaluating Expressions with Double-Angle Identities**

Find $\sin 2\theta$ and $\cos 2\theta$ if $\cos\theta = -\frac{3}{4}$ and $90° < \theta < 180°$.

Step 1 Find $\sin\theta$ to evaluate $\sin 2\theta = 2\sin\theta\cos\theta$.

Method 1 Use the reference angle.

In QII, $90° < \theta < 180°$, and $\cos\theta = -\frac{3}{4}$.



$(-3)^2 + y^2 = 4^2$ — *Use the Pythagorean Theorem.*

$y = \sqrt{16-9} = \sqrt{7}$ — *Solve for y.*

$$\sin\theta = \frac{\sqrt{7}}{4}$$

Method 2 Solve $\sin^2\theta = 1 - \cos^2\theta$.

$$\sin^2\theta = 1 - \cos^2\theta$$

$\sin\theta = \sqrt{1 - \left(-\frac{3}{4}\right)^2}$ — *Substitute $-\frac{3}{4}$ for $\cos\theta$.*

$= \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$ — *Simplify.*

$$\sin\theta = \frac{\sqrt{7}}{4}$$

Caution!
The signs of x and y depend on the quadrant for angle θ.

	sin	cos
QI	+	+
QII	+	−
QIII	−	−
QIV	−	+

Step 2 Find $\sin 2\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$ — *Apply the identity for $\sin 2\theta$.*

$= 2\left(\frac{\sqrt{7}}{4}\right)\left(-\frac{3}{4}\right)$ — *Substitute $\frac{\sqrt{7}}{2}$ for $\sin\theta$ and $-\frac{3}{4}$ for $\cos\theta$.*

$= -\frac{3\sqrt{7}}{8}$ — *Simplify.*

Step 3 Find $\cos 2\theta$.

$\cos 2\theta = 2\cos^2\theta - 1$ — *Select a double-angle identity.*

$= 2\left(-\frac{3}{4}\right)^2 - 1$ — *Substitute $-\frac{3}{4}$ for $\cos\theta$.*

$= 2\left(\frac{9}{16}\right) - 1$ — *Simplify.*

$= \frac{1}{8}$

CHECK IT OUT! **1.** Find $\tan 2\theta$ and $\cos 2\theta$ if $\cos\theta = \frac{1}{3}$ and $270° < \theta < 360°$.

You can use double-angle identities to prove trigonometric identities.

EXAMPLE 2 Proving Identities with Double-Angle Identities

Prove each identity.

A $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ — *Choose the right-hand side to modify.*

$= \frac{1}{2}\left(1 - (1 - 2\sin^2\theta)\right)$ — *Apply the identity for $\cos 2\theta$.*

$= \frac{1}{2}(2\sin^2\theta)$ — *Simplify.*

$\sin^2\theta = \sin^2\theta$

Helpful Hint

Choose to modify either the left side or the right side of an identity. Do not work on both sides at once.

B $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$

$(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$ — *Choose the left-hand side to modify.*

$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta =$ — *Expand the square.*

$(\cos^2\theta + \sin^2\theta) + (2\cos\theta\sin\theta) =$ — *Regroup.*

$1 + \sin 2\theta =$ — *Rewrite using $1 = \cos^2\theta + \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.*

$1 + \sin 2\theta = 1 + \sin 2\theta$

CHECK IT OUT! **Prove each identity.**

2a. $\cos^4\theta - \sin^4\theta = \cos 2\theta$

2b. $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

You can use double-angle identities for cosine to derive the *half-angle identities* by substituting $\frac{\theta}{2}$ for θ. For example, $\cos 2\theta = 2\cos^2\theta - 1$ can be rewritten as $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$. Then solve for $\cos\frac{\theta}{2}$.

Half-Angle Identities

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ $\quad$ $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$ $\quad$ $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

Choose + or − depending on the location of $\frac{\theta}{2}$.

Half-angle identities are useful in calculating exact values for trigonometric expressions.

Reading Math

In Example 3, the expressions $-\frac{\sqrt{2+\sqrt{3}}}{2}$ and $\frac{\sqrt{2-\sqrt{2}}}{2}$ are in reduced form and cannot be simplified further.

EXAMPLE 3 Evaluating Expressions with Half-Angle Identities

Use half-angle identities to find the exact value of each trigonometric expression.

A $\cos 165°$

$\cos\frac{330°}{2}$

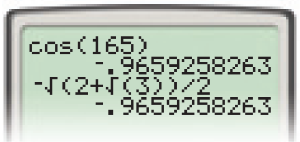
$-\sqrt{\frac{1+\cos 330°}{2}}$ *Negative in QII*

$-\sqrt{\frac{1+\left(\frac{\sqrt{3}}{2}\right)}{2}}$ $\quad$ *$\cos 330° = \frac{\sqrt{3}}{2}$*

$-\sqrt{\left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)}$ *Simplify.*

$-\frac{\sqrt{2+\sqrt{3}}}{2}$

Check Use your calculator.



B $\sin\frac{\pi}{8}$

$\sin\frac{1}{2}\left(\frac{\pi}{4}\right)$

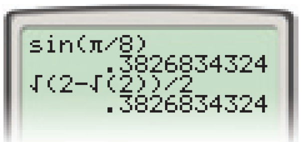
$+\sqrt{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}}$ *Positive in QI*

$\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$ $\quad$ *$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$*

$\sqrt{\left(\frac{2-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)}$ *Simplify.*

$\frac{\sqrt{2-\sqrt{2}}}{2}$

Check Use your calculator.



CHECK IT OUT! **Use half-angle identities to find the exact value of each trigonometric expression.**

3a. $\tan 75°$ $\qquad$ **3b.** $\cos\frac{5\pi}{8}$

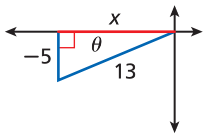
EXAMPLE 4 Using the Pythagorean Theorem with Half-Angle Identities

Find $\sin\frac{\theta}{2}$ and $\tan\frac{\theta}{2}$ if $\sin\theta = -\frac{5}{13}$ and $180° < \theta < 270°$.

Step 1 Find $\cos\theta$ to evaluate the half-angle identities.

Use the reference angle.

In QIII, $180° < \theta < 270°$, and $\sin\theta = -\frac{5}{13}$.



$x^2 + (-5)^2 = 13^2$ *Pythagorean Theorem*

$x = -\sqrt{169-25} = -12$ *Solve for the missing side x.*

Thus, $\cos\theta = -\frac{12}{13}$.

Step 2 Evaluate $\sin\frac{\theta}{2}$.

$\sin\frac{\theta}{2}$

$+\sqrt{\frac{1-\cos\theta}{2}}$ — *Choose + for $\sin\frac{\theta}{2}$ where $90° < \frac{\theta}{2} < 135°$.*

$\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}}$ — *Evaluate.*

$\sqrt{\left(\frac{25}{13}\right)\left(\frac{1}{2}\right)}$ — *Simplify.*

$\sqrt{\frac{25}{26}}$

$\frac{5\sqrt{26}}{26}$

Step 3 Evaluate $\tan\frac{\theta}{2}$.

$\tan\frac{\theta}{2}$

$-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ — *Choose − for $\tan\frac{\theta}{2}$ where $90° < \frac{\theta}{2} < 135°$.*

$-\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{1+\left(-\frac{12}{13}\right)}}$ — *Evaluate.*

$-\sqrt{\left(\frac{25}{13}\right)\left(\frac{13}{1}\right)}$ — *Simplify.*

$-\sqrt{25}$

-5

Caution!

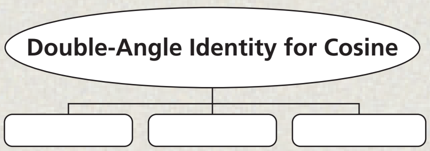
Be careful to choose the correct sign for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$. If $180° < \theta < 270°$, then $90° < \frac{\theta}{2} < 135°$.

CHECK IT OUT! **4.** Find $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ if $\tan\theta = \frac{4}{3}$ and $0° < \theta < 90°$.

THINK AND DISCUSS

Know it! Note

1. **EXPLAIN** which double-angle identity you would use to simplify $\frac{\cos 2\theta}{\sin\theta + \cos\theta}$.
2. **DESCRIBE** how to determine the sign of the value for $\sin\frac{\theta}{2}$ and for $\cos\frac{\theta}{2}$.
3. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, write one of the identities.



14-5 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 14-5
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

SEE EXAMPLE 1 p. 1020

Find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for each set of conditions.

1. $\cos\theta = -\frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$

2. $\sin\theta = \frac{4}{5}$ and $0° < \theta < 90°$

SEE EXAMPLE 2 p. 1021

Prove each identity.

3. $2\cos 2\theta = 4\cos^2\theta - 2$

4. $\sin^2\theta = 1 - \frac{\cos 2\theta + 1}{2}$

5. $\frac{1 + \cos 2\theta}{\sin 2\theta} = \cot\theta$

6. $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

SEE EXAMPLE 3 p. 1022

Use half-angle identities to find the exact value of each trigonometric expression.

7. $\cos 67.5°$

8. $\cos\frac{\pi}{12}$

9. $\tan\frac{3\pi}{8}$

10. $\sin 112.5°$

SEE EXAMPLE 4 p. 1022

Find $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ for each set of conditions.

11. $\sin\theta = -\frac{24}{25}$ and $180° < \theta < 270°$

12. $\cos\theta = \frac{1}{4}$ and $270° < \theta < 360°$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
13–14	1
15–18	2
19–22	3
23–24	4

Extra Practice

Skills Practice p. S31
Application Practice p. S45

Find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ for each set of conditions.

13. $\cos\theta = -\frac{7}{25}$ and $90° < \theta < 180°$

14. $\tan\theta = \frac{20}{21}$ and $0 \le \theta \le \frac{\pi}{2}$

Prove each identity.

15. $\frac{\sin 2\theta}{\sin\theta} = 2\cos\theta$

16. $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

17. $\tan\theta = \frac{1 - \cos 2\theta}{\sin 2\theta}$

18. $\tan\theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$

Use half-angle identities to find the exact value of each trigonometric expression.

19. $\sin\frac{7\pi}{12}$

20. $\cos\frac{5\pi}{12}$

21. $\sin 22.5°$

22. $\tan 15°$

Find $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, and $\tan\frac{\theta}{2}$ for each set of conditions.

23. $\tan\theta = -\frac{12}{35}$ and $\frac{3\pi}{2} < \theta < 2\pi$

24. $\sin\theta = -\frac{3}{5}$ and $180° < \theta < 270°$

Multi-Step **Rewrite each expression in terms of trigonometric functions of θ rather than multiples of θ. Then simplify.**

25. $\sin 3\theta$

26. $\sin 4\theta$

27. $\cos 3\theta$

28. $\cos 4\theta$

29. $\cos 2\theta + 2\sin^2\theta$

30. $\cos 2\theta + 1$

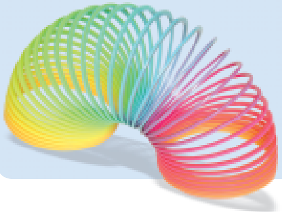
31. $\tan 2\theta(2 - \sec^2\theta)$

32. $\frac{\cos 2\theta}{\cos\theta + \sin\theta}$

33. $\frac{\cos\theta\sin 2\theta}{1 + \cos 2\theta}$

34. $\frac{\cos 2\theta - 1}{\sin^2\theta}$

MULTI-STEP TEST PREP



35. This problem will prepare you for the Multi-Step Test Prep on page 1034.

The displacement y of a mass attached to a spring is modeled by $y(t) = 3.1 \sin 2t$, where t is the time in seconds.

a. Rewrite the function by using a double-angle identity.

b. The displacement w of another mass attached to a spring is given by $w(t) = 3.8 \cos t$. The two masses are set in motion at $t = 0$. When do the masses have the same displacement for the first time?

c. What is the displacement at this time?

Multi-Step **Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ for each set of conditions.**

36. $\cos \theta = \frac{3}{8}$ and $\frac{\pi}{2} < \theta < \pi$

37. $\cos \theta = -\frac{\sqrt{5}}{3}$ and $180° < \theta < 270°$

38. $\sin \theta = \frac{2}{5}$ and $0° < \theta < 90°$

39. $\tan \theta = -\frac{1}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$

Use half-angle identities to find the exact value of each trigonometric expression.

40. $\cos \frac{7\pi}{8}$

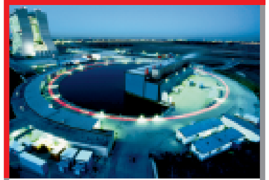
41. $\sin \frac{11\pi}{12}$

42. $\cos 105°$

43. $\sin(-15°)$

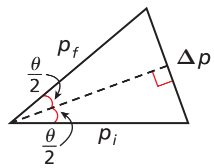
LINK

Physics



The Tevatron at Fermi National Accelerator Lab in Batavia, Illinois, uses superconducting magnets to study subatomic particles by colliding matter and antimatter inside of a ring with a diameter of 6.3 km.

44. **Physics** The change in momentum of a scattered nuclear particle is given by $\Delta P = P_f - P_i$, where P_f is the final momentum, and P_i is the initial momentum.



a. Use the diagram and the Pythagorean Theorem to write a formula for ΔP in terms of P_i. Then write a formula for ΔP in terms of P_f.

b. Compare your two answers to part **a.** What does this tell you about the magnitude, or size, of the momentum before and after the "collision"?

c. Write the formula for ΔP in terms of $\cos \theta$.

Prove each identity.

45. $\cos^2 \frac{\theta}{2} = \frac{\sin^2 \theta}{2(1 - \cos \theta)}$

46. $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

47. $\frac{\tan \theta + \sin \theta}{2 \tan \theta} = \cos^2 \frac{\theta}{2}$

48. **Graphing Calculator** Graph $y = \frac{(\cos x)(1 - \cos 2x)}{\sin 2x}$ to discover an identity. Then prove the identity.

49. **Multi-Step** A golf ball is hit with an initial velocity of v_0 in feet per second at an angle of elevation θ. The function $d(\theta) = \frac{v_0{}^2 \sin \theta \cos \theta}{16}$ gives the horizontal distance d in feet that the ball travels.

a. Rewrite the function in terms of the double angle 2θ.

b. Calculate the horizontal distance for an initial velocity of 80 ft/s for angles of 15°, 30°, 45°, 60°, and 75°.

c. For a given velocity, what angle gives the maximum horizontal distance?

d. **What if...?** If the initial velocity is 80 ft/s, through what approximate range of angles will the ball travel horizontally at least 175 ft?

50. **Critical Thinking** Explain how to find the exact value for $\sin 7.5°$.

51. **Write About It** How do you know when to use a double-angle or a half-angle identity?

TEST PREP

52. What is the value of $\sin 2\theta$ if $\cos\theta = -\frac{\sqrt{2}}{2}$ and $90° < \theta < 180°$?

Ⓐ $\frac{1}{2}$ Ⓑ $\frac{\sqrt{2}}{2}$ Ⓒ 1 Ⓓ -1

53. What is the value for $\cos 2\theta$ if $\sin\theta = \cos\theta$?

Ⓕ 0 Ⓖ 1 Ⓗ $2\sin^2\theta$ Ⓙ $2\cos^2\theta$

54. What is the value for $\sin\frac{\theta}{2}$ if $\cos\theta = -\frac{12}{13}$ and $90° < \theta < 180°$?

Ⓐ $\frac{\sqrt{26}}{26}$ Ⓑ $-\frac{\sqrt{26}}{26}$ Ⓒ $\frac{5\sqrt{26}}{26}$ Ⓓ $-\frac{5\sqrt{26}}{26}$

55. What is the exact value for $\sin 157.5°$?

Ⓕ $-\frac{\sqrt{2-\sqrt{2}}}{2}$ Ⓖ $\frac{\sqrt{2-\sqrt{2}}}{2}$ Ⓗ $-\frac{\sqrt{2+\sqrt{2}}}{2}$ Ⓙ $\frac{\sqrt{2+\sqrt{2}}}{2}$

56. **Short Response** Verify that $\frac{\cos 2\theta}{\sin\theta + \cos\theta} = \cos\theta - \sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Show each step in your justification process.

CHALLENGE AND EXTEND

57. Derive the double-angle formula for $\tan 2\theta$ by using the ratio identity for tangent and the double-angle identities for sine and cosine.

58. Derive the half-angle formula for $\tan\frac{\theta}{2}$ by using the ratio identity for tangent.

Use half-angle identities to find the exact value of each expression.

59. $\tan 7.5°$ **60.** $\tan\frac{\pi}{16}$ **61.** $\sin\frac{\pi}{24}$ **62.** $\cos 11.25°$

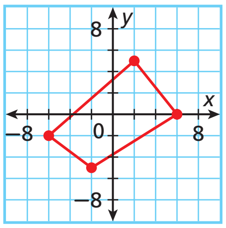
63. **Write About It** For what values of θ is $\sin 2\theta = 2\sin\theta$ true? Explain first by using graphs and then by solving the equation.

64. Derive the product-to-sum formulas $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ and $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$ by using the angle sum and difference formulas.

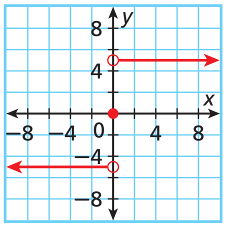
SPIRAL REVIEW

Use the vertical-line test to determine whether each relation is a function. *(Lesson 1-6)*

65.



66.



Add or subtract. Identify any x-values for which the expression is undefined. *(Lesson 8-3)*

67. $\frac{3x-2}{x+7} + \frac{2x+14}{x+7}$ **68.** $\frac{4x-1}{x} + \frac{6x-2}{2x}$

69. $\frac{7x+4}{x+1} - \frac{5x+8}{x-3}$ **70.** $\frac{x+9}{x^2} - \frac{x}{x+2}$

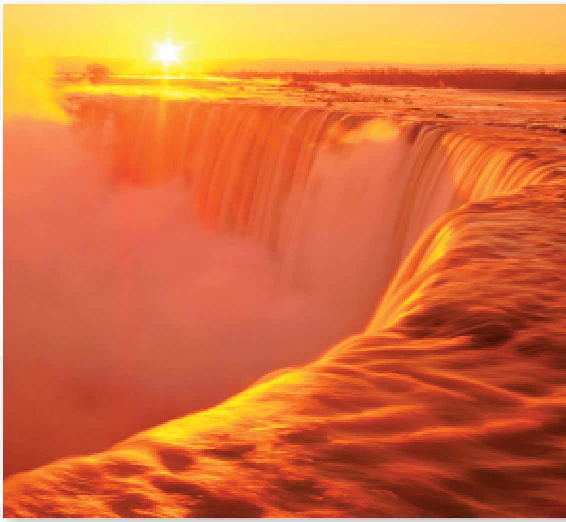
Find the exact value of each expression. *(Lesson 14-4)*

71. $\sin\left(-\frac{\pi}{12}\right)$ **72.** $\sin 105°$ **73.** $\cos\frac{7\pi}{12}$ **74.** $\cos 255°$

14-6 Solving Trigonometric Equations

Objectives
Solve equations involving trigonometric functions.

Why learn this?
You can use trigonometric equations to determine the day of the year that the sun will rise at a given time. (See Example 4.)



Unlike trigonometric identities, most trigonometric equations are true only for certain values of the variable, called *solutions*. To solve trigonometric equations, apply the same methods used for solving algebraic equations.

Helpful Hint

Compare Example 1 with this solution:

$3x = x + 2$
$3x - x = 2$
$2x = 2$
$x = 1$

EXAMPLE 1 **Solving Trigonometric Equations with Infinitely Many Solutions**

Find all of the solutions of $3\tan\theta = \tan\theta + 2$.

Method 1 Use algebra.

Solve for θ over one cycle of the tangent, $-90° < \theta < 90°$.

$3\tan\theta = \tan\theta + 2$

$3\tan\theta - \tan\theta = 2$ *Subtract $\tan\theta$ from both sides.*

$2\tan\theta = 2$ *Combine like terms.*

$\tan\theta = 1$ *Divide by 2.*

$\theta = \tan^{-1} 1$ *Apply the inverse tangent.*

$\theta = 45°$ *Find θ when $\tan\theta = 1$.*

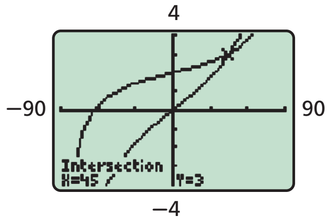
Find all real number values of θ, where n is an integer.

$\theta = 45° + 180°n$ *Use the period of the tangent function.*

Method 2 Use a graph.

Graph $y = 3\tan\theta$ and $y = \tan\theta + 2$ in the same viewing window for $-90° \le \theta \le 90°$.

Use the intersect feature of your graphing calculator to find the points of intersection.



The graphs intersect at $\theta = 45°$. Thus, $\theta = 45° + 180°n$, where n is an integer.

CHECK IT OUT! **1.** Find all of the solutions of $2\cos\theta + \sqrt{3} = 0$.

Some trigonometric equations can be solved by applying the same methods used for quadratic equations.

EXAMPLE 2 Solving Trigonometric Equations in Quadratic Form

Solve each equation for the given domain.

A $\sin^2\theta - 2\sin\theta = 3$ **for** $0 \le \theta < 2\pi$

$\sin^2\theta - 2\sin\theta - 3 = 0$ — *Subtract 3 from both sides.*

$(\sin\theta + 1)(\sin\theta - 3) = 0$ — *Factor the quadratic expression by comparing it with $x^2 - 2x - 3 = 0$.*

$\sin\theta = -1$ or $\sin\theta = 3$ — *Apply the Zero Product Property.*

$\sin\theta = 3$ has no solution because $-1 \le \sin\theta \le 1$.

$\theta = \frac{3\pi}{2}$ — *The only solution will come from $\sin\theta = -1$.*

B $\cos^2\theta + 2\cos\theta - 1 = 0$ **for** $0° \le \theta < 360°$

The equation is in quadratic form but cannot easily be factored. Use the Quadratic Formula.

$\cos\theta = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$ — *Substitute 1 for a, 2 for b, and −1 for c.*

$\cos\theta = -1 \pm \sqrt{2}$ — *Simplify.*

$-1 - \sqrt{2} < -1$ so $\cos\theta = -1 - \sqrt{2}$ has no solution.

$\theta = \cos^{-1}(-1 + \sqrt{2})$ — *Apply the inverse cosine.*

$\approx 65.5°$ or $294.5°$ — *Use a calculator. Find both angles for $0° \le \theta < 360°$.*

Caution!

A trigonometric equation may have zero, one, two, or an infinite number of solutions, depending on the equation and domain of θ.

CHECK IT OUT! **Solve each equation for $0 \le \theta < 2\pi$.**

2a. $\cos^2\theta + 2\cos\theta = 3$ **2b.** $\sin^2\theta + 5\sin\theta - 2 = 0$

You can often write trigonometric equations involving more than one function as equations of only one function by using trigonometric identities.

EXAMPLE 3 Solving Trigonometric Equations with Trigonometric Identities

Use trigonometric identities to solve each equation for $0 \le \theta < 2\pi$.

A $2\cos^2\theta = \sin\theta + 1$

$2(1 - \sin^2\theta) - \sin\theta - 1 = 0$ — *Substitute $1 - \sin^2\theta$ for $\cos^2\theta$ by the Pythagorean identity.*

$-2\sin^2\theta - \sin\theta + 2 - 1 = 0$ — *Simplify.*

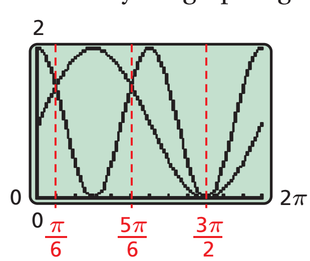
$2\sin^2\theta + \sin\theta - 1 = 0$ — *Multiply by −1.*

$(2\sin\theta - 1)(\sin\theta + 1) = 0$ — *Factor.*

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$ — *Apply the Zero Product Property.*

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\theta = \frac{3\pi}{2}$

Check Use the intersect feature of your graphing calculator. A graph supports your answer.



Use trigonometric identities to solve each equation for $0° \le \theta < 360°$.

B $\cos 2\theta + 3\cos\theta + 2 = 0$

$2\cos^2\theta - 1 + 3\cos\theta + 2 = 0$ — *Substitute $2\cos^2\theta - 1$ for $\cos 2\theta$ by the double-angle identity.*

$2\cos^2\theta + 3\cos\theta + 1 = 0$ — *Combine like terms.*

$(2\cos\theta + 1)(\cos\theta + 1) = 0$ — *Factor.*

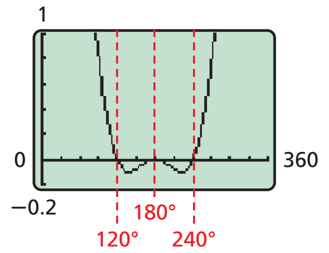
$\cos\theta = -\frac{1}{2}$ — *Apply the Zero Product Property.*

or

$\cos\theta = -1$

$\theta = 120°$ or $240°$ or $\theta = 180°$

Check Use the intersect feature of your graphing calculator. A graph supports your answer.



CHECK IT OUT! **Use trigonometric identities to solve each equation for the given domain.**

3a. $4\sin^2\theta + 4\cos\theta = 5$ for $0° \le \theta < 360°$

3b. $\sin 2\theta = -\cos\theta$ for $0 \le \theta < 2\pi$

EXAMPLE 4 *Problem-Solving Application*

PROBLEM SOLVING

The first sunrise in the United States each day is observed from Cadillac Mountain on Mount Desert Island in Maine. The time of the sunrise can be modeled by $t(m) = 1.665\sin\frac{\pi}{6}(m + 3) + 5.485$, where t is hours after midnight and m is the number of months after January 1. When does the sun rise at 7 A.M.?

1 Understand the Problem

The **answer** will be months of the year.

List the important information:

- The function model is $t(m) = 1.665\sin\frac{\pi}{6}(m + 3) + 5.485$.
- Sunrise is at 7 A.M., which is represented by $t = 7$.
- m represents the number of months after January 1.

2 Make a Plan

Substitute 7 for t in the model. Then solve the equation for m by using algebra.

3 Solve

$$7 = 1.665 \sin \frac{\pi}{6}(m + 3) + 5.485$$ *Substitute 7 for t.*

$$\frac{7 - 5.485}{1.665} = \sin \frac{\pi}{6}(m + 3)$$ *Isolate the sine term.*

$$\sin^{-1}(0.9\overline{099}) = \frac{\pi}{6}(m + 3)$$ *Apply the inverse sine.*

Sine is positive in Quadrants I and II. Compute both values.

QI: $\sin^{-1}(0.9\overline{099}) = \frac{\pi}{6}(m + 3)$

$$1.143 \approx \frac{\pi}{6}(m + 3)$$

$$\left(\frac{6}{\pi}\right)1.143 \approx m + 3$$

$$-0.817 \approx m$$

QII: $\pi - \sin^{-1}(0.9\overline{099}) = \frac{\pi}{6}(m + 3)$

$$\pi - 1.143 \approx \frac{\pi}{6}(m + 3)$$

$$\left(\frac{6}{\pi}\right)(\pi - 1.143) \approx m + 3$$

$$0.817 \approx m$$

The value $m = 0.817$ corresponds to late January and the value $m = -0.817$ corresponds to early December.

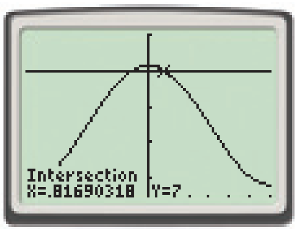
Caution!

Be sure to have your calculator in radian mode when working with angles expressed in radians.

4 Look Back

Check your answer by using a graphing calculator. Enter $y = 1.665 \sin \frac{\pi}{6}(x + 3) + 5.485$ and $y = 7$. Graph the functions on the same viewing window, and find the points of intersection.

The graphs intersect at about 0.817 and −0.817.



CHECK IT OUT!

4. The number of hours h of sunlight in a day at Cadillac Mountain can be modeled by $h(d) = 3.31 \sin \frac{\pi}{182.5}(d - 85.25) + 12.22$, where d is the number of days after January 1. When are there 12 hours of sunlight?

THINK AND DISCUSS

1. DESCRIBE the general procedure for finding all real-number solutions of a trigonometric equation.

2. GET ORGANIZED Copy and complete the graphic organizer. Write when each method is most useful, and give an example.

Method	Most useful when...	Example
Graphing		
Solving linear equations		
Factoring		
Quadratic Formula		
Identity substitution		

14-6 Exercises

go.hrw.com
Homework Help Online
KEYWORD: MB7 14-6
Parent Resources Online
KEYWORD: MB7 Parent

GUIDED PRACTICE

SEE EXAMPLE 1 p. 1027

Find all of the solutions of each equation.

1. $6\cos\theta - 1 = 2$
2. $2\sin\theta - \sqrt{3} = 0$
3. $\cos\theta = \sqrt{3} - \cos\theta$

SEE EXAMPLE 2 p. 1028

Solve each equation for the given domain.

4. $2\sin^2\theta + 3\sin\theta = -1$ for $0 \le \theta < 2\pi$
5. $\cos^2\theta - 4\cos\theta + 1 = 0$ for $0° \le \theta < 360°$

SEE EXAMPLE 3 p. 1028

Multi-Step Use trigonometric identities to solve each equation for the given domain.

6. $2\sin^2\theta - \cos 2\theta = 0$ for $0° \le \theta < 360°$
7. $\sin^2\theta + \cos\theta = -1$ for $0 \le \theta < 2\pi$

SEE EXAMPLE 4 p. 1029

8. **Heating** The amount of energy from natural gas used for heating a manufacturing plant is modeled by $E(m) = 350\sin\frac{\pi}{6}(m + 1.5) + 650$, where E is the energy used in dekatherms, and m is the month where $m = 0$ represents January 1. When is the gas usage 825 dekatherms? Assume an average of 30 days per month.

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
9–12	1
13–14	2
15–16	3
17	4

Extra Practice
Skills Practice p. S31
Application Practice p. S45

Find all of the solutions of each equation.

9. $1 - 2\cos\theta = 0$
10. $\sqrt{3}\tan\theta - 3 = 0$
11. $2\cos\theta + \sqrt{3} = 0$
12. $2\sin\theta + 1 = 2 + \sin\theta$

Solve each equation for the given domain.

13. $2\cos^2\theta + \cos\theta - 1 = 0$ for $0 \le \theta < 2\pi$
14. $\sin^2\theta + 2\sin\theta - 2 = 0$ for $0° \le \theta < 360°$

Multi-Step Use trigonometric identities to solve each equation for the given domain.

15. $\cos 2\theta + \cos\theta + 1 = 0$ for $0° \le \theta < 360°$
16. $\cos 2\theta = \sin\theta$ for $0 \le \theta < 2\pi$

17. **Multi-Step** The amount of energy used by a large office building is modeled by $E(t) = 100\sin\frac{\pi}{12}(t - 8) + 800$, where E is the energy in kilowatt-hours, and t is the time in hours after midnight.
    a. During what time in the day is the electricity use 850 kilowatt-hours?
    b. When are the least and greatest amounts of electricity used? Are your answers reasonable? Explain.

Solve each equation algebraically for $0° \le \theta < 360°$.

18. $2\sin^2\theta = \sin\theta$
19. $2\cos^2\theta = \sin\theta + 1$
20. $\cos 2\theta - 2\sin\theta + 2 = 0$
21. $2\cos^2\theta + 3\sin\theta = 3$
22. $\cos^2\theta + \sin\theta - 1 = 0$
23. $2\sin^2\theta + \sin\theta = 0$

Solve each equation algebraically for $0 \le \theta < 2\pi$.

24. $\sin^2\theta - \sin\theta = 0$
25. $\cos^2\theta - 3\cos\theta = 4$
26. $\cos\theta(0.5 + \cos\theta) = 0$
27. $2\sin^2\theta - 3\sin\theta = 2$
28. $\cos^2\theta + \frac{1}{2}\cos\theta = 5$
29. $\sin^2\theta + 3\sin\theta + 3 = 0$
30. $\cos^2\theta + 4\cos\theta - 3 = 0$
31. $\tan^2\theta = \sqrt{3}\tan\theta$

32. **Sports** A baseball is thrown with an initial velocity of 96 feet per second at an angle θ degrees with a horizontal.

a. The horizontal range R in feet that the ball travels can be modeled by $R(\theta) = \frac{v^2 \sin 2\theta}{32}$. At what angle(s) with the horizontal will the ball travel 250 feet?

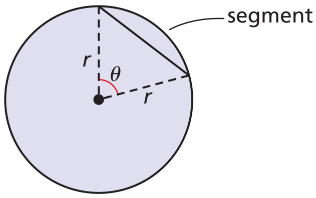
b. The maximum vertical height H_{max} in feet that the ball travels upward can be modeled by $H_{max}(\theta) = \frac{v^2 \sin^2 \theta}{64}$. At what angle(s) with the horizontal will the ball travel 50 feet?

LINK

Performing Arts

Traditional Japanese kabuki theaters were round and were able to be rotated to change scenes. The stages were also equipped with trapdoors and bridges that led through the audience.

33. **Performing Arts** A theater has a rotating stage that can be turned for different scenes. The stage has a radius of 18 feet, and the area in square feet of the segment of the circle formed by connecting two radii as shown is $A = \frac{r^2}{2}(\theta - \sin\theta)$, with θ in radians.



a. What angle gives a segment area of 92 square feet? How many such sets can simultaneously fit on the full rotating stage?

b. What angle gives a segment area of 50 square feet? About how many such sets can simultaneously fit on the full rotating stage?

34. **Oceanography** The height of the water on a certain day at a pier in Cape Cod, Massachusetts, can be modeled by $h(t) = 4.5 \sin \frac{\pi}{6.25}(t + 4) + 7.5$, where h is the height in feet and t is the time in hours after midnight.

a. On this particular day, when is the height of the water 5 feet?

b. How much time is there between high and low tides?

c. What is the period for the tide?

d. Does the cycle of tides fit evenly in a 24-hour day? Explain.

35. **///ERROR ANALYSIS///** Below are two solution procedures for solving $\sin^2\theta - \frac{1}{2}\sin\theta = 0$ for $0° \le \theta < 360°$. Which is incorrect? Explain the error.

A

$\sin^2\theta - \frac{1}{2}\sin\theta = 0$

$\sin\theta\left(\sin\theta - \frac{1}{2}\right) = 0$

$\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = 0°$ or $180°$ or $\theta = 30°$ or $150°$

B

$\sin^2\theta - \frac{1}{2}\sin\theta = 0$

$\sin^2\theta = \frac{1}{2}\sin\theta$

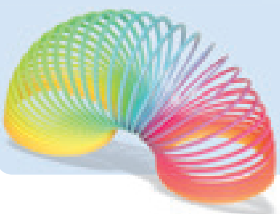
$\sin\theta = \frac{1}{2}$

$\theta = 30°$ or $150°$

36. **Critical Thinking** What is the difference between a trigonometric equation and a trigonometric identity? Explain by using examples.

37. **Graphing Calculator** Use your graphing calculator to find all solutions of the equation $2\cos x = 0.25x$.

MULTI-STEP TEST PREP



38. This problem will prepare you for the Multi-Step Test Prep on page 1034.

The displacement in centimeters of a mass attached to a spring is modeled by $y(t) = 2.9\cos\left(\frac{2\pi}{3}t + \frac{\pi}{4}\right) + 3$, where t is the time in seconds.

a. What are the maximum and minimum displacements of the mass?

b. The mass is set in motion at $t = 0$. When is the displacement of the mass equal to 1 cm for the first time?

c. At what other times will the displacement be 1 cm?

Estimation **Use a graphing calculator to approximate the solution to each equation to the nearest tenth of a degree for $0° \le \theta < 360°$.**

39. $\tan\theta - 12 = -1$

40. $\sin\theta + \cos\theta + 1.25 = 0$

41. $4\sin^2(2\theta - 30) = 4$

42. $\tan^2\theta + \tan\theta = 3$

43. $\sin^2\theta + 5\sin\theta = 3.5$

44. $\cos^2\theta - \cos 2\theta + 1 = 0$

45. **Write About It** How many solutions can a trigonometric equation have? Explain by using examples.

TEST PREP

46. Which values are solutions of $2\cos\theta + \sqrt{3} = 2\sqrt{3}$ for $0° \le \theta < 360°$?
    - (A) 30° or 150°
    - (B) 30° or 330°
    - (C) 60° or 120°
    - (D) 60° or 320°

47. Which gives an approximate solution to $5\tan\theta - \sqrt{3} = \tan\theta$ for $-90° \le \theta \le 90°$?
    - (F) −23.4°
    - (G) −19.1°
    - (H) 19.1°
    - (J) 23.4°

48. Which value for θ is NOT a solution to $\sin^2\theta = \sin\theta$?
    - (A) 0°
    - (B) 90°
    - (C) 180°
    - (D) 270°

49. Which gives all of the solutions of $\cos\theta - 1 = -\frac{1}{2}$ for $0 \le \theta < 2\pi$?
    - (F) $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$
    - (G) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$
    - (H) $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$
    - (J) $\frac{\pi}{3}$ or $\frac{5\pi}{3}$

50. Which gives the solution to $\sin^2\theta - \sin\theta - 2 = 0$ for $0° \le \theta < 360°$?
    - (A) 90°
    - (B) 270°
    - (C) 90° or 270°
    - (D) No solution

51. **Short Response** Solve $2\cos^2\theta + \cos\theta - 2 = 0$ algebraically. Show the steps in the solution process.

CHALLENGE AND EXTEND

Solve each equation algebraically for $0° \le \theta < 360°$.

52. $9\cos^3\theta - \cos\theta = 0$

53. $4\cos^3\theta - \cos\theta = 0$

54. $16\sin^4\theta - 16\sin^2\theta + 3 = 0$

55. $\sin^2\theta - 4.5\sin\theta = 2.5$

56. $|\sin\theta| = \frac{1}{2}$

57. $|\cos\theta| = \frac{\sqrt{3}}{2}$

SPIRAL REVIEW

Order the given numbers from least to greatest. *(Lesson 1-1)*

58. $\frac{\sqrt{3}}{2}, -1, 0.8\overline{6}, 1, \frac{5}{6}$

59. $2\sqrt{5}, \frac{19}{4}, 4.\overline{47}, \sqrt{21}, \frac{\pi}{0.65}$

60. **Technology** An e-commerce company constructed a Web site for a local business. Each time a customer purchases a product on the Web site, the e-commerce company receives 5% of the sale. Write a function to represent the e-commerce company's revenue based on total website sales per day. What is the value of the function for an input of 259, and what does it represent? *(Lesson 1-7)*

Simplify each expression by writing it only in terms of θ. *(Lesson 14-5)*

61. $\cos 2\theta - 2\cos^2\theta$

62. $\frac{\sin 2\theta}{2\sin\theta}$

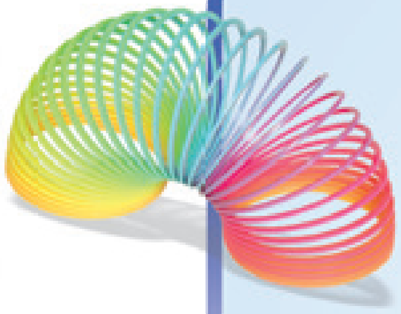
63. $\cos 2\theta + \sin^2\theta$

64. $\frac{\cos 2\theta + 1}{2}$

CHAPTER 14

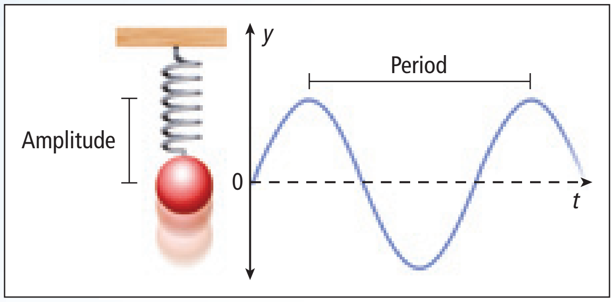
SECTION 14B

MULTI-STEP TEST PREP



Trigonometric Identities

Spring into Action Simple harmonic motion refers to motion that repeats in a regular pattern. The bouncing motion of a mass attached to a spring is a good example of simple harmonic motion. As shown in the figure, the displacement y of the mass as a function of time t in seconds is a sine or cosine function. The amplitude is the distance from the center of the motion to either extreme. The period is the time that it takes to complete one full cycle of the motion.



1. The displacement in inches of a mass attached to a spring is modeled by $y_1(t) = 3\sin\left(\frac{2\pi}{5}t + \frac{\pi}{2}\right)$, where t is the time in seconds. What is the amplitude of the motion? What is the period?

2. What is the initial displacement when $t = 0$ s? How long does it take until the displacement is 1.8 in.?

3. At what other times will the displacement be 1.8 in.?

4. Use trigonometric identities to write the displacement by using only the cosine function.

5. The displacement of a second mass attached to a spring is modeled by $y_2(t) = \sin\frac{2\pi}{5}t$. Both masses are set in motion at $t = 0$ s. How long does it take until both masses have the same displacement?

6. The displacement of a third mass attached to a spring is modeled by $y_3(t) = \cos\frac{\pi}{5}t$. The second and third masses are set in motion at $t = 0$ s. How long does it take until both masses have the same displacement?

READY TO GO ON?

CHAPTER 14

SECTION 14B

Quiz for Lessons 14-3 Through 14-6

14-3 Fundamental Trigonometric Identities

Prove each trigonometric identity.

1. $\sin^2\theta \sec\theta \csc\theta = \tan\theta$
2. $\sin(-\theta)\sec\theta\cot\theta = -1$
3. $\dfrac{\cot^2\theta - 1}{\cot^2\theta + 1} = 1 - 2\sin^2\theta$

Rewrite each expression in terms of a single trigonometric function.

4. $\cot\theta\sec\theta$
5. $\dfrac{1}{\cos(-\theta)}$
6. $\dfrac{\csc^2\theta}{\tan\theta + \cot\theta}$

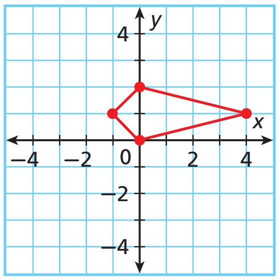
14-4 Sum and Difference Identities

Find the exact value of each expression.

7. $\cos\dfrac{5\pi}{12}$
8. $\sin(-75°)$
9. $\tan 75°$

Find each value if $\sin A = \frac{1}{4}$ with $90° < A < 180°$ and if $\cos B = \frac{12}{13}$ with $270° < B < 360°$.

10. $\sin(A + B)$
11. $\cos(A + B)$
12. $\cos(A - B)$
13. Find the coordinates, to the nearest hundredth, of the vertices of figure $ABCD$ with $A(0, 0)$, $B(4, 1)$, $C(0, 2)$, and $D(-1, 1)$ after a 120° rotation about the origin.



14-5 Double-Angle and Half-Angle Identities

Find each expression if $\cos\theta = -\frac{4}{5}$ and $180° < \theta < 270°$.

14. $\sin 2\theta$
15. $\cos 2\theta$
16. $\tan 2\theta$
17. $\sin\dfrac{\theta}{2}$
18. $\cos\dfrac{\theta}{2}$
19. $\tan\dfrac{\theta}{2}$
20. Use half-angle identities to find the exact value of cos 22.5°.

14-6 Solving Trigonometric Equations

21. Find all solutions of $1 + 2\sin\theta = 0$ where θ is in radians.

Solve each equation for $0° \le \theta < 360°$.

22. $\cos 2\theta + 2\cos\theta = 3$
23. $8\sin^2\theta - 2\sin\theta = 1$

Use trigonometric identities to solve each equation for $0 \le \theta < 2\pi$.

24. $\cos 2\theta = 3\cos\theta + 1$
25. $\sin^2\theta + \cos\theta + 1 = 0$
26. The average daily *minimum* temperature for Houston, Texas, can be modeled by $T(x) = -15.85\cos\frac{\pi}{6}(x - 1) + 76.85$, where T is the temperature in degrees Fahrenheit, x is the time in months, and $x = 0$ is January 1. When is the temperature 65°F? 85°F?

CHAPTER 14

Study Guide: Review

Vocabulary

Complete the sentences below with vocabulary words from the list above.

1. The shortest repeating portion of a periodic function is known as a(n) __?__.

2. The number of cycles in a given unit of time is called __?__.

3. The __?__ gives the length of a complete cycle for a periodic function.

4. A horizontal translation of a periodic function is known as a(n) __?__.

14-1 Graphs of Sine and Cosine *(pp. 990–997)*

EXAMPLES

■ **Using $f(x) = \cos x$ as a guide, graph $g(x) = -2\cos\frac{\pi}{2}x$. Identify the amplitude and period.**

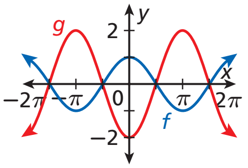
Step 1 Identify the period and amplitude.

Because $a = -2$, amplitude is $|a| = |-2| = 2$.

Because $b = \frac{\pi}{2}$, the period is $\frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{\pi}{2}\right|} = 4$.

Step 2 Graph.

The curve is reflected over the x-axis.

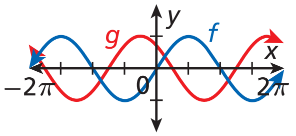


■ **Using $f(x) = \sin x$ as a guide, graph $g(x) = \sin\left(x - \frac{5\pi}{4}\right)$. Identify the x-intercepts and phase shift.**

The amplitude is 1. The period is 2π.

$-\frac{5\pi}{4}$ indicates a shift $\frac{5\pi}{4}$ units right.

The first x-intercept occurs at $\frac{\pi}{4}$. Thus, the intercepts occur at $\frac{\pi}{4} + n\pi$, where n is an integer.



EXERCISES

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the amplitude and period.

5. $f(x) = \cos 3x$

6. $g(x) = \cos\frac{1}{2}x$

7. $h(x) = -\frac{1}{3}\sin x$

8. $j(x) = 2\sin \pi x$

9. $f(x) = \frac{1}{2}\cos 2x$

10. $g(x) = \frac{\pi}{2}\sin \pi x$

Using $f(x) = \sin x$ or $f(x) = \cos x$ as a guide, graph each function. Identify the x-intercepts and phase shift.

11. $f(x) = \cos(x + \pi)$

12. $g(x) = \sin\left(x + \frac{\pi}{4}\right)$

13. $h(x) = \sin\left(x - \frac{3\pi}{2}\right)$

14. $j(x) = \cos\left(x + \frac{3\pi}{2}\right)$

Biology In photosynthesis, a plant converts carbon dioxide and water to sugar and oxygen. This process is studied by measuring a plant's carbon assimilation C (in micromoles of CO_2 per square meter per second). For a bean plant, $C(t) = 1.2\sin\frac{\pi}{12}(t - 6) + 7$, where t is time in hours starting at midnight.

15. Graph the function for two complete cycles.

16. What is the period of the function?

17. What is the maximum and at what time does it occur?

14-2 Graphs of Other Trigonometric Functions *(pp. 998–1003)*

EXAMPLE

- **Using $f(x) = \cot x$ as a guide, graph $g(x) = \cot \frac{\pi}{2}x$. Identify the period, x-intercepts, and asymptotes.**

Step 1 Identify the period.

Because $b = \frac{\pi}{2}$, the period is $\frac{\pi}{|b|} = \frac{\pi}{\left|\frac{\pi}{2}\right|} = 2$.

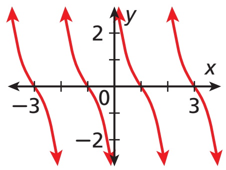
Step 2 Identify the x-intercepts.

The first x-intercept occurs at 1. Thus, the x-intercepts occur at $1 + 2n$, where n is an integer.

Step 3 Identify the asymptotes.

The asymptotes occur at $x = \frac{\pi n}{|b|} = \frac{\pi n}{\left|\frac{\pi}{2}\right|} = 2n$.

Step 4 Graph.



EXERCISES

Using $f(x) = \tan x$ or $f(x) = \cot x$ as a guide, graph each function. Identify the period, x-intercepts, and asymptotes.

18. $f(x) = \frac{1}{4}\tan x$

19. $g(x) = \tan \pi x$

20. $h(x) = \tan \frac{1}{2}\pi x$

21. $g(x) = 5\cot x$

22. $j(x) = -0.5\cot x$

23. $j(x) = \cot \pi x$

Using $f(x) = \cos x$ or $f(x) = \sin x$ as a guide, graph each function. Identify the period and asymptotes.

24. $f(x) = 2\sec x$

25. $g(x) = \csc 2x$

26. $h(x) = 4\csc x$

27. $j(x) = 0.2\sec x$

28. $h(x) = \sec(-x)$

29. $j(x) = -2\csc x$

14-3 Fundamental Trigonometric Identities *(pp. 1008–1013)*

EXAMPLES

- **Prove $\frac{\tan\theta}{1-\cos^2\theta} = \sec\theta\csc\theta$.**

$\frac{\left(\frac{\sin\theta}{\cos\theta}\right)}{(\sin^2\theta)} =$ *Modify the left side. Apply the ratio and Pythagorean identities.*

$\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\sin^2\theta}\right) =$ *Multiply by the reciprocal.*

$\left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right) =$ *Simplify.*

$\sec\theta\csc\theta$ *Reciprocal identities*

- **Rewrite $\frac{\cot\theta + \tan\theta}{\csc\theta}$ in terms of a single trigonometric function, and simplify.**

$(\cot\theta + \tan\theta)\sin\theta$ *Given.*

$\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)\sin\theta$ *Ratio identities*

$\frac{\cos^2\theta + \sin^2\theta}{\cos\theta}$ *Add fractions and simplify.*

$\frac{1}{\cos\theta} = \sec\theta$ *Pythagorean and reciprocal identities*

EXERCISES

Prove each trigonometric identity.

30. $\sec\theta\sin\theta\cot\theta = 1$

31. $\frac{\sin^2(-\theta)}{\tan\theta} = \sin\theta\cos\theta$

32. $(\sec\theta + 1)(\sec\theta - 1) = \tan^2\theta$

33. $\cos\theta\sec\theta + \cos^2\theta\csc^2\theta = \csc^2\theta$

34. $(\tan\theta + \cot\theta)^2 = \sec^2\theta + \csc^2\theta$

35. $\tan\theta + \cot\theta = \sec\theta\csc\theta$

36. $\sin^2\theta\tan\theta = \tan\theta - \sin\theta\cos\theta$

37. $\frac{\tan\theta}{1-\cos^2\theta} = \sec\theta\csc\theta$

Rewrite each expression in terms of a single trigonometric function, and simplify.

38. $\cot\theta\sec\theta$

39. $\frac{\sec\theta\sin\theta}{\cot\theta}$

40. $\frac{\tan(-\theta)}{\cot\theta}$

41. $\frac{\cos\theta\cot\theta}{\csc^2\theta - 1}$

14-4 Sum and Difference Identities *(pp. 1014–1019)*

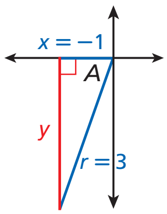
EXAMPLES

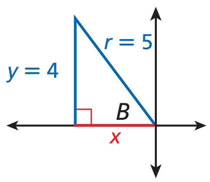
- **Find $\sin(A+B)$ if $\cos A = -\frac{1}{3}$ with $180° < A < 270°$ and if $\sin B = \frac{4}{5}$ with $90° < B < 180°$.**

Step 1 Find $\sin A$ and $\cos B$ by using the Pythagorean Theorem with reference triangles.

$180° < A < 270°$ $\quad\quad$ $90° < B < 180°$

$\cos A = -\frac{1}{3}$ $\quad\quad$ $\sin B = \frac{4}{5}$



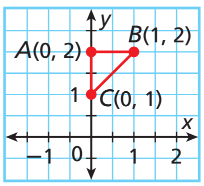


$y = -\sqrt{8}, \sin A = \frac{-\sqrt{8}}{3}$ $\quad\quad$ $x = -3, \cos B = \frac{-3}{5}$

Step 2 Use the angle-sum identity.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$= \left(\frac{-\sqrt{8}}{3}\right)\left(\frac{-3}{5}\right) + \left(-\frac{1}{3}\right)\left(\frac{4}{5}\right)$$
$$= \frac{3\sqrt{8} - 4}{15}$$

- **Find the coordinates to the nearest hundredth of the vertices of figure ABC with $A(0, 2)$, $B(1, 2)$, and $C(0, 1)$ after a 60° rotation about the origin.**



Step 1 Write matrices for a 60° rotation and for the points in the figure.

$$R_{60°} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \quad \textit{Rotation matrix}$$

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \quad \textit{Matrix of points}$$

Step 2 Find the matrix product.

$$R_{60°} \times S = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$
$$\approx \begin{bmatrix} -1.73 & -1.23 & -0.87 \\ 1 & 1.87 & 0.5 \end{bmatrix}$$

Step 3 The approximate coordinates of the points after a 60° rotation are $A'(-1.73, 1)$, $B'(-1.23, 1.87)$, and $C'(-0.87, 0.5)$.

EXERCISES

Find the exact value of each expression.

42. $\sin \frac{19\pi}{12}$ $\quad$ **43.** $\cos 165°$

44. $\cos 15°$ $\quad$ **45.** $\tan \frac{\pi}{12}$

Find each value if $\tan A = \frac{3}{4}$ with $0° < A < 90°$ and if $\tan B = -\frac{5}{12}$ with $90° < B < 180°$.

46. $\sin(A+B)$ $\quad$ **47.** $\cos(A+B)$

48. $\tan(A-B)$ $\quad$ **49.** $\tan(A+B)$

50. $\sin(A-B)$ $\quad$ **51.** $\cos(A-B)$

Find each value if $\sin A = \frac{\sqrt{7}}{4}$ with $0° < A < 90°$ and if $\cos B = -\frac{5}{13}$ with $90° < B < 180°$.

52. $\sin(A+B)$ $\quad$ **53.** $\cos(A+B)$

54. $\tan(A-B)$ $\quad$ **55.** $\tan(A+B)$

56. $\sin(A-B)$ $\quad$ **57.** $\cos(A-B)$

Find the coordinates, to the nearest hundredth, of the vertices of figure $ABCD$ with $A(0, 0)$, $B(3, 0)$, $C(4, 2)$, and $D(1, 2)$ after each rotation about the origin.

58. 30° rotation $\quad$ **59.** 45° rotation

60. 60° rotation $\quad$ **61.** 90° rotation

Find the coordinates, to the nearest hundredth, of the vertices of figure $ABCD$ with $A(0, 0)$, $B(5, 2)$, $C(0, 4)$, and $D(-5, 2)$ after each rotation about the origin.

62. 120° rotation $\quad$ **63.** 180° rotation

64. 240° rotation $\quad$ **65.** 270° rotation

14-5 Double-Angle and Half-Angle Identities *(pp. 1020–1026)*

EXAMPLES

Find each expression if $\sin\theta = \frac{1}{4}$ and $270° < \theta < 360°$.

- $\sin 2\theta$

For $\sin\theta = \frac{1}{4}$ in QIV, $\cos\theta = -\frac{\sqrt{15}}{4}$.

$\sin 2\theta = 2\sin\theta\cos\theta$ *Identity for sin 2θ*

$= 2\left(\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{15}}{8}$ *Substitute.*

- $\cos\frac{\theta}{2}$

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$ *Identity for $\cos\frac{\theta}{2}$*

$= -\sqrt{\frac{1+\left(-\frac{\sqrt{15}}{4}\right)}{2}}$ *Negative for $\cos\frac{\theta}{2}$ in QII*

$= -\sqrt{\left(\frac{4-\sqrt{15}}{4}\right)\left(\frac{1}{2}\right)} = -\frac{\sqrt{4-\sqrt{15}}}{\sqrt{8}}$

EXERCISES

Find each expression if $\tan\theta = \frac{4}{3}$ and $0° < \theta < 90°$.

66. $\sin 2\theta$

67. $\cos 2\theta$

68. $\tan\frac{\theta}{2}$

69. $\sin\frac{\theta}{2}$

Find each expression if $\cos\theta = \frac{3}{4}$ and $\frac{3\pi}{2} < \theta < 2\pi$.

70. $\tan 2\theta$

71. $\cos 2\theta$

72. $\cos\frac{\theta}{2}$

73. $\sin\frac{\theta}{2}$

Use half-angle identities to find the exact value of each trigonometric expression.

74. $\sin\frac{\pi}{12}$

75. $\cos 75°$

14-6 Solving Trigonometric Equations *(pp. 1027–1033)*

EXAMPLES

- **Find all of the solutions of $3\cos\theta - \sqrt{3} = \cos\theta$.**

$3\cos\theta - \sqrt{3} = \cos\theta$

$3\cos\theta - \cos\theta = \sqrt{3}$ *Subtract tan θ.*

$2\cos\theta = \sqrt{3}$ *Combine like terms.*

$\cos\theta = \frac{\sqrt{3}}{2}$ *Divide by 2.*

$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ *Apply the inverse cosine.*

$\theta = 30°$ or $330°$ *Find θ for $0° \le \theta < 360°$.*

$\theta = 30° + 360°n$

or $330° + 360°n$

- **Solve $6\sin^2\theta + 5\sin\theta = -1$ for $0° \le \theta < 360°$.**

$6\sin^2\theta + 5\sin\theta + 1 = 0$ *Set equal to 0.*

$(2\sin\theta + 1)(3\sin\theta + 1) = 0$ *Factor.*

$\sin\theta = -1$ or $\sin\theta = 3$ *Zero Product Property*

$\theta = 210°, 330°$ *$\sin\theta = 3$ has no solution since $-1 \le \sin\theta \le 1$.*

or $\approx 199.5°, 340.5°$

EXERCISES

Find all of the solutions of each equation.

76. $\sqrt{2}\cos\theta + 1 = 0$

77. $\cos\theta = 2 + 3\cos\theta$

78. $\tan^2\theta + \tan\theta = 0$

79. $\sin^2\theta - \cos^2\theta = \frac{1}{2}$

Solve each equation for $0 \le \theta < 2\pi$.

80. $2\cos^2\theta - 3\cos\theta = 2$

81. $\cos^2\theta + 5\cos\theta - 6 = 0$

82. $\sin^2\theta - 1 = 0$

83. $2\sin^2\theta - \sin\theta = 3$

Use trigonometric identities to solve each equation for $0 \le \theta < 2\pi$.

84. $\cos 2\theta = \cos\theta$

85. $\sin 2\theta + \cos\theta = 0$

86. **Earth Science** The number of minutes of daylight for each day of the year can be modeled with a trigonometric function. For Washington, D.C., S is the number of minutes of daylight in the model $S(d) = 180\sin(0.0172d - 1.376) + 720$, where d is the number of days since January 1.

a. What is the maximum number of daylight minutes, and when does it occur?

b. What is the minimum number of daylight minutes, and when does it occur?

CHAPTER 14

CHAPTER TEST

1. Using $f(x) = \cos x$ as a guide, graph $g(x) = \frac{1}{2}\cos 2x$. Identify the amplitude and period.
2. Using $f(x) = \sin x$ as a guide, graph $g(x) = \sin\left(x + \frac{\pi}{3}\right)$. Identify the x-intercepts and phase shift.
3. A torque τ in newton meters (N•m) applied to an object is given by $\tau(\theta) = Fr\sin\theta$, where r is the length of the lever arm in meters, F is the applied force in newtons, and θ is the angle between F and r in degrees. Find the amount and angle for the maximum torque and the minimum torque for a lever arm of 0.5 m and a force of 500 newtons, where $0° \leq \theta \leq 90°$.
4. Using $f(x) = \tan x$ as a guide, graph $g(x) = 2\tan \pi x$. Identify the period, x-intercepts, and asymptotes.
5. Using $f(x) = \cot x$ as a guide, graph $g(x) = \cot 4x$. Identify the period, x-intercepts, and asymptotes.
6. Using $f(x) = \sin x$ as a guide, graph $g(x) = \frac{1}{4}\csc x$. Identify the period and asymptotes.
7. Prove the trigonometric identity $\cot\theta = \cos^2\theta\sec\theta\csc\theta$.

Rewrite each expression in terms of a single trigonometric function.

8. $(\sec\theta + 1)(\sec\theta - 1)$

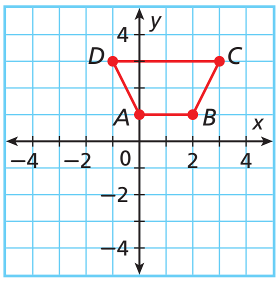
9. $\dfrac{\sin(-\theta)}{\cos(-\theta)}$

Find each value if $\tan A = \frac{3}{4}$ with $0° < A < 90°$ and if $\sin B = -\frac{12}{13}$ with $180° < B < 270°$.

10. $\sin(A + B)$

11. $\cos(A - B)$

12. Find the coordinates, to the nearest hundredth, of the vertices of figure $ABCD$ with $A(0, 1)$, $B(2, 1)$, $C(3, 3)$, and $D(-1, 3)$ after a 30° rotation about the origin.



Find each expression if $\tan\theta = -\frac{12}{5}$ and $90° < \theta < 180°$.

13. $\sin 2\theta$

14. $\cos 2\theta$

15. $\cos\dfrac{\theta}{2}$

16. Use half-angle identities to find the exact value of $\sin\dfrac{3\pi}{8}$.
17. Find all of the solutions of $\tan\theta + \sqrt{3} = 0$.
18. Solve $2\sin^2\theta = \sin\theta$ for $0° \leq \theta < 360°$.
19. Use trigonometric identities to solve $2\cos^2\theta + 3\sin\theta = 0$ for $0 \leq \theta < 2\pi$.
20. The voltage at a wall plug in a home can be modeled by $V(t) = 156\sin 2\pi(60t)$, where V is the voltage in volts and t is time in seconds. At what times is the voltage equal to 110 volts?

COLLEGE ENTRANCE EXAM PRACTICE

FOCUS ON SAT MATHEMATICS SUBJECT TESTS

To help decide which standardized tests you should take, make a list of colleges that you might like to attend. Find out the admission requirements for each school. Make sure that you register for and take the appropriate tests early enough for colleges to receive your scores.

You may want to time yourself as you take this practice test. It should take you about 6 minutes to complete.

HOT TIP! If your calculator malfunctions while you are taking an SAT Mathematics Subject Test, you may be able to have your score for that test canceled. To do so, you must inform a supervisor at the test center immediately when the malfunction occurs.

1. Identify the range of $f(x) = 3\sin x$.

(A) $-1 \le f(x) \le 1$

(B) $-3 < f(x) < 3$

(C) $0 \le f(x) \le 3$

(D) $-3 \le f(x) \le 3$

(E) $-\infty < f(x) < \infty$

2. If $2\sin^2\theta + 5\sin\theta = 3$, what could the value of θ be?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$

(D) $\frac{7\pi}{6}$

(E) $\frac{11\pi}{6}$

3. If $\sec\theta = 4$, what is $\tan^2\theta$?

(A) $\frac{1}{16}$

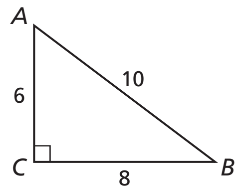
(B) 3

(C) 5

(D) 15

(E) 17

4. Given the figure, what is the value of $\cos(A - B)$?



Not to scale

(A) 0

(B) $\frac{7}{25}$

(C) $\frac{24}{25}$

(D) 1

(E) $\frac{28}{25}$

5. If $\sin\theta = \frac{7}{9}$, what is $\cos 2\theta$?

(A) $-\frac{8\sqrt{2}}{9}$

(B) $-\frac{17}{81}$

(C) $\frac{17}{81}$

(D) $\frac{56\sqrt{2}}{81}$

(E) $\frac{8\sqrt{2}}{9}$

TEST TACKLER

Standardized Test Strategies

Multiple Choice: Choose Answer Combinations

You may be given a test item in which you are asked to choose from a combination of statements. To answer these types of test items, try comparing each given statement with the question and determining whether the statement is true or false. If you determine that more than one of the statements is correct, choose the combination that contains each correct statement.

EXAMPLE 1

Which exact solution makes the equation $2\cos^2\theta - 3\cos\theta = 2$ true?

I. $\theta = 2°$
II. $\theta = 120°$
III. $\theta = 240°$

Look at each statement separately, and determine if it is true or false.

Ⓐ I only
Ⓑ II and III
Ⓒ II only
Ⓓ I, II, and III

As you consider each statement, mark it true or false.

Consider statement I: Substitute 2° for θ in the equation.

$2\cos^2(2°) - 3\cos(2°) \approx -1.0006$
$\neq 2$

Statement I is false.
So, the answer is ***not*** choice A or D.

Consider statement II: Substitute 120° for θ in the equation.

$2\cos^2(120°) - 3\cos(120°) = 2$

Statement II is true.
The answer *could be* choice B or C.

Consider statement III: Substitute 240° for θ in the equation.

$2\cos^2(240°) - 3\cos(240°) = 2$

Statement III is true.

Because both statements II and III are true, choice B is the correct response.

You can also use a table to keep track of whether the statements are true or false.

Statement	True/False
I	False
II	True
III	True

HOT TIP! As you eliminate a statement, cross out the corresponding answer choice(s).

Read each test item and answer the questions that follow.

Item A

Which expression is equivalent to $\tan^2\theta$?

I. $\sec^2\theta - 1$

II. $\sec^2\theta + 1$

III. $\dfrac{1}{\csc^2\theta - 1}$

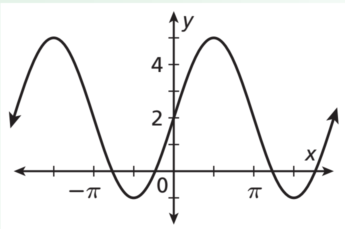
IV. $\dfrac{1-\cos^2\theta}{1-\sin^2\theta}$

Ⓐ I and II
Ⓑ II and III
Ⓒ I and III
Ⓓ I, III, and IV

1. What are some of the identities that involve the tangent function?
2. Determine whether statements I, II, III, and IV are true or false. Explain your reasoning.
3. Sally realized that statement III was true and selected choice B as her response. Do you agree? If not, what would you have done differently?

Item B

For the graph of $f(x) = 3\sin x + 2$, which of the statements are true?



I. The function has a period of $\dfrac{2\pi}{3}$.

II. The function has an amplitude of 3.

III. The function has a period of 2π.

Ⓕ I only
Ⓖ III only
Ⓗ II only
Ⓙ II and III

4. How do you determine the period of a trigonometric function?
5. How do you determine the amplitude of a trigonometric function?
6. Using your response to Problems 4 and 5, which of the three statements are true? Explain.

Item C

Which identities do you need to use to prove that $\tan\theta\csc\theta = \sec\theta$?

I. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

II. $\sec^2\theta = \tan^2\theta + 1$

III. $\csc\theta = \dfrac{1}{\sin\theta}$

Ⓐ I only
Ⓑ II only
Ⓒ I and II
Ⓓ I and III

7. Is statement I true or false? Can any answer choice be eliminated? Explain.
8. Is statement II true or false? Should you select the answer choice yet? Explain.
9. Is statement III true or false? Explain.
10. Which combination of statements is correct? How do you know?

Item D

For the graph of the function $f(x) = \sec 4x$, which are equations of some of the asymptotes?

I. $x = \dfrac{\pi}{8}$

II. $x = \dfrac{\pi}{2}$

III. $x = -\dfrac{3\pi}{4}$

Ⓕ I only
Ⓖ II and III
Ⓗ I, II, and III
Ⓙ I and III

11. Create a table, and determine whether each statement is true or false.
12. Using your table, which choice is the most accurate?

CHAPTER 14

STANDARDIZED TEST PREP

go.hrw.com
State Test Practice Online
KEYWORD: MB7 TestPrep

CUMULATIVE ASSESSMENT, CHAPTERS 1–14

Multiple Choice

1. What is the exact value of tan 15°?

Ⓐ $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

Ⓑ $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Ⓒ $2+\sqrt{3}$

Ⓓ $2-\sqrt{3}$

2. Where do the asymptotes occur in the given equation?

$y = \frac{1}{3}\cot 2x$

Ⓕ $2\pi n$

Ⓖ $\dfrac{\pi n}{2}$

Ⓗ $3\pi n$

Ⓙ $\dfrac{\pi n}{3}$

3. What is the period of the given equation?

$y = 5\cos\frac{1}{3}x$

Ⓐ $\dfrac{2\pi}{5}$

Ⓑ $\dfrac{5}{3}$

Ⓒ $\dfrac{2\pi}{3}$

Ⓓ 6π

4. A movie has 14 dialogue scenes and 10 action scenes. If these are the only two types of scenes, what is the probability that a randomly selected scene will be an action scene?

Ⓕ $\dfrac{5}{12}$

Ⓖ $\dfrac{7}{12}$

Ⓗ $\dfrac{5}{7}$

Ⓙ $\dfrac{7}{5}$

5. What is the value of $f(x) = 3x^3 + 4x^2 + 7x + 10$ for $x = -2$?

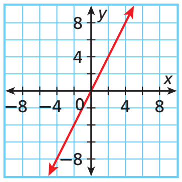
Ⓐ −44

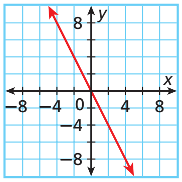
Ⓑ −12

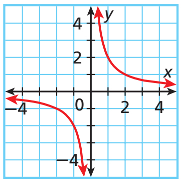
Ⓒ 0

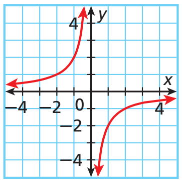
Ⓓ 36

6. Which is the graph of a function when $y = 2$ and $x = -1$ if y varies inversely as x?

Ⓕ


Ⓗ


Ⓖ


Ⓙ


7. What is the exact value of cos157.5° using half-angle identities?

Ⓐ $-\dfrac{\sqrt{2-\sqrt{2}}}{2}$

Ⓑ $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

Ⓒ $-\dfrac{\sqrt{2+\sqrt{2}}}{2}$

Ⓓ $\dfrac{\sqrt{2+\sqrt{2}}}{2}$

8. What are the coordinates of the vertex of the parabola given by the equation $f(x) = -x^2 + 6x - 4$?

Ⓕ (0, −4)

Ⓖ (−3, −13)

Ⓗ (−3, 5)

Ⓙ (3, 5)

9. Which is a solution of $2\cos\theta = 2\sin\theta$ for $\pi \leq \theta \leq 3\pi$?

Ⓕ $\frac{\pi}{4}$

Ⓖ π

Ⓗ $\frac{5\pi}{4}$

Ⓙ 3π

10. Which is the equation of a circle with center $(3, 2)$ and radius 5?

Ⓐ $25 = (x - 3)^2 + (y - 2)^2$

Ⓑ $5 = (x - 3)^2 + (y - 2)^2$

Ⓒ $25 = (x + 3)^2 + (y + 2)^2$

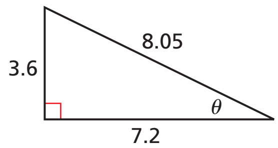
Ⓓ $5 = (x + 3)^2 + (y + 2)^2$

Gridded Response

11. What is the value of x?

$5\sqrt{2x - 7} + 4 = 9$

12. What is the value of $\cos\theta$? Round to the nearest thousandth.



In Item 13, the answer will be a *y*-value only. It will be quickest and most efficient to isolate *x* in one equation and substitute for *x* in the second equation because then the first variable for which you obtain a value will be *y*.

13. What is the y-value of the solution of the following system of nonlinear equations?

$$\begin{cases} x - 4 = \frac{1}{4}y^2 \\ \frac{(x+1)^2}{25} + \frac{y^2}{36} = 1 \end{cases}$$

14. Find the sum of the arithmetic series $\sum_{k=1}^{14}(3k - 5)$.

Short Response

15. The chart below shows the names of the students on the academic bowl team.

Robin	Drew	Jim
Greg	Sarah	Mindy
Ashley	Tina	Justin
David	Amy	Kevin

a. Only 2 students can be chosen for the final academic bowl. How many different ways can the students be selected?

b. Explain why you solved the problem the way that you did.

16. Given the sequence:

4, 12, 36, 108, 324, ...

a. Write the explicit rule for the nth term.

b. Find the 10th term.

Extended Response

17. The chart below shows the grades in Mr. Bradshaw's class.

90	85	72	86	94	96
85	95	94	68	71	85
93	98	84	83	80	89

Round each answer to the nearest tenth.

a. Find the mean.

b. Find the median.

c. Find the mode.

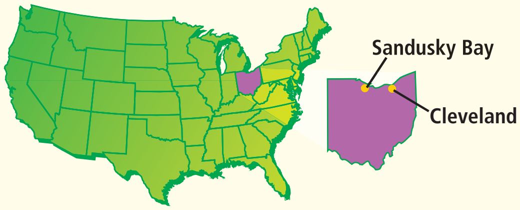
d. Find the variance.

e. Find the standard deviation.

f. Find the range.

Problem Solving on Location

OHIO

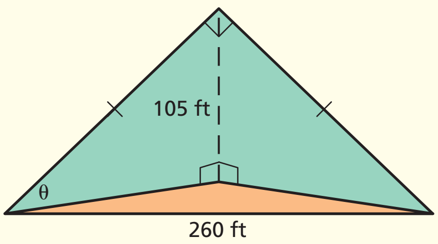


The Rock and Roll Hall of Fame

The Rock and Roll Hall of Fame in downtown Cleveland traces the history of rock music through live performances and interactive exhibits. Designed by renowned architect I. M. Pei, the 50,000-square-foot exhibition space houses everything from vintage posters to handwritten lyrics to John Lennon's report card.

Choose one or more strategies to solve each problem.
For 1 and 2, use the diagram.

1. Visitors enter the museum through an enormous glass entryway in the shape of a tetrahedron. The figure shows the dimensions of the tetrahedron. What is the pitch of the tetrahedron's slanted facade? (*Hint:* The pitch is shown in the figure by angle θ.)



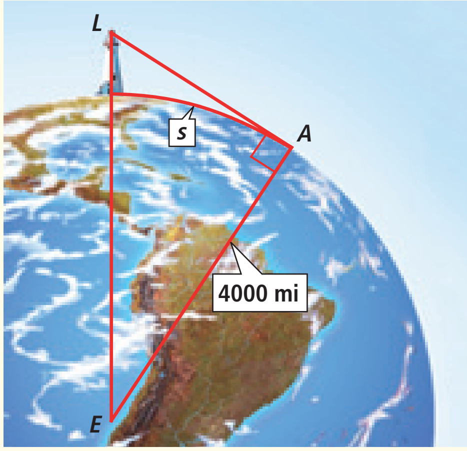
2. What is the area of the triangular floor space enclosed by the glass tetrahedron?
3. The Hall of Fame exhibits are displayed in an eight-story, 162-foot tower. Pei originally designed a 200-foot tower but had to reduce its height in order to meet the requirements of a nearby airport. From the top of the existing tower, an observer sights the entrance to the museum's plaza with an angle of depression of 18°. What would be the angle of depression to the entrance of the plaza from Pei's original tower?

Problem Solving Strategies

- Draw a Diagram
- Make a Model
- Guess and Test
- Work Backward
- Find a Pattern
- Make a Table
- Solve a Simpler Problem
- Use Logical Reasoning
- Use a Venn Diagram
- Make an Organized List

Marblehead Lighthouse

Since its construction in 1821, Marblehead Lighthouse has stood at the entrance to Sandusky Bay, guiding sailors along Lake Erie's rocky shores. The 65-foot tower is one of Ohio's best-known landmarks and the oldest continuously operating lighthouse on the Great Lakes.



Choose one or more strategies to solve each problem.

1. The range of a lighthouse is the maximum distance at which its light is visible. In the figure, point A is the farthest point from which it is possible to see the light at the top of the lighthouse L. The distance along Earth s is the range. Assuming that the radius of Earth is 4000 miles, find the range of Marblehead Lighthouse.

2. In 1897, a new lighting system was installed in the lighthouse. A set of descending weights rotated the tower's lantern to produce a flashing light. The rotation could be modeled by the function $f(x) = \sin\frac{\pi}{5}x$, where x is the time in seconds since the weights were released. The light briefly flashed on whenever $f(x) = 1$. How many times per minute did the light flash?

3. Today the flashing light of Marblehead Lighthouse can be modeled by $g(x) = \sin\frac{\pi}{3}x$. How many seconds are there between each flash? Does the light flash more or less frequently than in 1897?